b-Coloring of the Product of Paths and Cycles

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Abstract: *In this paper, we study the b-coloring of the product of paths and cycles. Let G be a graph with vertex set V(G) and edge E(G). The b-coloring is nothing but the b-chromatic number. The b-chromatic number is the largest integer k colors such that every color class has b-vertex. The b-vertex is the color dominating vertex that has an adjacent in all other color class. The b-chromatic number of a graph is denoted by φ(G).*

Keywords: *b-coloring, b-chromatic number, b-vertex.*

I. INTRODUCTION

Let G be a graph containing no loops or multiple edges with vertex set V(G) and edge set E(G). A coloring of the vertices of G is a function $c:V(G)\rightarrow\{1,2,...k\}$. Then the integer c(v) is called the color of v. A coloring is proper if no two adjacent vertices have the same color. The chromatic number $\chi(G)$of a graph G is that the least integer k such that G has a proper coloring using k-colors. Several interesting concepts of the coloring and related parameter are studied in [6,7,8,9].

Motivated by these concepts, W.Iriving and F.Manlove[1] introduced a new concept called b-coloring. A b-coloring of G by k colors is a proper coloring of the vertices of G such that in each color class there is a vertex having neighbours in all the other k-1 color classes. We call any such vertex a b-vertex. The b-chromatic number of a graph G is the greatest integer k such that G has a b-coloring with k-colors. Kouider and Manlove[3] proved some lower and upper bounds for the b-chromatic number of the cartesian product of two graphs. S.K.Vaidya and Rakhimol V.Issac[5] discussed the b-chromatic number of regular graphs, path related graphs, shell and gear graph. More results on the b-chromatic number of a graph can be found in [2,4].

In this paper, we prove the b-chromatic number of the product paths and cycles. The definition of the product of graphs are as follows:

Definition 1.1: A graph G that has one vertex distinguish as the root node, then G is called the rooted graph. The rooted product of a graph G_1 and a rooted graph G_2 is defined as follows :

(i) Draw |V(G)| copies of G_2

(ii) For each vertex v_i of G_1, join v_i with the basis node of the i^{th} copy of G_2.

It is denoted by G_1 o G_2.

Definition 1.2: Let G_1 and G_2 be two graphs. Then the cartesian product of G_1 and G_2 is defined as follows:

(i) Vertex set : $V(G_1)\times V(G_2) = \{(u,v): u\epsilon G_1, v\epsilon G_2\}$

(ii) Edge set : Join (u,v) and (u',v') if u=u' and $vv'\epsilon E(G_2)$ or v=v' and $uu'\epsilon E(G_1)$.

It is denoted by $G_1\square G_2$.

II. MAIN RESULTS

In this section, we prove the b-chromatic number for the product of paths and cycles. Before proving the theorem, let us state an important result on the bounds of b-chromatic number of G, which is frequently used in our main results (see [3]).

Theorem 2.1 [3] : For any graph G, $\chi(G) \leq \varphi(G) \leq \Delta(G) + 1$.

Theorem 2.2: For any $m \geq 3$, $(P_n \circ C_m) = \begin{cases} 3, & if\ n = 2,3 \\ 4, & if\ n = 4,5,6 \\ 5, & if\ n \geq 7. \end{cases}$

Proof: Let $P_n \circ C_m$ be the rooted product of path P_n and cycle C_m. Then $P_n \circ C_m$ is a connected graph which is obtained from the path P_n and cycle C_m such that attach cycle C_m in each vertex of P_n.

Denote v_i, i =1,2,...,n the vertices of P_n and u_{ij}, i = 1,2,...,n, j = 1,2,...,m-1, the vertices in the i^{th} copy of C_m.

So, $V(G) = \{v_i : i = 1,2,...,n\} \cup \{u_{ij} : 1 \leq i \leq n, 1 \leq j \leq m-1\}$ and $E(G) = \{(v_i, u_{ij}) : 1 \leq i \leq n, j = 1, m-1\} \cup \{(u_{ij}, u_{ij+1}) : 1 \leq i \leq n, 1 \leq j \leq m-1\} \cup \{(v_i, v_{i+1}) : 1 \leq i \leq n-1\}$. From the definition of $P_n \circ C_m$, v_1 and v_n have degree 3, v_i, i = 2,...,n-1 has degree 4, u_{ij}, i = 1,2,...n and j = 1,2,...m-1 has degree 2. The proof consists of so many cases.

Case 1: n = 2

Then $V(P_2 \circ C_m) = \{v_1, v_2, u_{11}, u_{12}, ..., u_{1(m-1)}, u_{21}, u_{22}, ..., u_{2(m-1)}\}$ and $|V(P_2 \text{ o } C_m)| = 2m$.

Clearly $P_2 \circ C_m$ has (2m-2) vertices of degree 2 and two vertices of degree 3.

Since the maximum degree of $P_2 \circ C_m$ is 3, $\varphi(P_2 \circ C_m) \leq 4$.

Subcase 1:

Suppose $\varphi(P_2 \circ C_m) = 4$.

Then $P_2 \circ C_m$ must have four vertices of degree 3.

This is impossible because $P_2 \circ C_m$ has only two vertices of degree 3.

Hence $\varphi(P_2 \circ C_m) \neq 4$ and so $\varphi(P_2 \circ C_m) \leq 3$.

Subcase 2:

We have $P_2 \circ C_m$ has (2m-2) vertices of degree 2. So we color the vertices as follows : $c(v_1) = 1$, $c(v_2) = 2$, $c(u_{11}) = 3$, $c(u_{1(m-1)}) = 2$, $c(u_{21}) = 3$, $c(u_{2(m-1)}) = 1$ and color the remaining vertices such that no adjacent vertices receive the same color. Then v_1, v_2, u_{11} are the b-vertices for the color classes c_1, c_2 and c_3, where c_i denotes the set of all vertices receive the i^{th} color. Therofore $\varphi(P_2 \circ C_m)=3$.

Case 2: n = 3

Then $V(P_3 \circ C_m) = \{v_1, v_2, v_3, u_{11}, u_{12}, ..., u_{1(m-1)}, u_{21}, ..., u_{2(m-1)}, u_{31}, ..., u_{3(m-1)}\}$ and $|V(P_3 \circ C_m)| = 3m$.

Clearly $P_3 \circ C_m$ has (3m-3) vertices of degree 2 and two vertices of degree 3 and one vertex of degree 4.

Since the maximum degree of $P_3 \circ C_m$ is 4, $\varphi(P_3 \circ C_m) \leq 5$.

Subcase 1: Suppose $\varphi(P_3 \circ C_m) = 5$.

Then $P_3 \circ C_m$ must have five vertices of degree 4. This is impossible since $P_3 \circ C_m$ has only one vertex of degree 4.

Hence $\varphi(P_3 \circ C_m) \neq 5$ and so $\varphi(P_3 \circ C_m) \leq 4$.

Subcase 2: Suppose $\varphi(P_3 \circ C_m) = 4$.

Then $P_3 \circ C_m$ must have four vertices of degree 3. This subcase is also not possible since $P_3 \circ C_m$ has only two vertex of degree 3. Hence $\varphi(P_3 \circ C_m) \neq 4$ and so $\varphi(P_3 \circ C_m) \leq 3$.

Subcase 3:

Clearly $P_3 \circ C_m$ has (3m-3) vertices of degree 2.

So we color the vertices as follows : $c(v_1) = 1$, $c(v_2) = 2$, $c(v_3) = 3$, $c(u_{11}) = 3$, $c(u_{1(m-1)}) = 2$, $c(u_{21}) = 3$, $c(u_{2(m-1)}) = 1$, $c(u_{31}) = 1$, $c(u_{3(m-1)}) = 2$ and color the remaining vertices such that no adjacent vertices have the same color. Then v_1, v_2, v_3 are the b-vertices for the color classes c_1, c_2 and c_3 where c_i denotes the set of all vertices have the color i. Therefore φ $(P_3 \circ C_m) = 3$.

Case 3: n = 4.

Then $V(P_4 \circ C_m) = \{v_1, v_2, v_3, v_4, u_{11}, u_{12}, ..., u_{1(m-1)}, u_{21}, ..., u_{2(m-1)}, u_{31}, ..., u_{3(m-1)}, u_{41}, ..., u_{4(m-1)}\}$ and $|V(P_4 \circ C_m)| = 4m$.

Clearly $P_4 \circ C_m$ has (4m-4) vertices of degree 2 and two vertices of degree 3 and two vertices of degree 4.

Since the maximum degree of $P_4 \circ C_m$ is 4, φ $(P_4 \circ C_m) \leq 5$.

Subcase 1: Suppose φ $(P_4 \circ C_m) = 5$

Then $P_4 \circ C_m$ must have five vertices of degree 4. This is impossible since $P_4 \circ C_m$ has only two vertices of degree 4. Hence φ $(P_4 \circ C_m) \neq 5$ and so φ $(P_4 \circ C_m) \leq 4$.

Subcase 2:

Clearly $P_4 \circ C_m$ has two vertices of degree 3 and two vertices of degree 4.

So we color the vertices as follows : $c(v_1) = 1$, $c(v_2) = 2$, $c(v_3) = 3$, $c(v_4) = 4$, $c(u_{11}) = 3$, $c(u_{1(m-1)}) = 4$, $c(u_{21}) = 4$, $c(u_{2(m-1)}) = 1$, $c(u_{31}) = 4$, $c(u_{3(m-1)}) = 4$, $c(u_{41}) = 1$, $c(u_{4(m-1)}) = 2$ and we color the remaining vertices such that no adjacent vertices have the same color. Then v_1, v_2, v_3, v_4 are the b-vertices for the color classes c_1, c_2, c_3 and c_4. Therefore φ $(P_4 \circ C_m) = 4$.

Case 4: n=5.

Then $V(P_5 \circ C_m) = \{v_1, v_2, v_3, v_4, v_5, u_{11}, u_{12}, ..., u_{1(m-1)}, u_{21}, ..., u_{2(m-1)}, u_{31}, ..., u_{3(m-1)}, u_{41}, ..., u_{4(m-1)}, u_{51}, ..., u_{5(m-1)}\}$ and $|V(P_5 \circ C_m)| = 5m$. Clearly $P_5 \circ C_m$ has (5m-5) vertices of degree 2, two vertices of degree 3 and three vertices of degree 4. Since the maximum degree of $P_5 \circ C_m$ is 4, φ $(P_5 \circ C_m) \leq 5$.

Subcase 1: Suppose φ $(P_5 \circ C_m) =5$

Then $P_5 \circ C_m$ must have five vertices of degree 4. This is impossible since $P_5 \circ C_m$ has only three vertices of degree 4. Hence φ $(P_5 \circ C_m) \neq 5$ and so φ $(P_5 \circ C_m) \leq 4$.

Subcase 2:

Since $P_4 \circ C_m$ is a subgraph $P_5 \circ C_m$ and φ $(P_5 \circ C_m) = 4$, φ $(P_5 \circ C_m) = 4$.

Case 5: n = 6.

Then $V(P_6 \circ C_m) = \{v_1, v_2, v_3, v_4, v_5, v_6, u_{11}, ..., u_{1(m-1)}, u_{21}, ..., u_{2(m-1)}, u_{31}, ..., u_{3(m-1)}, u_{41}, ..., u_{4(m-1)}, u_{51}, ..., u_{5(m-1)}, u_{61}, ..., u_{6(m-1)}\}$ and $|V(P_6 \circ C_m)| = 6m$. Clearly $P_6 \circ C_m$ has (6m-6) vertices of degree 2, two vertices of degree 3 and four vertices of degree 4. Since the maximum degree of $P_6 \circ C_m$ is 4, φ $(P_6 \circ C_m) \leq 5$.

Subcase 1: Suppose φ $(P_6 \circ C_m) = 5$

Then $P_6 \circ C_m$ must have five vertices of degree 4. This is impossible since $P_6 \circ C_m$ has only four vertices of degree 4. Hence φ $(P_6 \circ C_m) \neq 5$ and so φ $(P_6 \circ C_m) \leq 4$.

Subcase 2:

Since $P_4 \circ C_m$ is a subgraph $P_6 \circ C_m$ and φ $(P_4 \circ C_m) = 4$, φ $(P_6 \circ C_m) = 4$.

Case 6: n = 7.

Then $V(P_7 \circ C_m) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, u_{11}, ..., u_{1(m-1)}, u_{21}, ..., u_{2(m-1)}, u_{31}, ..., u_{3(m-1)}, u_{41}, ..., u_{4(m-1)}, u_{51}, ..., u_{5(m-1)}, u_{61}, ... , u_{6(m-1)}, u_{71}, ..., u_{7(m-1)}\}$ and $|V(P_7 \circ C_m)| = 7m$.

Clearly $P_7 \circ C_m$ has (7m - 7) vertices of degree 2, two vertices of degree 3 and five vertices of degree 4.

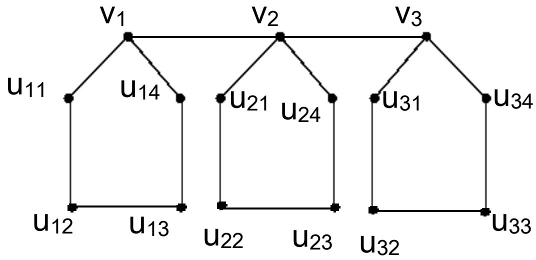
Since the maximum degree of $P_7 \circ C_m$ is 4, $\varphi(P_7 \circ C_m) \leq 5$.

Now, we color the vertices as follows : $c(v_1) = 1, c(v_2) = 2$, $c(v_3) = 3$, $c(v_4) = 4$, $c(v_5) = 5$, $c(v_6) = 1$, $c(v_7) = 2$, $c(u_{11}) = 3$, $c(u_{1(m-1)}) = 4$, $c(u_{21}) = 4$, $c(u_{2(m-1)}) = 5$, $c(u_{31}) = 5$, $c(u_{3(m-1)}) = 1$, $c(u_{41}) = 1$, $c(u_{4(m-1)}) = 2$, $c(u_{51}) = 2$, $c(v_{5(m-1)}) = 3$, $c(v_{61}) = 3$, $c(v_{6(m-1)}) = 4$ and color the remaining vertices such that no adjacent vertices have the same color.

Then v_2, v_3, v_4, v_5, v_6 are the b-vertices for the color classes c_2, c_3, c_4, c_5 and c_1. Therefore $\varphi(P_7 \circ C_m) = 5$.

Case 7: n>7

Since the maximum degree of $P_n \circ C_m$ is 4, $\varphi(P_n \circ C_m) \leq 5$. Also $P_7 \circ C_m$ is a subgraph of $P_n \circ C_m$, for all n >7. Therefore $\varphi(P_n o C_m) = 5$ where $n \geq 7$.



Example: n =3, m =5

Figure 1

$$\therefore \varphi(P_3 \circ C_5) = 3$$

Here $c(v_1) = 1$, $c(v_2) = 2$, $c(v_3) = 3$, $c(u_{11}) = 3$, $c(u_{12}) = 2$, $c(u_{13}) = 3$, $c(u_{14}) = 2$, $c(u_{21}) = 3$, $c(u_{22}) = 1$, $c(u_{23}) = 3$, $c(u_{24}) = 1$, $c(u_{31}) = 1$, $c(u_{32}) = 2$, $c(u_{33}) = 1$, $c(u_{34}) = 2$.

Theorem 2.3: For any $m \geq 3$, $\varphi(P_n \square C_m) = 5$, if $n \geq 5$.

Proof : Let G be the cartesian product of path P_n and cycle C_m.

Denote $u_1, u_2, ..., u_n$ the vertices of P_n and $v_1, v_2, ..., v_m$ the vertices of C_m.

Then $V(G) = \{w_{ij} = (u_i, v_j) : 1 \leq i \leq n, 1 \leq j \leq m\}$ and $E(G) = \{(w_{pq}, w_{rs})$: either $u_p = u_r$ and $v_p v_s \in E(C_m)$ or $v_p = v_s$ and $u_p u_r \in E(P_n)$.

Case 1: n = 5, m =3.

Then $V(P_5 \square C_3) = \{w_{11}, w_{12}, w_{13}, w_{21}, w_{22}, w_{23}, w_{31}, w_{32}, w_{33}, w_{41}, w_{42}, w_{43}, w_{51}, w_{52}, w_{53}\}$ and $|V(P_5 \square C_3)| = 15$.

Clearly $P_5 \square C_3$ has six vertices of degree 3 and nine vertices of degree 4.

Since the maximum degree of $P_5 \square C_3$ is 4, $\varphi(P_5 \square C_3) \leq 5$.

Now, we color the vertices as follows: $c(w_{11}) = 2$, $c(w_{12}) = 4$, $c(w_{13}) = 1$, $c(w_{21}) = 5$, $c(w_{22}) = 1$, $c(w_{23}) = 3$, $c(w_{31}) = 4$, $c(w_{32}) = 2$, $c(w_{33}) = 5$, $c(w_{41}) = 1$, $c(w_{42}) = 3$, $c(w_{43}) = 4$, $c(w_{51}) = 4$, $c(w_{52}) = 5$, $c(w_{53}) = 2$ and no adjacent vertices have the same color [see figure 2].

Then $w_{21}, w_{22}, w_{32}, w_{42}, w_{43}$ are the b-vertices for the color classes c_5, c_1, c_2, c_3 and c_4. Therefore, $\varphi(P_5 \square C_3) = 5$.

Case 2: n > 5, m > 3.

Since the maximum degree of $P_n \square C_m$ is 4, $\varphi(P_n \square C_m) \leq 5$.

Also $P_5 \square C_3$ is subgraph $P_n \square C_m$, for all n > 5, m > 3.

Therefore φ ($P_n \square C_m$) = 5 where n ≥ 5, m ≥ 3.

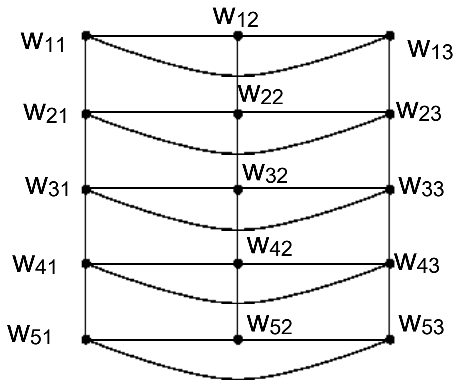


Figure 2

CONCLUSION

In this paper, we found the b-chromatic number for rooted and cartesian product of paths and cycles.

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