On Hadamard Powers of Non-Negative Matrices

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Abstract – *New upper bounds on spectral radius of Hadamard product of Hadmard powers of non-negative matrices are proposed.*

Keywords – *Hadamard product, Hadamard power, non-negative matrix, spectral radius, irreducible matrix.*

I. Introduction

Let M_n denote the set of all real matrices of order n, N is the set of positive integers $\leq$ n, for any positive integer n. According to Berman and Plemmons [1], a matrix $A = [a_{ij}] \in M_n$ is called a non-negative matrix and write $A \geq 0$,if all $a_{ij} \geq 0$ for $i, j = 1,2, \ldots, n$. Let $\sigma(A)$ be the set of all eigen values of A. The spectral radius of A is denoted by $\rho(A)$and defined by $\rho(A)$=Max $\{|\lambda| : \lambda \in \sigma(A)\}$. If A is non-negative matrix, the Perron-Frobenius theorem guarantees that $\rho(A) \in \sigma(A)$.

In the paper of Fang [3], a matrix $A \in M_n$ is said to be reducible if there exists a permutation matrix Q such that$Q^T AQ = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$, where $A_{11}$is $r \times r$ submatrix and A_{22} is$(n-r) \times (n-r)$ matrix, for $r = 1,2, \ldots, n$. If no such permutation matrix exists, then A is called irreducible. If A is a 1×1 complex matrix, then A is irreducible if and only if its single entry is non-zero.

From a paper of Liu et.al. [6], a matrix$A = [a_{ij}] \in M_n$ is said to be diagonally dominant by its row elements (or column elements) if $|a_{ii}| \geq |a_{ij}|$ (or $|a_{ii}| \geq |a_{ji}|$) for each $i, j = 1,2, \ldots n$.

For$A = [a_{ij}] \in M_n$, $B = [b_{ij}] \in M_n$, the Hadamard product of A and B is the matrix defined by

$$AoB = [a_{ij} b_{ij}] \in M_n.$$

If A,B are nonnegative matrices, then Horn and Johnson [5] proved that AoB is also a nonnegative matrix.

From Fang [3], the Hadamard product of matricesarise in a wide variety of ways, such as products of integral equation Kernels, trigonometric moments of convolutions of periodic functions, characteristic functions in probability theory, the study of association schemes in combinational theory and the weak minimum principle in partial differential equations.

From Horn and Johnson [5], if $A = [a_{ij}] \in M_n$ and A is nonnegative matrix, then for $\alpha \geq 0$, we can write $A^{(\alpha)} = [a_{ij}{}^{\alpha}]$. $A^{(\alpha)}$is called α^{th}Hadamard power (entry wise power) of A.

For example if $A = \begin{bmatrix} 5 & 1 & 1 & 4 \\ 0 & 3 & 2 & 2 \\ 9 & 1 & 4 & 2 \\ 6 & 0 & 2 & 5 \end{bmatrix}$, then $A^{(3)} = \begin{bmatrix} 5^3 & 1^3 & 1^3 & 4^3 \\ 0 & 3^3 & 2^3 & 2^3 \\ 9^3 & 1^3 & 4^3 & 2^3 \\ 6^3 & 0 & 2^3 & 5^3 \end{bmatrix}$.

In Hadamard power, each minor of $A^{(\alpha)}$ is an exponential polynomial

$$f(t) = \sum_{i=1}^{n} a_i \, e^{\lambda_i t},$$

where $a_1, a_2, \dots, a_n$ are real numbers and $\lambda_1 < \lambda_2 < \cdots < \lambda_n$.

In this paper, we will give some new upper bounds on spectral radius of the Hadamard product of Hadamard powers for non-negative matrices A and B. In Section 2 we give some previous results which are useful to establish our main result. In section 3 we prove our main results.

II. Some Lemmas

In order to prove our result, we first give some lemmas.

Lemma 2.1[1].Let $A \in M_n$ be a nonnegative matrix. If A_r is a principal submatrix of A, then $\rho(A_r) \le \rho(A)$. If A is irreducible and $A_r \ne A$, then $\rho(A_r) < \rho(A)$.

Lemma 2.2[1]. (Perron Frobenius Theorem) If A is an irreducible non-negative matrix, then there exists a positive vector X such that $AX = \rho(A)X$.

Lemma 2.3[5].If A, B are two complex matrices of order n and D_1, D_2 are positive diagonal matrices, then

$$D_1(A \circ B)D_2 = (D_1 A D_2) \circ B = (D_1 A) \circ (B D_2) = (A D_2) \circ (D_1 B) = A \circ (D_1 B D_2).$$

Lemma 2.4 [8].Let $A = [a_{ij}]$ be a complex matrix of order n, then all the eigen values of A lie in the region

$$\cup_{i,j=1, i\ne j}^{n} \{z \in C: |z - a_{ii}||z - a_{jj}| \le (\sum_{k\ne i} |a_{ki}|)(\sum_{k\ne j} |a_{kj}|)\}.$$

Lemma 2.5[5].If $A \in M_n$, $A \ge 0, \alpha \ge 1$, then

$$\rho(A^{(\alpha)}) \le \rho(A)^{\alpha}.$$

Lemma 2.6[5]. Let $A, B \in M_n$, $A, B \ge 0$ and α is any real number such that $0 \le \alpha \le 1$, then

$$\rho(A^{(\alpha)} \circ B^{(1-\alpha)}) \le \rho(A)^{\alpha} \rho(B)^{1-\alpha}.$$

Theorem 2.1[5]. If $A_1, A_2, \dots, A_k \in M_n$ and all $A_i \ge 0$ for all $i = 1,2,\dots,k$ and if $\alpha_i \ge 0$ for $i = 1,2,\dots,k$ satisfy $\alpha_1 + \alpha_2 + \cdots + \alpha_k = 1$, then

$$\rho({A_1}^{(\alpha_1)} \circ \dots \circ {A_k}^{(\alpha_k)}) \le \rho(A_1)^{\alpha_1} \dots \rho(A_k)^{\alpha_k}.$$

Proof.To prove above theorem, we use induction on k.

If $k = 1$ then result follows from Lemma 2.5.

If $k = 2$ then result follows from Lemma 2.6, Now if $k > 2$ and let assume $\alpha_k < 1$ and define $B \geq 0$ by $B^{(1-\alpha_k)} = {A_1}^{(\alpha_1)} o \ldots o {A_{k-1}}^{(\alpha_k-1)}$.

Let us assume

$$\beta_i = \frac{\alpha_i}{1-\alpha_k}, i = 1,2, \ldots, k-1,$$

then

$$\sum_{i=1}^{k-1} \beta_i = 1$$

and

$$B = {A_1}^{(\beta_1)} o \ldots o {A_{k-1}}^{(\beta_{k-1})}.$$

Now

$$\rho\left({A_1}^{(\alpha_1)} o \ldots o {A_k}^{(\alpha_k)}\right) = \rho\left(B^{(1-\alpha_k)} o {A_k}^{(\alpha_k)}\right)$$

$$\leq \rho(B)^{1-\alpha_k} \rho(A_k)^{\alpha_k} = \rho[{A_1}^{(\beta_1)} o \ldots o {A_{k-1}}^{(\beta_{k-1})}]^{1-\alpha_k} \rho(A_k)^{\alpha_k}$$

$$\leq [\rho(A_1)^{\beta_1} \ldots \rho(A_{k-1})^{\beta_{k-1}}]^{1-\alpha_k} \rho(A_k)^{\alpha_k} = \rho(A_1)^{\alpha_1} \ldots \rho(A_k)^{\alpha_k},$$

i.e., the result follows.

III. Main Results

We prove our main results here.

Theorem 3.1. Let $A, B \in M_n$, $A, B \geq 0$ and if DBD^{-1} is diagonally dominant by its column(or row) for any positive diagonal matrix D and for any real number α such that $0 \leq \alpha \leq 1$, then

$$\rho\left(A^{(\alpha)} o\, B^{(1-\alpha)}\right) \leq \rho(A)^{\alpha} \max_{1 \leq i \leq n} {b_{ii}}^{1-\alpha}.$$

Proof. Since for any diagonal matrix$D$, we can write

$$A^{(\alpha)} o\, (DB^{(1-\alpha)}D^{-1}) = D\left(A^{(\alpha)} o\, B^{(1-\alpha)}\right)D^{-1}.$$

Then

$$\rho(A^{(\alpha)} o\, (DB^{(1-\alpha)}D^{-1})) = \rho(D\left(A^{(\alpha)} o\, B^{(1-\alpha)}\right)D^{-1})$$

and diagonal elements of $DB^{(1-\alpha)}D^{-1}$ and $B^{(1-\alpha)}$are the same. So, we assume that $B^{(1-\alpha)}$is diagonally dominant by column (or row) elements.

Case I. If $B^{(1-\alpha)}$is diagonally dominant by its column elements, then

$$A^{(\alpha)} o\, B^{(1-\alpha)} \le A^{(\alpha)} diag({b_{11}}^{1-\alpha}, \dots, {b_{nn}}^{1-\alpha})$$

$$\le A^{(\alpha)} \max_{1\le i\le n} {b_{ii}}^{1-\alpha},$$

then from monotonicity of the Perron eigen value (see [4]), we deduce that

$$\rho\left(A^{(\alpha)} o\, B^{(1-\alpha)}\right) \le \rho(A^{(\alpha)} diag({b_{11}}^{1-\alpha}, \dots, {b_{nn}}^{1-\alpha})) \le \rho(A)^{\alpha} \max_{1\le i\le n} {b_{ii}}^{1-\alpha}.$$

Case II. If $B^{(1-\alpha)}$ is diagonally dominant by its row elements, then we know that

$$\left(A^{(\alpha)} o\, B^{(1-\alpha)}\right)^{T} = (A^{(\alpha)})^{T} o\, (B^{(1-\alpha)})^{T}.$$

So, the result follows by the fact that spectral radius is invariant under transpose.

Theorem 3.2. Let $A, B \in M_n$, $A, B \ge 0$ and any real number α such that $0 \le \alpha \le 1$, then

$$\rho\left(A^{(\alpha)} o\, B^{(1-\alpha)}\right) \le \max_{i\ne j} \frac{1}{2}\Big\{{a_{ii}}^{\alpha} {b_{ii}}^{1-\alpha} + {a_{jj}}^{\alpha} {b_{jj}}^{1-\alpha} + \Big[({a_{ii}}^{\alpha} {b_{ii}}^{1-\alpha} - {a_{jj}}^{\alpha} {b_{jj}}^{1-\alpha})^2 +$$

$$4\left(\rho\left(A^{(\alpha)}\right) - {a_{ii}}^{\alpha}\right)\left(\rho\left(A^{(\alpha)}\right) - {a_{jj}}^{\alpha}\right)\left(\rho\left(B^{(1-\alpha)}\right) - {b_{ii}}^{1-\alpha}\right)\left(\rho\left(B^{(1-\alpha)}\right) - {b_{jj}}^{1-\alpha}\right)\Big]^{1/2}\Big\}. \quad (3.1)$$

Proof. For $n = 1$ inequality (3.1) turns into equality, so we can suppose that $n \ge 2$ and divide the problem into two cases.

Case I. Suppose that $A^{(\alpha)} o\, B^{(1-\alpha)}$ is irreducible, then $A^{(\alpha)}$ and $B^{(1-\alpha)}$ are also irreducible, then by Lemma 2.1, we have

$$\rho\left(A^{(\alpha)}\right) > {a_{ii}}^{\alpha} \quad \text{for } i \in N.$$

Similarly

$$\rho\left(B^{(1-\alpha)}\right) > {b_{ii}}^{1-\alpha} \text{ for } i \in N.$$

Since $A^{(\alpha)}$ and $B^{(1-\alpha)}$ are nonnegative irreducible matrices, then from Lemma 2.2, there exists two positive vectors $u = (u_1, \dots, u_n)^T$ and $v = (v_1, \dots, v_n)^T$ such that $A^{(\alpha)} u = \rho\left(A^{(\alpha)}\right) u$ and $B^{(1-\alpha)} v = \rho\left(B^{(1-\alpha)}\right) v$.

Then we can write

$${a_{ii}}^{\alpha} + \sum_{j\ne i} \frac{{a_{ij}}^{\alpha} u_j}{u_i} = \rho\left(A^{(\alpha)}\right) \text{for } i \in N$$

and

$${b_{ii}}^{1-\alpha} + \sum_{j\ne i} \frac{{b_{ij}}^{1-\alpha} v_j}{v_i} = \rho\left(B^{(1-\alpha)}\right) \text{for } i \in N.$$

Now define $U = \text{diag}(u_1, \dots, u_n)$ and $V = \text{diag}(v_1, \dots, v_n)$ and let $\breve{A} = U^{-1} A^{(\alpha)} U$ and $\breve{B} = V^{-1} B^{(1-\alpha)} V$, then we see that

$$\breve{A} = U^{-1}A^{(\alpha)}U = \begin{bmatrix} \frac{1}{u_1} & & & \\ & \frac{1}{u_2} & & \\ & & \ddots & \\ & & & \frac{1}{u_n} \end{bmatrix} \begin{bmatrix} a_{11}{}^{\alpha} & a_{12}{}^{\alpha} & \cdots & a_{1n}{}^{\alpha} \\ a_{21}{}^{\alpha} & a_{22}{}^{\alpha} & \dots & a_{2n}{}^{\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}{}^{\alpha} & a_{n2}{}^{\alpha} & \cdots & a_{nn}{}^{\alpha} \end{bmatrix} \begin{bmatrix} u_1 & & & \\ & u_2 & & \\ & & \ddots & \\ & & & u_n \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}{}^{\alpha} & \frac{a_{12}{}^{\alpha} u_2}{u_1} & \cdots & \frac{a_{1n}{}^{\alpha} u_n}{u_1} \\ \frac{a_{21}{}^{\alpha} u_1}{u_2} & a_{22}{}^{\alpha} & \dots & \frac{a_{2n}{}^{\alpha} u_n}{u_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}{}^{\alpha} u_1}{u_n} & \frac{a_{n2}{}^{\alpha} u_2}{u_n} & \cdots & a_{nn}{}^{\alpha} \end{bmatrix}$$

and

$\breve{B} = V^{-1}B^{(1-\alpha)}V$

$$= \begin{bmatrix} \frac{1}{v_1} & & & \\ & \frac{1}{v_2} & & \\ & & \ddots & \\ & & & \frac{1}{v_n} \end{bmatrix} \begin{bmatrix} b_{11}{}^{\alpha} & b_{12}{}^{1-\alpha} & \cdots & b_{1n}{}^{\alpha} \\ b_{21}{}^{1-\alpha} & b_{22}{}^{1-\alpha} & \dots & b_{2n}{}^{1-\alpha} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}{}^{1-\alpha} & b_{n2}{}^{1-\alpha} & \cdots & b_{nn}{}^{1-\alpha} \end{bmatrix} \begin{bmatrix} v_1 & & & \\ & v_2 & & \\ & & \ddots & \\ & & & v_n \end{bmatrix}$$

$$= \begin{bmatrix} b_{11}{}^{1-\alpha} & \frac{b_{12}{}^{1-\alpha} v_2}{v_1} & \cdots & \frac{b_{1n}{}^{1-\alpha} v_n}{v_1} \\ \frac{b_{21}{}^{1-\alpha} v_1}{v_2} & b_{22}{}^{1-\alpha} & \dots & \frac{b_{2n}{}^{1-\alpha} v_n}{v_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{b_{n1}{}^{1-\alpha} v_1}{v_n} & \frac{b_{n2}{}^{1-\alpha} v_2}{v_n} & \cdots & b_{nn}{}^{1-\alpha} \end{bmatrix}.$$

Then

$$\breve{A} o \breve{B} = (c_{ij}) = \begin{bmatrix} a_{11}{}^{\alpha} b_{11}{}^{1-\alpha} & \frac{a_{12}{}^{\alpha} b_{12}{}^{1-\alpha} u_2 v_2}{u_1 v_1} & \cdots & \frac{a_{1n}{}^{\alpha} b_{1n}{}^{1-\alpha} u_n v_n}{u_1 v_1} \\ \frac{a_{21}{}^{\alpha} b_{21}{}^{1-\alpha} u_1 v_1}{u_2 v_2} & a_{22}{}^{\alpha} b_{22}{}^{1-\alpha} & \dots & \frac{a_{2n}{}^{\alpha} b_{2n}{}^{1-\alpha} u_n v_n}{u_2 v_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{a_{n1}{}^{\alpha} b_{n1}{}^{1-\alpha} u_1 v_1}{u_n v_n} & \frac{a_{n2}{}^{\alpha} b_{n2}{}^{1-\alpha} u_2 v_2}{u_n v_n} & \cdots & a_{nn}{}^{\alpha} b_{nn}{}^{1-\alpha} \end{bmatrix}.$$

We also assume that$D = VU$, then D is a nonsingular diagonal matrix. Also, from Lemma 2.3

$$D^{-1}\left(A^{(\alpha)} o\, B^{(1-\alpha)}\right)D = (VU)^{-1}\left(A^{(\alpha)} o\, B^{(1-\alpha)}\right)VU$$

$$= U^{-1}V^{-1}\left(A^{(\alpha)} o\, B^{(1-\alpha)}\right)VU$$

$$= U^{-1}[A^{(\alpha)} o \left(V^{-1}B^{(1-\alpha)}V\right)]U$$

$$= \left(U^{-1}A^{(\alpha)}U\right) o \left(V^{-1}B^{(1-\alpha)}V\right) = \breve{A}\ o\ \breve{B}.$$

Then

$$\rho\left(A^{(\alpha)} o\, B^{(1-\alpha)}\right) = \rho(\breve{A}\; o\; \breve{B}).$$

Next let us assume that $\rho(\breve{A}\; o\; \breve{B}) = \lambda$ and since $\rho(\breve{A}\; o\; \breve{B}) = \lambda \geq c_{ij}$ for $i \in N$, then by Lemma 2.4,there is a pair (i, j) of positive integer $i \neq j$ such that

$$\left|\lambda - a_{ii}{}^{\alpha} b_{ii}{}^{1-\alpha}\right|\left|\lambda - a_{jj}{}^{\alpha} b_{jj}{}^{1-\alpha}\right| \leq \left(\sum_{k\neq i}|c_{ik}|\right)\left(\sum_{l\neq j}|c_{jl}|\right)$$

$$= \left(\sum_{k\neq i}\left|\frac{a_{ik}{}^{\alpha}\, b_{ik}{}^{1-\alpha} u_k v_k}{u_i v_i}\right|\right)\left(\sum_{l\neq j}\left|\frac{a_{jl}{}^{\alpha}\, b_{jl}{}^{1-\alpha} u_l v_l}{u_j v_j}\right|\right)$$

$$= \left(\sum_{k\neq i}\frac{a_{ik}{}^{\alpha} u_k}{u_i}\left(\rho\left(B^{(1-\alpha)}\right) - b_{ii}{}^{1-\alpha}\right)\right)\left(\sum_{l\neq j}\frac{a_{jl}{}^{\alpha}\, u_l}{u_j}\left(\rho\left(B^{(1-\alpha)}\right) - b_{jj}{}^{1-\alpha}\right)\right)$$

$$= \left(\rho\left(A^{(\alpha)}\right) - a_{ii}{}^{\alpha}\right)\left(\rho\left(A^{(\alpha)}\right) - a_{jj}{}^{\alpha}\right)\left(\rho\left(B^{(1-\alpha)}\right) - b_{ii}{}^{1-\alpha}\right)\left(\rho\left(B^{(\alpha)}\right) - b_{jj}{}^{1-\alpha}\right).$$

Then

$$\left(\lambda - a_{ii}{}^{\alpha} b_{ii}{}^{1-\alpha}\right)\left(\lambda - a_{jj}{}^{\alpha}\, b_{jj}{}^{1-\alpha}\right) \leq \left(\rho\left(A^{(\alpha)}\right) - a_{ii}{}^{\alpha}\right)\left(\rho\left(A^{(\alpha)}\right) - a_{jj}{}^{\alpha}\right)$$

$\left(\rho\left(B^{(1-\alpha)}\right) - b_{ii}{}^{1-\alpha}\right)\left(\rho\left(B^{(\alpha)}\right) - b_{jj}{}^{1-\alpha}\right),$

i.e., $\lambda^2 - \left(a_{ii}{}^{\alpha}\, b_{ii}{}^{1-\alpha} + a_{jj}{}^{\alpha}\, b_{jj}{}^{1-\alpha}\right)\lambda + a_{ii}{}^{\alpha}\, b_{ii}{}^{1-\alpha} a_{jj}{}^{\alpha} b_{jj}{}^{1-\alpha}$

$\leq \left(\rho\left(A^{(\alpha)}\right) - a_{ii}{}^{\alpha}\right)\left(\rho\left(A^{(\alpha)}\right) - a_{jj}{}^{\alpha}\right)\left(\rho\left(B^{(1-\alpha)}\right) - b_{ii}{}^{1-\alpha}\right)\left(\rho\left(B^{(\alpha)}\right) - b_{jj}{}^{1-\alpha}\right).$

Then

$$\lambda \leq \frac{1}{2}\Big\{\Big\{a_{ii}{}^{\alpha}\, b_{ii}{}^{1-\alpha} + a_{jj}{}^{\alpha}\; b_{jj}{}^{1-\alpha} + \left[\left(a_{ii}{}^{\alpha}\, b_{ii}{}^{1-\alpha} - a_{jj}{}^{\alpha}\, b_{jj}{}^{1-\alpha}\right)^2 + 4\left(\rho\left(A^{(\alpha)}\right) - a_{ii}{}^{\alpha}\right)\left(\rho\left(A^{(\alpha)}\right) - a_{jj}{}^{\alpha}\right)\left(\rho\left(B^{(1-\alpha)}\right) - b_{ii}{}^{1-\alpha}\right)\left(\rho\left(B^{(1-\alpha)}\right) - b_{jj}{}^{1-\alpha}\right)\right]^{1/2}\Big\}$$

$\leq \max_{i\neq j}\frac{1}{2}\Big\{\Big\{a_{ii}{}^{\alpha}\, b_{ii}{}^{1-\alpha} + a_{jj}{}^{\alpha}\, b_{jj}{}^{1-\alpha} + \left[\left(a_{ii}{}^{\alpha} b_{ii}{}^{1-\alpha} - a_{jj}{}^{\alpha}\, b_{jj}{}^{1-\alpha}\right)^2 +\right.$ $4\left(\rho\left(A^{(\alpha)}\right) -\right.$ *aiiαρAα−ajjαρB1−α−bii 1−α (ρB1−α−bjj1− α)1/2,*

i.e., inequality (3.1)holds.

Case II. If any one$A$ or B is reducible, then defined a permutation matrix $Z = [z_{ij}]$ of order n with $z_{12}=z_{23}= \ldots =z_{n-1,n}=z_{n1} = 1$, the remaining $z_{ij} = 0$, then both $A+\varepsilon Z$ and $B + \varepsilon Z$ are non-negative irreducible matrices, for sufficiently very small positive real number ε. Noω replace A and B by $A+\varepsilon Z$ and $B + \varepsilon Z$respectively in case I, and letting $\varepsilon \to 0$, the result follows by continuity.

Corollary 3.1. Let $A, B \in M_n$, $A, B \geq 0$ and any real number α such that $0 \leq \alpha \leq 1$, then ρ

$$\rho\left(A^{(\alpha)} o\, B^{(1-\alpha)}\right) \leq \max_{i \neq j} \frac{1}{2}\Big\{a_{ii}{}^{\alpha} b_{ii}{}^{1-\alpha} + a_{jj}{}^{\alpha}\, b_{jj}{}^{1-\alpha} + \big[\big(a_{ii}{}^{\alpha} b_{ii}{}^{1-\alpha} - a_{jj}{}^{\alpha}\, b_{jj}{}^{1-\alpha}\big)^2 + 4\, (\rho(A)^{\alpha} - \alpha a_{ii})$$

$$(\rho(A)^{\alpha} - \alpha a_{jj})(\rho(B)^{1-\alpha} - (1-\alpha) b_{ii})((\rho(B)^{1-\alpha} - (1-\alpha) b_{jj})\big]^{1/2}\Big\}.$$

Proof. Since for any α**,** $0 \leq \alpha \leq 1$ and for $a_{ii} \geq 0$, then we have

$$a_{ii}{}^{\alpha} > \alpha a_{ii}$$

and from Lemma 2.5

$$\rho\left(A^{(\alpha)}\right) \leq \rho(A)^{\alpha},$$

we can write

$$\rho\left(A^{(\alpha)}\right) - a_{ii}{}^{\alpha} \leq \rho(A)^{\alpha} - \alpha a_{ii}.$$

Then inequality (3.1) reduces into the following form

$$\rho\left(A^{(\alpha)} o B^{(1-\alpha)}\right) \leq \max_{i \neq j} \frac{1}{2}\Big\{a_{ii}{}^{\alpha} b_{ii}{}^{1-\alpha} + a_{jj}{}^{\alpha}\, b_{jj}{}^{1-\alpha} + \big[\big(a_{ii}{}^{\alpha} b_{ii}{}^{1-\alpha} - a_{jj}{}^{\alpha}\, b_{jj}{}^{1-\alpha}\big)^2 + 4\, (\rho(A)^{\alpha} - \alpha a_{ii})$$

$$(\rho(A)^{\alpha} - \alpha a_{jj})(\rho(B)^{1-\alpha} - (1-\alpha) b_{ii})((\rho(B)^{1-\alpha} - (1-\alpha) b_{jj})\big]^{1/2}\Big\}.$$

This completes the proof.

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