Non-Coprime Graph of Integers

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Abstract-*In this paper, we introduce a new concept of graph named as non-coprime graph of integers. A non-coprime graph of integers, denoted by* $\Gamma^{(n)}$, *is arrived from an integer set* $X = \{1, 2, \ldots, n\}$ *whereas the vertex set* $V(G) = X \backslash Y$ *where* $Y = \{x : gcd(x, y) = 1 \text{ for every } y \in X\}$ *and the edge set* $E(G) = \{(x, y) : x, y \in X \text{ and } gcd(x, y) \neq 1\}$. *In this paper, we analyzed some basic properties of the non-coprime graph of integers such as circumference, girth, clique, chromatic number and also prove that the bounds of the domination number, independence number and independent domination number is sharp.*

Keywords — *non-coprime graph, domination, Hamiltonian cycle, semi perfect*

I. INTRODUCTION

Nowadays many investigations of different graphs based on integers have been carried out by several researchers [1,7]. Among all those various graphs on integers the coprime graph is the most popular and dynamic study even if there are many alluring problems and interesting results on divisor graphs [5, 6, 8, 9]. The first problem on special subgraphs of coprime graphs was raised by Paul Erdos in 1962[4]. In this paper we consider only simple, connected undirected graph $G(V, E)$ with $|V|$ vertices and $|E|$ edges. Paul Erdos and Sarkozy[2] have studied the cycles related problem in the coprime graph of integers and also defined coprime graph in 1996. The coprime graph of integers $G = (V, E)$ is constructed from an integer set $X = \{1, 2, \ldots, n\}$, and the vertex set $V = X$ and the edge set $E = \{(x, y) : x, y \in X \text{ and } gcd(x, y) = 1\}$. On the other hand, G. N. Sarkozy[3] has extended their studies from cycles to complete tripartite subgraphs in the coprime graph of integers. Further, this concept was generalized by S. Mutharasu, et. al., [1] in 2014. Let $n \geq 2$, $X = \{1, 2, \ldots, n\}$ and $A \subseteq X$: Then the generalized coprime graph on n and A, denoted by $CP(n, A) = (V, E)$, where $V = X$ and $E = \{(x, y): x, y \in X \text{ and } gcd(x, y) \in A\}$. In [1], mentioned clearly the coprime graph need not be a subgraph of a generalized coprime graph. Also find out some basic properties, perfectness of the same have been studied. In this aspect we define a new graph and named as non-coprime graph. A non-coprime graph of integers is a graph constructed from an integer set $X = \{1, 2, \ldots, n\}$ with $V(G) = X \backslash Y$ where $Y = \{x : gcd(x, y) = 1 \text{ for every } y \in X\}$ and $E(G) = \{(x, y): x, y \in X \text{ and } gcd(x, y) \neq 1\}$ and it is denoted by $\Gamma^{(n)}$. Note that in [11] studied about non-coprime graph of finite group but which is different from non-coprime graph of integers. Once we may easily convert such integers as a graph in this way we can study the relation between graphs in terms of integers.

A (p, q)-graph G is said to be *complete* if and only if every vertices in G are adjacent to every other vertices in G, denoted by K_n. A graph only with vertices and no edges is called an *empty graph* and the empty graph with zero vertices is called a *null graph*. A cycle through all the vertices is called a spanning cycle or *Hamiltonian cycle*. The length of the shortest (longest) cycle called as *girth (circumference)* and they simply noted as $gr(G)$

and $c(G)$ respectively. The *clique number* $\omega(G)$is the order of the maximum complete subgraph of G. The *chromatic number* $\chi(G)$ is minimum number of colours need to colour all the vertices in G at the same time adjacent vertices receives different colour. In some cases the chromatic and clique are same, then the graph is said to be *semi perfect*. A *dominating set* for a graph $G = (V, E)$ is a subset D of V such that every vertex not in D is adjacent to atleast one vertex in D. A *dominating set* D is said to be a *minimum dominating set* if no proper subset of D is a dominating set and the *domination number* $\gamma(G)$ is the cardinality of the minimum dominating set of G. A set of vertices I is called *independent set* if no two vertices in the set I are adjacent to each other or in other words the set of non adjacent vertices is called *independent set.* A *maximum independent set* is an independent set of largest possible size for a given graph G and is called *independence number* of G, denoted by $\alpha(G)$. An *independent dominating set* of G is a set that is both dominating and independent. The *independent domination number* of G, denoted by $i(G)$ is the minimum cardinality of an independent dominating set.

Throughout this paper we use the following notations in short. Let $\Gamma^{(\mathrm{n})}$ be the non-coprime graph of integers constructed from the set $X = \{1, 2, \dots, n\}$ (n positive integers) and let $\mathcal{M} = \{2, p_2, p_3, \dots, p_m\}$ such that $2 < p_2 < p_3 < \cdots < p_m < n$ be the set contains only the prime vertices of the non-coprime graph of integers $\Gamma^{(\mathrm{n})}$.

The domination in non-coprime graph of integers has applications in several fields such as computer communication networks, radio stations, locating radar stations, nuclear power plants, modeling biological networks, modeling social networks, facility locating problems and coding theory. In this paper, we analyzed some basic properties of the non-coprime graph of integers such as circumference, girth, clique, chromatic number and also prove that the domination number, independent domination number and the independence number is sharp for the same.

II. SOME BASIC PROPERTIES OF NON-COPRIME GRAPH OF INTEGERS

In this section we discussed about some basic properties of non-coprime graph of integers. The following lemmas are immediate from the observation.

Lemma 2.1. Let $\Gamma^{(\mathrm{m})}$ be the non-coprime graph of integers corresponding to $\mathcal{M}$, then the graph is null.

Lemma 2.2. Suppose the set X contains only even positive integers. Then the non-coprime graph $\Gamma^{(\mathrm{n})}$ is isomorphic to complete graph of order $\left\lfloor \frac{|X|}{2} \right\rfloor$.

Lemma 2.3. The non-coprime graph $\Gamma^{(\mathrm{n})}$ is always not Hamiltonian.

Lemma 2.4 . The non-coprime graph $\Gamma^{(\mathrm{x})}$ is a subgraph of $\Gamma^{(\mathrm{y})}$ whenever $\mathrm{x} \leq \mathrm{y}$.

Lemma 2.5. The non-coprime graph $\Gamma^{(\mathrm{n})}$ is of the form $\cup K_{\left\lfloor \frac{n}{\mathcal{M}} \right\rfloor}$ where $\mathcal{M} = \{2, p_2, \dots, p_m\}$ such that $2 < p_2 < p_3 < \cdots < p_m < n$ be the set contains only the prime vertices of $\Gamma^{(\mathrm{n})}$.

Lemma 2.6. The non-coprime graph $\Gamma^{(\mathrm{n})}$ is null graph or bipartite graph when $n \leq 5$.

Clearly K_3 is a subgraph contained in $\Gamma^{(\mathrm{n})}$ for all $n \geq 6$. The following lemma is true.

Lemma 2.7. Let $\Gamma^{(\mathrm{n})}$ be a non-coprime graph. Then the girth $gr\left(\Gamma^{(\mathrm{n})}\right) = 3$ for all $n \geq 6$.

Lemma 2.8. Let $\Gamma^{(n)}$ be a non-coprime graph. Then the circumference $c\left(\Gamma^{(n)}\right)$, for all $n \geq 6$,

$$c\left(\Gamma^{(n)}\right) = \begin{cases} \frac{n}{2} & \text{if n is even} \\ \frac{n-1}{2} & \text{if n is odd} \end{cases}$$

Proof. Suppose n is even. Then $gcd(n, 2) = 2$ and $gcd(2x, 2(x + 1)) = 2$ where $1 \leq x \leq \frac{n}{2} - 1$. Suppose n is odd. Then $gcd(n - 1, 2) = 2$ and $gcd\left(2x, 2(x + 1)\right) = 2$ where $1 \leq x \leq \frac{n-3}{2}$. So there is a longest cycle formed by $\frac{n}{2} - 1$ edges and $\frac{n-3}{2}$ edges along with one edge respectively in both cases.

Remark 2.9. Let $X = \{1,2, \ldots, n\}$ and $S_1, S_2 \subseteq X$, where $S_1 = \left\{2k : 1 \leq k \leq \frac{n}{2}\right\}$ when n is even and $S_2 = \left\{2k : 1 \leq k \leq \frac{n-1}{2}\right\}$ when n is odd Then $gcd(x, y) \geq 2$ for every elements x, y contained in S_1 and S_2.

Proposition 2.10. Let $\Gamma^{(n)}$ be a non-coprime graph. Then the clique $\omega\left(\Gamma^{(n)}\right)$, for all $n \geq 4$,

$$\omega\left(\Gamma^{(n)}\right) = \begin{cases} \frac{n}{2} & \text{if n is even} \\ \frac{n-1}{2} & \text{if n is odd} \end{cases}$$

Proof. By Remark 2.9, It is clear that $\Gamma^{(n)}, n \geq 4$ contains maximal induced subgraph of order $|S_1|$ and $|S_2|$ respectively in both cases.

Proposition 2.11. Let $\Gamma^{(n)}$ be a non-coprime graph. Then the chromatic number $\chi\left(\Gamma^{(n)}\right)$, for all $n \geq 4$,

$$\chi\left(\Gamma^{(n)}\right) = \begin{cases} \frac{n}{2} & \text{if n is even} \\ \frac{n-1}{2} & \text{if n is odd} \end{cases}$$

Proof. By Lemma 2.5, the non-coprime graph $\Gamma^{(n)}$ is of the form $K_{\left\lfloor\frac{n}{2}\right\rfloor} \cup K_{\left\lfloor\frac{n}{p_2}\right\rfloor} \cup K_{\left\lfloor\frac{n}{p_3}\right\rfloor} \cup \ldots \cup K_{\left\lfloor\frac{n}{p_m}\right\rfloor}$ where $2 < p_2 < p_3 < \cdots < p_m$. Since 2 is the smallest prime p. Then complete graph $K_{\left\lfloor\frac{n}{2}\right\rfloor}$ is a subgraph of $\Gamma^{(n)}$. So we must need $\frac{n}{2}$ different colour when n is even and $\frac{n-1}{2}$ different colour to colour the graph $\Gamma^{(n)}$ when n is odd.

Theorem 2.12. The non-coprime graph $\Gamma^{(n)}$ is semi perfect.

Proof. The result is easily verified from the Propositions 2.10. and 2.11.

III. DOMINATION, INDEPENDENCE AND INDEPENDENT DOMINATION NUMBER OF NON-COPRIME GRAPH OF INTEGERS

In this section we obtained independence, domination and independent domination number of non-coprime graph of integers.

Proposition 3.1. Let $\Gamma^{(n)}$ be a non-coprime graph of integers. Then $\gamma\left(\Gamma^{(n)}\right) = 1$ for $4 \leq n \leq 9$.

Proof. Suppose $n = 4,5$. Then the graph $\Gamma^{(n)}$ is isomorphic to K_2 and when $n = 6,7,8,9$. Then the graph $\Gamma^{(n)}$ has a common vertex between two complete graphs.

Theorem 3.2. Let p, q be two consecutive prime numbers and $p \geq 5$ such that $2p \leq n \leq 2q - 1$ and let $\Gamma^{(n)}$ be a non-coprime graph. Then the domination number $\gamma(\Gamma^{(n)}) = \left\lfloor \frac{2q-1}{6} \right\rfloor$.

Proof. Let $S = \{v_1, v_2, \dots, v_k\}$ where $k = \left\lfloor \frac{2q-1}{6} \right\rfloor$ be a minimum dominating set of $\Gamma^{(n)}$. Note that $|S| = \left\lfloor \frac{2q-1}{6} \right\rfloor$. By Lemma 2.5, $\Gamma^{(n)}$ is of the form $K_{\lfloor \frac{n}{2} \rfloor} \cup K_{\lfloor \frac{n}{p_2} \rfloor} \cup K_{\lfloor \frac{n}{p_3} \rfloor} \cup \dots \cup K_{\lfloor \frac{n}{p_m} \rfloor}$. Let Y be a set formed by choosing a vertex which contains in both $K_{\lfloor \frac{n}{2} \rfloor}$ and $K_{\lfloor \frac{n}{\mathcal{M}'} \rfloor}$ where $\mathcal{M}' = \mathcal{M} \setminus \{2\}$. Note that $Y = \{2p_i : 2 \leq i \leq m\}$. Clearly Y dominates all other vertices in $\Gamma^{(n)}$. Suppose Y is not a minimum dominating set then there exist a dominating set Y' such that $|Y'| < |Y|$. Suppose a vertex $2p_i \notin Y'$ for any i. Clearly Y' does not a dominate the vertex p_i. Therefore Y' is not dominating set and hence Y is a minimum dominating set and $|Y| = |S|$.

Theorem 3.3. Let $\Gamma^{(n)}$ be a non-coprime graph. Then the independence number $\alpha(\Gamma^{(n)}) = |\mathcal{M}|$, where $\mathcal{M} = \{2, p_2, p_3, \dots, p_m\}$ such that $2 < p_2 < p_3 < \dots < p_m < n$ be the set contains only the prime vertices of $\Gamma^{(n)}$.

Proof. Let $\mathcal{M} = \{2, p_2, p_3, \dots, p_m\}$ such that $2 < p_2 < p_3 < \dots < p_m < n$ be the set contains only the prime vertices of $\Gamma^{(n)}$.Since every vertices in $\mathcal{M}$ are relatively prime ,by the definition of non-coprime graph, each p_i's are non adjacent in $\mathcal{M}$. Therefore $\mathcal{M}$ is an independent set. Suppose $\mathcal{M}$ is not maximum then there exist an independent set $\mathcal{M}_1$ such that $|\mathcal{M}_1| > |M|$. Let $v \in V(\Gamma^{(n)}) \setminus \mathcal{M}$. Suppose $v \in \mathcal{M}_1$. Since $\mathcal{M}$ is the set of all prime vertices of $\Gamma^{(n)}$ the vertex v in $\mathcal{M}_1$ must be a multiple of atleast one of the prime factors in $\mathcal{M}_1$. Clearly the vertex v is adjacent to atleast any one of the prime vertices in $\mathcal{M}_1$. Therefore $\mathcal{M}_1$ is not an independent set and hence $\mathcal{M}$ is the maximum independent set of $\Gamma^{(n)}$.

Proposition 3.4. Let $\Gamma^{(n)}$ be a non-coprime graph of integers. Then $i(\Gamma^{(n)}) = 1$ for $4 \leq n \leq 9$.

Theorem 3.5. Let p, q be two consecutive prime numbers and $p \geq 5$ such that $2p \leq n \leq 2q - 1$ and let $\Gamma^{(n)}$ be a non-coprime graph. Then the independence domination number $i(\Gamma^{(n)}) = \left\lfloor \frac{2q-1}{6} \right\rfloor$.

Proof. Let X be the set formed by removing the vertices $2, p_2$ from $\mathcal{M}$ and adding the vertex $2p_2$, i.e., $X = \{2p_2, p_3, \dots, p_m\}$, where $m = \left\lfloor \frac{2q-1}{6} \right\rfloor$. Note that $|X| = \left\lfloor \frac{2q-1}{6} \right\rfloor$. Since every vertices in X are relatively prime, by the definition of non-coprime graph, each p_i's are non adjacent in X and also each p_i is adjacent to its multiples. Hence each p_i dominates all the other vertices in $V(\Gamma^{(n)}) \setminus X$. Therefore the set X is both independent and dominating. Hence X is an independent dominating set. Suppose X is not minimum then there exist an independent dominating set X' such that $|X'| < |X|$. Suppose a vertex $p_i \notin X'$ for any i. Clearly X' is not dominating set. Therefore X' is not an independent dominating set and hence X is a minimum independent dominating set and $i(\Gamma^{(n)}) = |X| = \left\lfloor \frac{2q-1}{6} \right\rfloor$.

IV. CONCLUSIONS

In this research we introduce a new concept of graph named as non-coprime graph of integers. We may construct many graphs in such a way that it may represent different types of graphs. One of our main aims to study and analyses the properties of such graphs in graphic theoretical way. If suppose we got solution of such graphs we may discuss about to choices integers may be in different. Still lot of research is pending in this area.

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