Fair Secure Domination in Graphs

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Abstract: *Let G be a connected simple graph. A dominating set* $S \subseteq V(G)$ *is a fair dominating set in* G *if for every two distinct vertices* u *and* v *from* $V(G) \setminus S$, $|N(u) \cap S| = |N(v) \cap S|$, *that is, every two distinct vertices not in* S *have the same number of neighbors from* S. *A fair dominating set* $S \subseteq V(G)$ *is a fair secure dominating set if for each* $u \in V(G) \setminus S$, *there exists* $v \in S$ *such that* $uv \in E(G)$ *and the set* $(S \setminus \{v\}) \cup \{u\}$ *is a dominating set of* G. *The minimum cardinality of a fair secure dominating set of* G, *denoted by* $\gamma_{fsd}(G)$, *is called the fair secure domination number of* G. *In this paper, we initiate the study of the concept and give some realization problems. In particular, we show that given positive integers* k, m, *and* $n \geq 2$ *such that* $1 \leq k \leq m \leq n-1$, *there exists a connected nontrivial graph* G *with* $|V(G)| = n$ *such that* $\gamma_{fd}(G) = k$ *and* $\gamma_{fsd}(G) = m$. *Further, we show the characterization of the fair secure dominating set in the join of two nontrivial connected graphs.*

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I. INTRODUCTION

Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [1]. Following an article [2] by Ernie Cockayne and Stephen Hedetniemi in 1977, the domination in graphs became an area of study by many researchers. A subset S of $V(G)$ is a dominating set of G if for every $v \in V(G) \setminus S$, there exists $x \in S$ such that xv \in $E(G)$, that is, $N[S] = V(G)$. The domination number $\gamma(G)$ of G is the smallest cardinality of a dominating set of G. Some studies on domination in graphs were found in the papers [3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20].

One variant of domination is the secure domination in graphs. A dominating set S of $V(G)$ is a secure dominating set of G if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. The minimum cardinality of a secure dominating set of G, denoted by $\gamma_s(G)$, is called the secure domination number of G. A secure dominating set of cardinality $\gamma_s(G)$ is called a γ_s-set of G. Secure dominating set was introduced by E.J. Cockayne et.al [21]. Secure dominating sets can be applied as protection strategies by minimizing the number of guards to secure a system so as to be cost effective as possible. Some variants of secure domination in graphs were found in the papers [22,23,24,25,26,27,28,29].

In 2011, Caro, Hansberg and Henning [20] introduced fair domination and k-fair domination in graphs. A dominating subset S of $V(G)$ is a fair dominating set in G if all the vertices not in S are dominated by the same number of vertices from S, that is, $|N(u) \cap S| = |N(v) \cap S|$ for every two distinct vertices u and v from $V(G) \setminus S$ and a subset S of $V(G)$ is a k-fair dominating set in G if for every vertex $v \in V(G) \setminus S$, $|N(v) \cap S| = k$. The minimum cardinality of a fair dominating set of G, denoted by $\gamma_{fd}(G)$, is called the fair domination number of G. A fair dominating set of cardinality $\gamma_{fd}(G)$ is called γ_{fd}-set. Some studies on fair domination in graphs were found in the paper [31,32].

In this paper, we introduce the study of fair secure dominating set. A fair dominating set $S \subseteq V(G)$ is a fair secure dominating set if for each $u \in V(G) \setminus S$, there exists $v \in S$ such that $uv \in E(G)$ and the set $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of G. The minimum cardinality of a fair secure dominating set of G, denoted by $\gamma_{fsd}(G)$, is called the fair secure domination number of G. A fair secure dominating set of cardinality $\gamma_{fsd}(G)$ is called γ_{fsd}-set.

For the general terminology in graph theory, readers may refer to [33]. A graph G is a pair $(V(G), E(G))$, where $V(G)$ is a finite nonempty set called the vertex-set of G and $E(G)$ is a set of unordered pairs $\{u, v\}$ (or simply uv) of distinct elements from $V(G)$ called the edge-set of G. The elements of $V(G)$ are called vertices and the cardinality $|V(G)|$ of $V(G)$ is the order of G. The elements of $E(G)$ are called edges and the cardinality $|E(G)|$ of $E(G)$ is the size of G. If $|V(G)| = 1$, then G is called a trivial graph. If $E(G) = \emptyset$, then G is called an empty graph. The open neighborhood of a vertex $v \in V(G)$ is the set $N_G(v) = \{u \in V(G): uv \in E(G)\}$. The elements of $N_G(v)$ are called neighbors of v. The closed neighborhood of $v \in V(G)$ is the set $N_G[v] = N_G(v) \cup \{v\}$. If $X \subseteq V(G)$, the open neighborhood of X in G is the set $N_G(X) = \cup_{v\in X} N_G(v)$. The closed neighborhood of X in G is the set $N_G[X] = \cup_{v\in X} N_G[v] = N_G(X) \cup X$. When no confusion arises, $N_G[x]$ $[resp. N_G(x)]$ will be denoted by $N[x]$ [resp. $N(x)$].

II. RESULTS

Remark 2.1 [29] If $G \neq \overline{K}_n$, then $\gamma_{fd}(G) = min\{\gamma_{kfd}(G)\}$, where the minimum is taken over all integers k where $1 \leq k \leq |V(G)| - 1$.

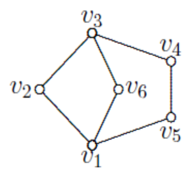

Figure 1: A graph G with $\gamma_{1fd}(G) = 3$, $\gamma_{2fd}(G) = 3$ and $\gamma_{3fd}(G) = 4$

Example 2.2 Consider the graph in Figure [1]. Then the set $S_1 = \{v_1, v_5, v_6\}$ is a γ_{1fd}-set of G, the set $S_2 = \{v_1, v_3, v_5\}$ is a γ_{2fd}-set of G and the set $S_3 = \{v_2, v_4, v_5, v_6\}$ is a γ_{3fd}-set of G. Hence, $\gamma_{1fd}(G) = 3$, $\gamma_{2fd}(G) = 3$, $\gamma_{3fd}(G) = 4$ and $\gamma_{fd}(G) = 3$. Further, observed that S_1 is also a γ_s-set of G. Therefore, S_1 is a γ_{fsd}-set of G. Hence, $\gamma_{fsd}(G) = 3$.

Remark 2.3 A fair secure dominating set of a graph G is a fair dominating set and a secure dominating set of G.

From the definition of a fair secure domination number $\gamma_{frd}(G)$ of G, the following result is immediate.

Remark 2.4 Let G be any connected graph of order $n \geq 2$. Then

(i) $1 \leq \gamma_{fsd}(G) \leq n - 1$ and
(ii) $\gamma(G) \leq \gamma_{fd}(G) \leq \gamma_{fsd}(G)$.

A complete graph of order n, denoted by K_n, is the graph in which every pair of its distinct vertices are joined by an edge.

Remark 2.5 Let $n \geq 2$. The $\gamma_{fsd}(K_n) = 1$.

The complement of a graph K_n, denoted by $\overline{K}_n$, is a graph with $V(\overline{K}_n) = V(K_n)$ such that two vertices in $\overline{K}_n$ are adjacent if and only they are not adjacent in K_n.

The path P_n of order n is the graph with distinct vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3, \ldots, v_{n-1}v_n$. In this case, P_n is also called a v_1-v_n path or the path $P(v_1, v_n)$.

Remark 2.6 Let $n \geq 2$. Then $\gamma_{fsd}(P_n) = \begin{cases} \frac{n}{2} & if \quad n \text{ is even} \\ \frac{n+1}{2} & if \quad n \text{ is odd} \end{cases}$

The next result says that the value of the parameter $\gamma_{frd}(G)$ ranges over all positive integers from $1, 2, \ldots, n$ where n is the order of G.

Theorem 2.7 Let k, m, and $n \geq 2$ be positive integers such that $1 \leq k \leq m \leq n-1$. Then there exists a connected nontrivial graph G with $|V(G)| = n$ such that $\gamma_{fd}(G) = k$ and $\gamma_{fsd}(G) = m$.

Proof: Consider the following cases:

Case 1. Suppose that $1 = k = m \leq n-1$.

Let $G = K_n$. Clearly, $|V(G)| = n$ and $\gamma_{fd}(G) = 1 = \gamma_{fsd}(G)$.

Case 2. Suppose that $1 = k < m \leq n-1$.

Let $G = \langle v \rangle + \langle V(K_{n-r-1}) \cup V(\overline{K}_r) \rangle$ such that $m = r + 1$. Then $A = \{v\}$ is a fair dominating set of G and $B = \{v\} \cup V(\overline{K}_r)$ is the minimum fair secure dominating set of G. Thus, $|V(G)| = 1 + (n - r - 1) + r = n$, $\gamma_{fd}(G) = |A| = 1 = k$, and $\gamma_{fsd}(G) = |B| = 1 + r = m$.

Case 3. Suppose that $1 < k = m \leq n-1$.

Let $G = H \circ I$ where H is a connected graph (of order $k \geq 1$) and I is a complete graph (of order $r \geq 1$). If $n = k(r+1)$, then the set $V(H)$ is a minimum fair dominating set and a minimum fair secure dominating set of a graph G. Hence $\gamma_{fd}(G) = |V(H)| = k = m = \gamma_{fsd}(G)$ and

$$|V(G)| = |V(H \circ I)| = |V(H)| + \sum_{v \in V(H)} |V(I^v)| = |V(H)| + |V(H)||V(I)| = k + kr = k(1+r) = n$$

Case 4. Suppose that $1 < k < m < n-1$.

Let $V(P_r) = \{x_1, x_2, \ldots, x_r\}$, $V(P_s) = \{y_1, y_2, \ldots, y_s\}$ with $k = 2$, $n - 2 = r + s = m$, $r \geq 1$, $s \geq 1$, and $r = s$. Then let G be a graph obtained from $H_1 = \langle v_1 \rangle + P_r$ and $H_2 = \langle v_2 \rangle + P_s$ with $v_1v_2 \in E(G)$ (see Figure 2).

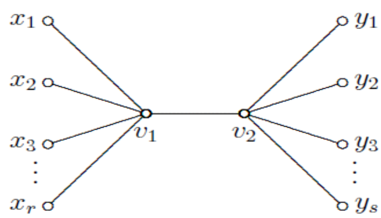

Figure 2: A graph G with $1 < k < m < n-1$.

The set $A = \{v_1, v_2\}$ is a γ_{fd}-set and $B = V(P_r) \cup V(P_s)$ is a γ_{fsd}-set of G. Thus, $|V(G)| = r + s + 2 = n$, $\gamma_{fd}(G) = |A| = 2 = k$, and $\gamma_{fsd}(G) = |B| = r + s = n - 2 = m$.

This proves the assertion. ■

The following result is a direct consequence of Theorem 2.7.

Corollary 2.8 The difference $\gamma_{fsd}(G) - \gamma_{fd}(G)$ can be made arbitrarily large.

Proof: Let $n = 2r + 2$ and $k = n - 4$ where r is a positive integer and k is a nonnegative integer. By Theorem 2.7\ref, there exists a connected graph G such that $\gamma_{fsd}(G) = 2r$ and $\gamma_{fd}(G) = 2$. Thus, $\gamma_{fsd}(G) - \gamma_{fd}(G) = 2r - 2 = (n-2) - 2 = n - 4 = k$, showing that $\gamma_{fsd}(G) - \gamma_{fd}(G)$ can be made arbitrarily large. ■

Theorem 2.9 Let G be a connected graph of order $n \geq 2$. Then $\gamma_{fsd}(G) = 1$ if and only if $G = K_n$.

Proof: Clearly, $\gamma_{fsd}(K_n) = 1$. Suppose now that $\gamma_{fsd}(G) = 1$. Let $S = \{x\}$ be a secure dominating set in G. Suppose that G is not complete. Then there exist $y, z \in V(G)$ such that $yz \neq E(G)$. It follows that $(S \setminus \{x\}) \cup \{y\} = \{y\}$ is not a dominating set of G. This implies that S is not a secure dominating set, contrary to our assumption. Thus, $G = K_n$. ■

The cycle C_n of order n, $n \geq 3$, is the graph with distinct vertices $v_1, v_2, \ldots, v_n$ and edges $v_1v_2, v_2v_3\}, \ldots, v_{n-1}v_n, v_nv_1$.

Remark 2.10 Let $n \geq 3$. Then $$\gamma_{fsd}(C_n) = \begin{cases} \frac{n}{2} & if \quad n \text{ is even} \\ \frac{n+1}{2} & if \quad n \text{ is odd} \end{cases}$$

A graph G is called a bipartite graph if its vertex-set $V(G)$ can be partitioned into two nonempty subsets V_1 and V_2 such that every edge of G has one end in V_1 and one end in V_2. The sets V_1 and V_2 are called the partite sets of G. If each vertex in V_1 is adjacent to every vertex in V_2, then G is called a complete bipartite graph. If $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m,n}$.

Remark 2.11 Let $m \geq 2$ and $n \geq 2$. Then $$\gamma_{fsd}(K_{m,n}) = \begin{cases} m & if \quad m \leq n \text{ and } m \leq 4 \\ n & if \quad m < n \leq 4 \\ 4 & if \quad m \geq 4 \text{ and } n \geq 4 \end{cases}$$

A star graph $S_n = K_1 + \overline{K}_n$ is a complete bipartite $K_{1,n}$ where $n \geq 1$.

Remark 2.12 $\gamma_{fsd}(S_n) = n$ for all $n \geq 1$.

Let $n \geq 1$. The fan of order $n + 1$, denoted by F_n, is the graph $K_1 + P_n$.

Remark 2.13 Let $n \geq 1$. Then $$\gamma_{fsd}(F_n) = \begin{cases} 1 & if \quad n = 1 \text{ or } n = 2 \\ \frac{n+3}{3} & if \quad n \equiv 0 (mod\ 3) \\ \frac{n+5}{3} & if \quad n \equiv 1 (mod\ 3) \\ \frac{n+4}{3} & if \quad n \equiv 2 (mod\ 3) \end{cases}$$

Let $n \geq 3$. The wheel of order $n+1$, denoted by W_n, is the graph $K_1 + C_n$.

Remark 2.14 Let $n \geq 3$. Then

$$\gamma_{fsd}(W_n) = \begin{cases} n-2 & if \quad n = 3 \; or \; n = 4 \\ \frac{n+3}{3} & if \quad n \equiv 0 (mod\ 3) \\ \frac{n+5}{3} & if \quad n \equiv 1 (mod\ 3) \\ \frac{n+7}{3} & if \quad n \equiv 2 (mod\ 3) \end{cases}$$

The join of two graphs G and H is the graph $G+H$ with vertex-set $V(G+H) = V(G) \cup V(H)$ and edge-set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

We are needing the following results for the characterization of the fair secure dominating set in the join of two connected graphs.

Remark 2.15 Let G and H be connected graphs. Then $V(G)$ and $V(H)$ are fair dominating sets of $G+H$.

Lemma 2.16 Let G and H be connected non-complete graphs. If $S = V(G)$ or S is a k-fair dominating set of G where $k = |S| \geq 2$, then a nonempty proper subset S of $V(G+H)$ is a fair secure dominating set of $G+H$.

Proof: If $S = V(G)$, then S is a fair dominating set of $G+H$ by Remark 2.15. Since G is connected non-complete graph, $|S| \geq 2$. Let $v, z \in S$ and $u \in V(G+H) \setminus S$. Then $u \in V(H)$ and $uv \in E(G+H)$. Since $z \in S = V(G)$ and $u \in V(H)$, $z, u \in (S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G+H$. Thus, S is a fair secure dominating set of $G+H$.

Suppose that S is a k-fair dominating set of G where $k = |S| \geq 2$. Let $u \in V(G+H) \setminus S$. If $u \in V(G) \setminus S$, then $|N_{G+H}(u) \cap S| = |N_G(u) \cap S| = k = |S|$. If $u \in V(H)$, then $|N_{G+H}(u) \cap S| = |S|$. Thus, for any $u, v \in V(G+H) \setminus S$, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$, that is, S is a fair dominating set of $G+H$. Let $x \in V(G+H) \setminus S$. If $x \in V(G) \setminus S$, then $|N_G(x) \cap S| = k$ since S is a k-fair dominating set of G. Since $k = |S|$, $|N_G(x) \cap S| = |S|$. This implies that $N_G(x) \cap S = S$, that is, $xv \in E(G)$ for all $v \in S$. Since $k = |S| \geq 2$, it follows that $S \setminus \{v\} \neq \emptyset$ and for every $x \in V(G) \setminus S$, there exists $z \in (S \setminus \{v\})$ such that $xz \in E(G)$. This means that $S \setminus \{v\}$ is a dominating set of G, that is, $(S \setminus \{v\}) \cup \{x\}$ is a dominating set of G and hence a dominating set of $G+H$. If $x \in V(H)$, then $xv \in E(G+H)$ for all $v \in S$. Since $S \setminus \{v\}$ is a nonempty subset of $V(G)$ and $\{x\} \subset V(H)$, it follows that $(S \setminus \{v\}) \cup \{x\}$ is a dominating set of $G+H$. Thus, S is a fair secure dominating set of $G+H$. ■

Lemma 2.17 Let G and H be connected non-complete graphs. If $S = V(H)$ or S is a k-fair dominating set of H where $k = |S| \geq 2$, then a nonempty proper subset S of $V(G+H)$ is a fair secure dominating set of $G+H$.

Proof: If $S = V(H)$, then S is a fair dominating set of $G+H$ by Remark 2.15. Since H is connected non-complete graph, $|S| \geq 2$. Let $v, z \in S$ and $u \in V(G+H) \setminus S$. Then $u \in V(G)$ and $uv \in E(G+H)$. Since $z \in S = V(H)$ and $u \in V(G)$, $z, u \in (S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G+H$. Thus, S is a fair secure dominating set of $G+H$.

Suppose that S is a k-fair dominating set of H where $k = |S| \geq 2$. Let $u \in V(G+H) \setminus S$. If $u \in V(H) \setminus S$, then $|N_{G+H}(u) \cap S| = |N_H(u) \cap S| = k = |S|$. If $u \in V(G)$, then $|N_{G+H}(u) \cap S| = |S|$. Thus, for any $u, v \in V(G+H) \setminus S$, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$, that is, S is a fair dominating set of $G+H$. Let $x \in V(G+H) \setminus S$. If $x \in V(H) \setminus S$, then $|N_H(x) \cap S| = k$ since S is a k-fair dominating set of H. Since $k = |S|$, $|N_H(x) \cap S| = |S|$. This implies that $N_H(x) \cap S = S$, that is, $xv \in E(H)$ for all $v \in S$. Since $k = |S| \geq 2$, it follows that $S \setminus \{v\} \neq \emptyset$ and for every $x \in V(H) \setminus S$, there exists $z \in (S \setminus \{v\})$ such that $xz \in E(H)$. This means that $S \setminus \{v\}$ is a dominating set of H, that is, $(S \setminus \{v\}) \cup \{x\}$ is a dominating set of H and hence a dominating set of $G+H$. If $x \in V(G)$, then $xv \in E(G+H)$ for all $v \in S$. Since $S \setminus \{v\}$ is a nonempty subset of $V(H)$ and $\{x\} \subset V(G)$, it follows that $(S \setminus \{v\}) \cup \{x\}$ is a dominating set of $G + H$. Thus, S is a fair secure dominating set of $G + H$. ■

Lemma 2.18 Let G and H be connected non-complete graphs. If $S_G \subset V(G)$ is an r-fair dominating set of G, $S_H \subset V(H)$ is an s-fair dominating set of H, and $r - s = |S_G| - |S_H|$, then a nonempty proper subset $S = S_G \cup S_H$ of $V(G + H)$ is a fair secure dominating set of $G + H$.

Proof: Since S_G is an r-fair dominating set of G, for every $u \in V(G) \setminus S_G$, $|N_G(u) \cap S_G| = r$. Since S_H is an s-fair dominating set of H, for every $v \in V(H) \setminus S_H$, $|N_H(v) \cap S_H| = s$. Now, $S_G \subset V(G)$ implies that $V(G) \setminus S_G \neq \emptyset$. Let $u \in V(G) \setminus S_G$. Then $u \in V(G + H) \setminus S$, $|(N_{G+H}(u) \cap S_H)| = |S_H|$, and

$$\begin{aligned} |N_{G+H}(u) \cap S| &= |N_{G+H}(u) \cap (S_G \cup S_H)| \\ &= |(N_{G+H}(u) \cap S_G) \cup (N_{G+H}(u) \cap S_H)| \\ &= |(N_{G+H}(u) \cap S_G)| + |(N_{G+H}(u) \cap S_H)| \\ &= |(N_G G(u) \cap S_G)| + |S_H| \\ &= r + |S_H|. \end{aligned}$$

Similarly, since $S_H \subset V(H)$, $V(H) \setminus S_H \neq \emptyset$. Let $v \in V(H) \setminus S_H$. Then $v \in V(G + H) \setminus S$, $|(N_{G+H}(v) \cap S_G)| = |S_G|$, and

$$\begin{aligned} |N_{G+H}(v) \cap S| &= |N_{G+H}(v) \cap (S_G \cup S_H)| \\ &= |(N_{G+H}(v) \cap S_G) \cup (N_{G+H}(v) \cap S_H)| \\ &= |(N_{G+H}(v) \cap S_G)| + |(N_{G+H}(v) \cap S_H)| \\ &= |S_G| + |(N_H(v) \cap S_H)| \\ &= |S_G| + s \\ &= |S_G| + (r - |S_G| + |S_H|) \\ &= r + |S_H|. \end{aligned}$$

Thus, for every $u, v \in V(G + H) \setminus S$, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$. Hence, S is a fair dominating set of $G + H$. Now, let $u \in V(G + H) \setminus S$. If $u \in V(G) \setminus S_G$, then there exists $v \in S_H$ such that $uv \in E(G + H)$. Since S_G is a dominating set of G, $S \setminus \{v\} = (S_G \cup S_H) \setminus \{v\}$ is a dominating set of $G + H$. Thus, $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G + H$. If $u \in V(H) \setminus S_H$, then there exists $v \in S_G$ such that $uv \in E(G + H)$. Since S_H is a dominating set of H, $S \setminus \{v\} = (S_G \cup S_H) \setminus \{v\}$ is a dominating set of $G + H$. Thus, $(S \setminus \{v\}) \cup \{u\}$ is a dominating set of $G + H$. Hence, S is a secure dominating set of $G + H$. Accordingly, S is a fair secure dominating set of $G + H$. ■

The following result is the characterization of the fair secure domination in the join of two connected graphs.

Theorem 2.19 Let G and H be connected non-complete graphs. Then a nonempty proper subset S of $V(G + H)$ is a fair secure dominating set of $G + H$ if and only if one of the following statement is satisfied.

(i) $S = V(G)$ or S is a k-fair dominating set of G where $k = |S| \geq 2$.
(ii) $S = V(H)$ or S is a k-fair dominating set of H where $k = |S| \geq 2$.
(iii) $S = S_G \cup S_H$ where $S_G \subset V(G)$ is an r-fair dominating set of G, and $S_H \subset V(H)$ is an s-fair dominating set of H, and $r - s = |S\_G| - |S\_H|$.

Proof: Suppose a nonempty proper subset S of $V(G + H)$ is a fair secure dominating set of $G + H$. Consider the following cases:

Case 1. Consider that $S \cap V(H) = \emptyset$. Then $S \subseteq V(G)$. If $S = V(G)$, then the proof of statement (i) is satisfied. Suppose that $S \neq V(G)$. Let $u \in V(G) \setminus S$ and $v \neq u$ such that $u, v \in V(G + H) \setminus S$. Then, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$ since S is a fair dominating set of $G + H$. If $v \in V(G) \setminus S$, then $|N_G(u) \cap S| = |N_G(v) \cap S| = k$ for some positive integer k. This implies that S is a k-fair dominating set of G. If $v \in V(H)$, then $|N_{G+H}(v) \cap S| = |S|$. Since $|N_G(u) \cap S| = |N_{G+H}(u) \cap S|$, it follows that

$$k = |N_G(u) \cap S| = |N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S| = |S|.$$

Now, suppose $k = 1$. Since H is a connected non-complete graph, there exist distinct vertices $x, y \in V(H)$ such that $xy \notin E(H)$. Let $S = \{v\}$. Then $x \in V(G+H) \setminus S$ and $(S \setminus \{v\}) \cup \{x\} = \{x\}$ is not a dominating set of $G+H$ since $xy \notin E(G+H)$. This contradict to our assumption that S is a secure dominating set of $G+H$. Thus, $k \neq 1$ and so $k \geq 2$. This complete the proof of statement (i).

Case 2. Consider that $S \cap V(G) = \emptyset$. Then $S \subseteq V(H)$. If $S = V(H)$, then the proof of statement (ii) is satisfied. Suppose that $S \neq V(H)$. Let $u \in V(H) \setminus S$ and $v \neq u$ such that $u, v \in V(G+H) \setminus S$. Then, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$ since S is a fair dominating set of $G+H$. If $v \in V(H) \setminus S$, then $|N_H(u) \cap S| = |N_H(v) \cap S| = k$ for some positive integer k. This implies that S is a k-fair dominating set of H. If $v \in V(G)$, then $|N_{G+H}(v) \cap S| = |S|$. Since $|N_H(u) \cap S| = |N_{G+H}(u) \cap S|$, it follows that $k = |N_H(u) \cap S| = |N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S| = |S|$. Now, suppose $k = 1$. Since G is a connected non-complete graph, there exist distinct vertices $x, y \in V(G)$ such that $xy \notin E(G)$. Let $S = \{v\}$. Then $x \in V(G+H) \setminus S$ and $(S \setminus \{v\}) \cup \{x\} = \{x\}$ is not a dominating set of $G+H$ since $xy \notin E(G+H)$. This contradict to our assumption that S is a secure dominating set of $G+H$. Thus, $k \neq 1$ and so $k \geq 2$. This complete the proof of statement (ii).

Case 3. Consider that $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$. Let $S_G = S \cap V(G)$ and $S_H = S \cap V(H)$. Then $S = S_G \cup S_H$ where $S_G \subset V(G)$ and $S_H \subset V(H)$. Suppose that to the contrary, S_G is not a fair dominating set of G. Then there exists distinct vertices u and v in $V(G) \setminus S_G$ such that $|N_G(u) \cap S_G| \neq |N_G(v) \cap S_G|$. Thus,

$$\begin{aligned}
|N_{G+H}(u) \cap S| &= |N_{G+H}(u) \cap (S_G \cup S_H)| \\
&= |(N_{G+H}(u) \cap S_G) \cup (N_{G+H}(u) \cap S_H)| \\
&= |(N_G(u) \cap S_G) \cup S_H|, \text{ since } u \in V(G) \setminus S \\
&= |N_G(u) \cap S_G| + |S_H| \\
&\neq |N_G(v) \cap S_G| + |S_H| \\
&= |(N_G(v) \cap S_G) \cup S_H| \\
&= |(N_{G+H}(v) \cap S_G) \cup (N_{G+H}(v) \cap S_H)|, \text{ since } v \in V(G) \setminus S \\
&= |N_{G+H}(v) \cap (S_G \cup S_H)| \\
&= |N_{G+H}(v) \cap S|.
\end{aligned}$$

This contradict to our assumption that S is a fair dominating set of $G+H$. Therefore, S_G must be a fair dominating set of G. Similarly, S_H is a fair dominating set of H. Thus, for every vertex $u \in V(G) \setminus S_G$, $|N_G(u) \cap S_G| = r$ for some positive integer r, and for every vertex $v \in V(H) \setminus S_H$, $|N_H(v) \cap S_H| = s$ for some positive integer s. This implies that S_G is an r-fair dominating set of G and S_H is an s-fair dominating set of H. Now, let $u \in V(G) \setminus S_G$ and $v \in V(H) \setminus S_H$. Then,

$$\begin{aligned}
|N_{G+H}(u) \cap S| &= |N_{G+H}(u) \cap (S_G \cup S_H)| \\
&= |(N_{G+H}(u) \cap S_G) \cup (N_{G+H}(u) \cap S_H)| \\
&= |(N_G(u) \cap S_G) \cup S_H| \\
&= |(N_G(u) \cap S_G| + |S_H| \\
&= r + |S_H| \text{ and,}
\end{aligned}$$

$$\begin{aligned}
|N_{G+H}(v) \cap S| &= |N_{G+H}(v) \cap (S_G \cup S_H)| \\
&= |(N_{G+H}(v) \cap S_G) \cup (N_{G+H}(v) \cap S_H)| \\
&= |S_G \cup (N_H(v) \cap S_H)| \\
&= |S_G| + |N_H(v) \cap S_H| \\
&= |S_G| + s.
\end{aligned}$$

Since S is a fair dominating set of $G+H$, $|N_{G+H}(u) \cap S| = |N_{G+H}(v) \cap S|$, that is, $r + |S_H| = |S_G| + s$. Hence, $r - s = |S_G| - |S_H|$ proving statement (iii).

For the converse, if statement (i) $[(ii)$ or $(iii)]$ is satisfied, then S is a fair secure dominating set of $G+H$ by Lemma 2.16 [Lemma 2.17 or Lemma 2.18]. ∎

Corollary 2.20 Let G and H be a connected non-complete graphs. If S_G is an r-fair dominating set of G or S_H is an s-fair dominating set of H with $|S_G| - |S_H| = r - s$, then $\gamma_{fsd}(G+H) \leq min\{r, s, r+s\}$.

Proof: Suppose that S_G is an r-fair dominating set of G. If $r = |S_G| \neq 1$, then S_G is a fair secure dominating set of $G+H$ by Theorem 2.19(i). Thus, $\gamma_{fsd}(G+H) \leq |S_G| = r$.

If $r = |S_G| = 1$, then consider that S_H is an s-fair dominating set of H. If $s = |S_H| \neq 1$, then S_H is a fair secure dominating set of $G+H$ by Theorem 2.19(ii). Thus, $\gamma_{fsd}(G+H) \leq |S_H| = s$.

If $(r = |S_G| = 1$ and $s = |S_H| = 1)$ or $(r \neq |S\_G|$ and $s \neq |S_H|)$, then let $S = S_G \cup S_H$. Since $S_G \subset V(G)$ is an r-fair dominating set of G, $S_H \subset V(H)$ is an s-fair dominating set of H, and $r - s = |S_G| - |S_H|$, it follows that S is a fair secure dominating set of $G+H$ by Theorem 2.19(iii). Thus, $\gamma_{fsd}(G+H) \leq |S| = |S_G \cup S_H| = |S_G| + |S_H| = r + s$.

Hence, $\gamma_{fsd}(G+H) \leq min\{r, s, r+s\}$. ∎

III. CONCLUSIONS

In this work, we introduced a new domination in graphs - the fair secure domination in graphs. The fair secure domination in the join of two connected non-complete graphs was characterized. The exact fair secure domination number resulting from the join of two connected non-complete graphs was computed. This study will pave a way to new research such as bounds and other binary operations of two connected graphs. Other parameters involving fair secure domination in graphs may also be explored. Finally, the characterization of a fair secure domination in graphs and its bounds is a promising extension of this study.

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