The Existence of Stationary Solution for Nonlinear Random Reaction-Diffusion Equation in Banach Spaces

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Abstract

We study a nonlinear random reaction-diffusion problem in abstract Banach spaces, driven by a real noise, with random diffusion coefficient and random initial condition. The reaction-diffusion equation belongs to the class of parabolic stochastic partial differential equations. Given a Gelfand triplet $\boldsymbol{V} \subset \boldsymbol{H} = \boldsymbol{H}' \subset \boldsymbol{V}'$ *with dense embeddings .Let A (t) be a family of nonlinear random operations, acting from* $\boldsymbol{V}$ ***to*** $\boldsymbol{V}'$, $\boldsymbol{t} \in \boldsymbol{R}^+$, *which satisfies the following assumptions: strong measurability, continuity, monotony, and coercivity. We assume that the initial condition is an element of Hilbert space. We construct a suitable stochastic basis and define the solution of reaction-diffusion problem in the weak sense. We define the stationary process in abstract Banach spaces in the strong sense of Doob-Rozanov. That is, the probability density function of the stochastic process is independent of time shift. In other words, we define the invariant measure for random dynamical system, associated with random reaction-diffusion problem. We prove the existence of an invariant measure and the existence of a stationary solution for nonlinear random reaction-diffusion problem. The obtained theoretical results have several applications in Quantum Physics, Biology, Medicine, and Economic Sciences. Especially, we can study the existence of stationary solution for the stochastic models of tumor growth.*

Keywords — *random reaction-diffusion problem, real noise stationary solution*

Introduction

Random reaction-diffusion equations form an important part of the theory of random partial differential equations that is both very rich changelings mathematically and is related in physics, chemistry, biology, medicine, astronomy, and Economic Sciences. To be specific, in this paper we consider a random reaction-diffusion equation with a polynomial nonlinearity. Of course, we can consider more general mathematical models. For example, we can investigate the random nerve equations, random Lotka- Volterra equations, random Boussinesq- Glover equations, random superfluid equations, random Belousov-Zhabotinsky reaction equations in chemical dynamics, etc.

Let $(\Omega, \mathrm{F}, (\mathrm{F}_t)_{t\geq 0}, P)$ be a stochastic basis, and let $w(t) = (w_1(t), w_2(t), \dots\dots, w_m(t))$ be a standard m-dimensional Wiener process defined on $(\Omega, \mathrm{F}, (\mathrm{F}_t)_{t\geq 0}, P)$. Let $\widehat{\xi(t)}$ be a stationary solution of the Ito equation in R^k

$$d\xi(t) = a\xi(t)\,dt + b\xi(t)\,dw(t), \qquad t \geq 0, \tag{1}$$

where a $(\cdot)$ and b $(\cdot)$ satisfy the assumptions of section 2.1 below. We look at the process $(\Omega, \mathrm{F}, (\mathrm{F}_t)_{t\geq 0}, P, (\xi(t))_{t\geq 0})$ as a model of real noise, stationary in time. Having assumed that the noise process is given, we consider the random nonlinear evolution equation in Hilbert spaces driven by the real noise $\widehat{\xi(t)}$:

$$\frac{du(t,\omega)}{dt} + A\left(\hat{\xi}(t,\omega), u(t,\omega)\right) = f\left(\hat{\xi}(t,\omega)\right), \quad t \geq 0, \quad \omega \in \Omega, \tag{2}$$

Where $\{A(\xi,\cdot), \xi \in R^k\}$ is a family of monotone operators in a Gelfand triplet $\subset H \subset V'$,
and f is a function from R^k *to* V' (see section 2.2.1 for detailed assumptions on A and f). The aim of this paper is to prove the existence of a stationary solution of equation (2). Note that this equation does not contain Ito differential.

In the last few years a lot of papers appeared on invariant measures and stationary solutions for Ito type equations in Hilbert spaces. The case of real noise is not treated. In comparison with the existing literature, we mention two aspects of this paper.

The first one is that we want to consider the real noise $\hat{\xi}(t)$ as a given Markov process, stationary in time. Corresponding to this process $\hat{\xi}(t)$, we would find a stationary solution of equation (2). This fact motivates some technical details of the following analysis, like the choice of a special stochastic basis (see section 3.1) and Theorem 3.1, which are novel with respect to the literature conceding with Ito equations.

The second point is that we shall not assume any compactness. At our knowledge, all the methods know in the literature to prove the existence of invariant measures or stationary solutions use some compactness, coming from the topologies of the function spaces involved. It is well-know that the structure of the monotonicity allows to prove existence of solutions without any compactness assumption. However, a similar result for the existence of invariant measures and stationary solutions is not known, because the usual approach to construct invariant measures, so it uses some compactness arguments.
At the end of the paper we give some applications, which contributed to motivate our analysis.

1. *Preliminaries*

1.1 *The Noise Equation*

Denote by $L(R^m, R^k)$ the space of linear operators from R^m to R^k and by $|\cdot|_{L_2(R^m,R^k)}$ the Hilbert-Schimdt norm of linear operator from R^m to R^k.
Assume that:
(a.1) the mappings $a(\cdot): R^m \to R^k$ and $b(\cdot): R^k \to L(R^m, R^k)$ are locally Lipchitz continuous and satisfy the dissipativity condition
$2\langle a(x), x\rangle_{R^k} + b(x)^2_{L^2(R^m,R^k)} \leq -\eta|x|^2_{R^k} + \rho, \quad \forall x \in R^k$
for some real constants η and ρ;
(a.2) $\eta > 0$
Consider the stochastic differential equation

$$d\xi(t) = a(\xi(t))dt + b(\xi(t))dw(t), \quad t \geq 0, \tag{3}$$

with the initial condition

$$\xi(0) = \xi_0 \tag{4}$$

The following two facts are well –known:

(i) under the assumption (a.1), for every p≥ 2 and for every $\xi_0 \in L^p(\Omega, F_0, P, R^k)$, the equation (3)-(4) has a unique progressively measurable solution $\xi(t)$ defines for $t \geq 0$ with $\xi(t) \in L^p(\Omega, C[0,T], R^k))$, for each T>0

This solution if a Markov process with the Feller property;

(ii) if in addition (a.2) holds true , there exists an invariant measure m_0 for equation (3) with finite moments of all orders.

These results can be found in [9] and [14] for instance. We recall the basic steps of the proof, for future reference. Let us start with (1). The local Lipschitz condition on the coefficients implies existence and uniqueness of a maximal solution. The solution is global, and satisfies $\xi(t) \in L^p(\Omega, C[0,T], R^k))$, by the following estimate.
By Ito formula and the dissipativity condition in (a.1), we have:

$$d|\xi(t)|^p_{R^k} \leq \left(2\langle a(\xi(t)), \xi(t)\rangle_{R^k} + |b(\xi(t))|^2_{L^2(R^m,R^k)}\right) \cdot |\xi(t)|^{p-2}_{R^k} \cdot dt$$
$$+2|\xi(t)|^{p-2}_{R^k} \cdot \langle b(\xi(t))dw(t), \xi(t)\rangle_{R^k} \leq -\eta|\xi(t)|^p_{R^k} \cdot dt + \rho \cdot |\xi(t)|^{p-2}_{R^k} \cdot dt$$
$$+2|\xi(t)|^{p-2}_{R^k} \cdot \langle b(\xi(t))dw(t), \xi(t)\rangle_{R^k} \leq -\eta|\xi(t)|^p_{R^k} \cdot dt + C_0(p)dt$$
$$+2|\xi(t)|^{p-2}_{R^k} \cdot \langle b(\xi(t))dw(t), \xi(t)\rangle_{R^k}$$

for some constant $C_0(p)$, given by Young inequality. Since $\xi_0 \in L^p(\Omega, F_0, P, R^k)$, we have:

$$E|\xi(t)|^p_{R^k} \leq e^{\frac{-\eta t}{2}} \cdot E|\xi_0|^p_{R^k} + \int_0^t e^{\frac{-\eta(t-s)}{2}} C_0(p) \cdot ds, \qquad t \geq 0 \tag{5}$$

In fact, since we deal with a local solution defined a priori on a random time interval, one should use stopping times; moreover, instead of (5) we could prove, by means of classical martingale inequalities, an estimate for the expectation of the supremmum in time of $|\xi(t)|^p_{R^k}$. The details can be found in [9]. These estimates give the a priori bound which implies the global existence of the solution and its integrability. The Markov and Feller properties are classical.
The proof of (ii) also follow from (5), under the assumption (a.2). Indeed, choose $\xi_0 = 0$. Then, (5) and (a.2) imply that for every p≥ 2 there exists a constant $C(p) > 0$ such that

$$|\xi(t)|^p_{R^k} \leq C(p), \quad \forall t \geq 0 \tag{6}$$

This implies the existence of an invariant measure m_0, by the standard Krylov- Bogoliubov method. Moreover (see [14], for every p≥ 2,

$$\int_{R^k} |x|^p \cdot dm_0(x) \leq C(p) \tag{7}$$

This proves (ii).

The uniqueness of invariant measures for equation (3) has been studied in detail. For instance, if $b(\xi)b^*(\xi)$ is positive definite, uniformly in $\xi \in R^k$, the invariant measure is unique. We don't require uniqueness in the sequel, but just fix one invariant measure m_0 with all finite moments (moments up to order r, given the assumption (A4) below, would suffice).
With appropriate minor modifications, we could study the case when the noise $\xi(t)$ is also a stochastic process in a Hilbert space, defined by an infinite dimensional stochastic evolution equation (see [4], [11], [13]). For sake of simplicity, we restrict the attention to the finite-dimensional case.

2.2. The Random Evolution Equation with Monotone Operators

2.2.1 Assumptions

Let H be a real separable Hilbert space and $V \subset H$ be a real reflexive separable Banach space (norms $|\cdot|, \|\cdot\|$respectively , inner product $\langle \cdot, \cdot \rangle$in H),$V$ *dense in* H,and $V \subset\to H$. Let H' and V' be their dual spaces. Identifying H with its dual H', and $H'$with a subspace of V', we can write $V \subset H \subset V'$, with dense embeddings.We shall denote also the dual pairing between $V'$and V by $\langle \cdot, \cdot \rangle$, since its restriction to $V \times H$ coincides with inner product of H.
Let $\{A(\xi,\cdot), \xi \in R^k\}$ be a family of nonlinear operators from be to V' which satisfy the following assumptions.
(A.1) for every $\in V$, $\xi \to A(\xi, u)$ is a strong measurable mapping from R^k to V' ,bounded in bounded sets,

(A.2) for every $\xi \in R^k$ $u, v, z \in V$, the function
$s \to \langle A(\xi, v + sz), u \rangle$ is continuous on R.

(A.3) there is a constant $\lambda_0 \in R$ such that for every $\in R^k$, $u, v \in V$
$2\langle A(\xi, u) - A(\xi, v), u - v\rangle + \lambda_0 |u - v|^2 \geq 0,$

(A.4) there are a function $\rho{:}R^k \to [0, \infty)$ and constants $p \geq 2,\ \lambda \in R,\ \alpha > 0,\ r \geq 0, C_\rho > 0,$ such that for every $\xi \in R^k, u \in V$,

$2\langle A(\xi, u), u\rangle + \lambda |u|^2 \rho(\xi) \geq \alpha \|u\|^p$, and

$\rho(\xi) \leq C_\rho(1 + |\xi|^r_{R^k})$

(A.5) with p as above, there is a constant $C_A > 0$ such that for every $\in R^k$, $u \in V$,

$\|A(\xi, u)\|_{V'} \leq C_A(1 + \|u\|^{p-1})$

Finally, let $f = f(\xi){:} R^k \to V'$ be a given strong measurable function which satisfies the assumption

(f.1) there is constant $C_f > 0$, such that for all $\xi \in R^k$

$\|f(\xi)\|^{p'}_{V'} \leq C_f(1 + |\xi|^r_{R^k})$

Where p' is the conjugate exponent of p (i.e $\frac{1}{p} + \frac{1}{p'} = 1$)
Remark 1. In some application it is natural to assume that the noise processes entering in A and f are different; but this can be accomplished just by considering as $\xi(t)$ the joint noise process.

Remark 2. One can incorporate f into A without restriction, but we leave them separated to emphasize the role of the coercively of A.

2.2.2 The Equation

Let a stochastic basis$(\Omega, \mathrm{F}, (\mathrm{F}_t)_{t\geq 0}, \mathrm{P})$ be given, with a standard m-dimensional.
Wiener process $w(t) = (w_1(t), w_2(t), \dots \dots, w_m(t)), t \geq 0$ under the assumption (a.1) of section 2.1, given$\xi_0 \in L^r(\Omega, \mathrm{F}_0, P, R^k)$, let $\xi = \xi(t,\omega)$ be the corresponding solution of the problem (3) - (4). Here r is the same as in assumption (A.4).
We consider the random evolution equation

$$\frac{du(t,\omega)}{dt} + A(\xi(t,\omega), \mathrm{u}(t,\omega)) = f(\xi(t,\omega)), t \geq 0, \ \omega \in \Omega, \tag{8}$$

With the initial condition

$$u(0,\omega) = u_0(w), \qquad \omega \in \Omega \tag{9}$$

We assume that

$$u_0 \in L^2(\Omega, \mathrm{F}_0, P, H) \tag{10}$$

We interpret the previous equation in the weak form:

$$\langle u(t,\omega), \phi\rangle + \int_0^1 \langle A(\xi(s,\omega), \mathrm{u}(s,\omega)), \phi\rangle \cdot ds = \langle u_0(\omega), \phi\rangle + \int_0^1 \langle f(\xi(s,\omega)), \phi\rangle \cdot ds, \ \forall t \geq 0, \forall \phi \in V \tag{11}$$

By solution of the equation (8) – (9) we mean a progressively measurable process $u = u(t,\omega)$, defined for $t \geq 0$, with the property

We have the following fact:

(iii) Under the assumption (a.1) of section 2.1, given $\xi_0 \in L^p(\Omega, \mathrm{F}_0, P, R^k))$, given the corresponding solution $\xi(t,\omega)$ of equation (3) – (4), and under the assumptions (A.1) – (A.5), (f.1), and (10), there exists a unique solution of the problem (8) – (9). We outline the proof. Since the solution $\xi(t,\omega)$ of (3) is given, one can take an arbitrary given $\omega \in \Omega$ (P- a. s.) and apply to problem (8) – (9) (which is now deterministic) the result of [12], to have a unique solution

$u(t,\omega) \in C([0,T]; H) \cap L^P(0,T,V)$
(it is a solution in the sense of equation (11)). The progressive measurability easily follows from the construction of the solution as a limit of progressively measurable solutions of finite dimensional Galerkin approximations. Finally, the property (12) follows from estimates like those of Lemma 2.1 below.

2.2.3 An a Priori Estimate

The a priori estimate proved in this section will be used in Theorem 3.1 below. We prove it here since it is not based on the additional hypotheses of Theorem 3.1, but it depends only the framework introduced up to now.
Let $e_1, \dots, e_n, \dots$ be a complete orthonormal system in H formed by elements of V. Denote by $H_n \in V$ the finite dimensional space spanned by $e_1, \dots, e_n$, and let P_n be the linear operator from V' to H_n defined as

$$P_n u = \sum_{i=1}^{n} \langle u, e_i\rangle e_i, \quad u \in V'$$

Restricted to H, it is the orthogonal projector onto H_n.
Let $A_n(\xi,\cdot)$ be the operator in H_n. Defined as

$$A_n(\xi,u) = P_n(A_n(\xi,u)), \quad u \in H_n, \ \xi \in R^k$$

and let

$f_n(\xi) = P_n(f_n(\xi)), \quad \xi \in R^k$

Moreover, let a sequence $\{u_{0n}\}_{n\in N}$ of initial conditions be given, where $\forall n, \ u_{0n} \in L^2(\Omega, \mathrm{F}_0, P, H_n)$ and $u_{0n} \to u_0$ in H.
given

$$\xi_0 \in L^r(\Omega, \mathrm{F}_0, P, R^k) \tag{13}$$

Let as consider the Galerkin approximation of equation (8) coupled with equation (3):

$$\frac{du_n(t,\omega)}{dt} + A_n(\xi(t,\omega), u_n(t,\omega)) = f_n(\xi(t,\omega)), t \geq 0, \ \ \omega \in \Omega \tag{14}$$

$$d\xi(t) = a\big(\xi(t)\big)dt + b\big(\xi(t)\big)dw(t)\,, \quad t \geq 0 \tag{15}$$

With initial conditions

$$u_n(0,\omega) = u_{0n}(w), \quad \xi(0,\omega) = \xi_0(\omega) \tag{16}$$

Let $\beta > 0$ be a constant satisfying the (Poincare type) inequality

$|v|^2 \leq \beta \|v\|^2, \quad \forall v \in V$

Lemma 2.1 Consider system (14)-(15)-(16) under the hypotheses (a.1), (A.1)-(A.5), (f.1), (13), and under assumption that there exists a constant C> 0, independent of n, such that

$$E(|u_{0n}|^2) \leq C, \quad \forall n \in N \tag{17}$$

Then the following properties hold true:

(i) System (14)-(15)-(16) has a unique progressively measurable and continuous path solution (u_n, ξ) taking values in $H_n \times R^k$. Moreover, system (14)-(15) defines a Markov process with the Feller property in $H_n \times R^k$.

(ii) For every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$ such that

$|u_n(t,\omega)|^2 \leq e^{\gamma t}|u_{0n}(\omega)|^2 + \int_0^1 e^{\gamma(t-s)} \cdot [a - \varepsilon + (C_f C(\varepsilon) + C_\rho)(1 + |\xi(s)|^r_{R^k})] \cdot ds\,,$ for all

$$n \in N\,, t > 0\,, P - a.e.\, \omega \in \Omega \tag{18}$$

Where $\gamma \overset{d}{=} \lambda - \frac{\alpha - \varepsilon}{\beta}$.

(iii) For every T> 0 there exist constants $C_1(T), \ldots\ldots, C_4(T)$ independent of n, such that

$$E\big(\sup_{t\in[0,T]} |u_n(t,\omega)|^2\big) \leq C_1(T), \qquad \forall n \in N \tag{19}$$

$$E\int_0^T \|u_n(t,\omega)\|^p dt \leq C_2(T)\,, \qquad \forall n \in N \tag{20}$$

$$E\int_0^T \|A_n(\xi_n(t,\omega), u_n(t,\omega)\|_{V'}{}^{p} dt \leq C_3(T), \qquad \forall n \in N \tag{21}$$

$$E\int_0^T \|u'_n(t,\omega)\|_{V'}{}^{p} dt \leq C_4(T), \qquad \forall n \in N \tag{22}$$

<u>Proof</u>

Part (i). First, the equation (15) is considered independently, with given initial condition ξ_0, and the unique progressively measurable and continuous solution $\xi(t)$ is found. Then, we can take a given $\omega \in \Omega$ (P-a.s), apply to equation (14) with the initial condition u_{0n} (which is now a deterministic problem) the results of [12], to have a unique solution

$u_n(t,\omega) \in C([0,T], H_n)$.

The progressive measurability of $u_\square(t,\omega)$ easily follows from the construction of $u_n(t,\omega)$ as a limit of progressively measurable Peano-type approximations. The proof of Markov and Feller properties is classical.

Part (ii). From equation (14) we have

$$\frac{d|u_n(t)|^2}{dt} + 2\langle A_n\big(\xi(t), u_n(t)\big), u_n(t)\rangle = 2\langle f_n\big(\xi(t)\big), u_n(t)\rangle \tag{23}$$

Therefore, from assumptions we have

$$\frac{d|u_n(t)|^2}{dt} + \alpha\|u_n(t)\|^p \leq \lambda|u_n(t)|^2 + \rho\big(\xi(t)\big) + 2\big\|f_n\big(\xi(t)\big)\big\|_{V'} \cdot \|u_n(t)\|$$

Whence , using also assumption (f.1), for every $\varepsilon > 0$ we have

$$\frac{d|u_n(t)|^2}{dt} + (\alpha - \varepsilon)\|u_n(t)\|^p \le \lambda|u_n(t)|^2 + C(\varepsilon)\|f_n(\xi(t))\|^{p'}{}_{V'} \cdot \|u_n(t)\| + \rho(\xi(t))$$
$$\le \lambda|u_n(t)|^2 + (C_f C(\varepsilon) + C_\rho)\cdot(1 + |\xi(s)|^r_{R^k}) \quad \textbf{(24)}$$

Where $C(\varepsilon) = \frac{1}{p' \varepsilon^{\frac{p}{p'}}}$ by Young inequality. Recalling the definition of β, the assumption p≥ 2, and using the fact that $x^{\square} \ge x^2 - 1 \; for \; x \ge 0$, we have

$$\frac{d|u_n(t)|^2}{dt} \le \left(\lambda - \frac{\alpha-\varepsilon}{\beta}\right)|u_n(t)|^2 + \alpha - \varepsilon + (C_f C(\varepsilon) + C_\rho)\cdot(1 + |\xi(s)|^r_{R^k}) \quad \textbf{(25)}$$

Inequality (18) follows now from the Gromwell lemma.
Part (iii). Given $T > 0$, denoted by $C_0(T)$ the supreme of $e^{\gamma t}$ $over$ $[0, T]$, from (18) we have

$$|u_n(t,\omega)|^2 \le C_0(T)|u_{0n}(w)|^2 + \int_0^T C_0(T)\left[\alpha - \varepsilon + (C_f C(\varepsilon) + C_\rho)\cdot(1 + |\xi(s)|^r_{R^k})\right]\cdot ds, \quad \forall t \in [0, T] \quad \textbf{(26)}$$

Recall (5), take the supremum in$t \in [0, T]$, and the expectation, to have (19)
Going back to inequality (24), we integrate it over $[0, T]$, and take the expectation with $\varepsilon = \frac{\alpha}{2}$ we get

$$E\int_0^T \|u_n(t)\|^p dt \le E(|\square_{0n}|^2) + E\int_0^T [\lambda|u_n(t)|^2 + C_f \cdot C\left(\frac{\alpha}{2}\right) + C_p)\cdot(1 + |\xi(s)|^r_{R^k})]dt$$

Using (17), (19), and (5), we obtain (20).
From (20) and assumption (A.5) we also have (22). The proof is complete.

Conclusion

In this paper we investigate the existence and uniqueness of stationary solution for nonlinear random reaction-diffusion equation in Banach spaces, driven by a real noise. We assume that diffusion coefficient is a random variable and the initial condition is a random function. The real noise process is defined as a stationary solution of ito stochastic differential equation in finite dimensional Euclidian space or Hilbert space. We consider a multiplicative noise term, which is more general than additive noise term introduced by P.L Chow or other researchers, see Kuttler [12], M. Metivier [14]. To be specific, we consider a random reaction – diffusion equation with a polynomial nonlinearity. Of course, we can investigate more general mathematical models, and suggest several applications, especially Boussinesq –Glover equation.

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