On Vector Space Ordered by Reflexive Cones

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Abstract — *Let X be a vector space and P be a cone in X, then (X, P) is an ordered vector space. In this paper, we assumed the cone P to be a reflexive cone and show that (X, P) is an Archimedean space. Among other things, we also show that if an ordered Banach space (X, P) with normal, generating and reflexive cone P has a Riesz decomposition property, then X is a Riesz space.*

Keywords — *Vector space, Ordered Vector space, Cone, Reflexive cone, Riesz Space, Riesz Decomposition Property.*

I. INTRODUCTION

Given a vector space X, a non-empty subset P of a vector space X is called a convex cone (wedge) if $P + P \subseteq P$ and $\gamma P \subseteq P$ for all $\gamma \geq 0$. A convex cone P is said to be pointed if $P \cap (-P) = \{0\}$. Apart from the norm structure, a given vector space can also be equipped with a partial order structure, such a partial order can be naturally induced by a cone P. A partial order "$\leq$" on X induced by a cone P is defined such that $x \leq y$ if and only if $y - x \in P$ for all $x, y \in X$, thus, (X, P) is said to be an ordered vector space. We also called $(X, \leq)$ as ordered vector space only if the translation invariance (i.e., $x \leq y \Rightarrow x + z \leq y + z$ *for all* $x, y, z \in X$) and positive homogeneity (i.e. $x \leq y \Rightarrow \lambda x \leq \lambda y$ *for all* $x, y \in X$ *and* $\lambda \geq 0$) properties are satisfied. Cone has a lot of applications in both pure and applied mathematics. For instance, due to the natural order of a cone, it plays a vital role in the study of partially ordered spaces and vector optimization (see, [1,6]). Similarly, in mathematical economics (see, [2]), the theory of partially ordered spaces are used in general equilibrium theory. In [8], Cones that are locally isomorphic to the positive cone of $l_1(\Gamma)$ were studied, where the author gives a necessary and sufficient conditions for an infinite dimensional, closed cone P of a Banach space X to be locally isomorphic to the positive cone $l_1^+(\Gamma)$ of $l_1(\Gamma)$. This result help so much in the development of another class of cones called Reflexive cones (cones with weakly compact intersection with the unit ball or in other words, cones that coincide with their second dual P^{**} i.e., $P = P^{**}$), (see, e.g., [4,5,9]). In this paper, the concept of reflexive cones and the natural order induced by the cone P on X is being employed and some properties of the space (X, P) have been studied.

In the following, we always represent a pointed cone P as just a cone P, X a vector space and "$\leq$" a partial ordering with respect to P.

II. PRELIMINARIES

Definition 2.1 (Partial ordering) A partial ordering is a binary relation "$\leq$"over a non-empty set X, such that for all $x, y, z \in X$, the following are satisfied:

1. $x \leq x$ (reflexive),
2. $x \leq y$ and $y \leq x \Rightarrow x = y$ (anti-symmetric),
3. $x \leq y$ and $y \leq z \Rightarrow x \leq z$ (transitive).

The pair $(X, \leq)$ is called a partially ordered set.

Definition 2.2 (ordered vector space) A vector space X over $\mathbb{R}$ endowed with a partial ordering "$\leq$", is called an ordered vector space if:

1. $x \leq y \Rightarrow x + z \leq y + z$ for all $x, y, z \in X$, (translation invariance),
2. $x \leq y \Rightarrow \lambda x \leq \lambda y$ for all $x, y \in X$ and $\lambda \in \mathbb{R}^+$, (positive homogeneity),

are satisfied.

Definition 2.3 [10] (Riesz space) A Riesz space is an ordered vector space E, such that for any pair $x, y \in E$, $sup\{x, y\}$ and $inf\{x, y\}$ denoted by $x \vee y$ and $x \wedge y$ respectively, exist in E.

Definition 2.4 [10] (Riesz Norm) A norm $\|\cdot\|$on a Riesz space (vector lattice) E, is called a Riesz norm (lattice norm) if

$$|x| \leq |y| \text{ Implies } \|x\| \leq \|y\| \text{ for all } x, y \in E, \text{ where } |x| = \sup\{x, -x\}$$

The pair (E,$\|\cdot\|$) is called a normed Riesz space (normed lattice). A complete normed Riesz space is a called a Banach lattice.

Definition 2.5 [2](Cone) A non-empty subset P of a vector space is said to be a convex cone if the following conditions are satisfied

i) $P + P \subseteq P$ Closure under vector addition,

ii) $\lambda P \subseteq P \, \forall \lambda \geq 0$ Closure under scalar multiplication with non-negative real number.

If in addition, $P \cap \{-P\} = \{0\}$, then P is called a pointed cone.

Some authors, used to refer to a convex cone as wedge and a pointed cone as just a cone respectively. For a given cone $P \subseteq X$, where X is a vector space, we define a partial ordering "$\leq$" with respect to P by $x \leq y$ if and only if $y - x \in P$.

Definition 2.6 (Generating Cone): A cone P of a vector space X is said to be generating or reproducing if $P - P = X$.

Definition 2.7 (Normal Cone): A cone P of a Banach space X is said to be normal, if there exist a constant $c > 0$ such that for all $x, y \in X$

$0 \leq x \leq y \; implies \; \|x\| \leq c\|y\|$

Definition 2.8 [5] (Reflexive Cone) A Cone P of a Banach space X is reflexive, if the positive part $B_X^+ = B_X \cap P$ is weakly compact. Where B_X is the unit ball defined by

$B_X = \{x \in X \mid \|x\| \leq 1\}$

In other words, a cone is reflexive if $P = I$.

Definition 2.9 (Open Decomposition): A cone P of a Banach space X is said to give an open decomposition if there exist $\alpha > 0$ so that $\alpha B_X \subseteq B_X^+ - B_X^+$.

Definition 2.10 [10] (Riesz Decomposition Property): An ordered vector space is said to have the Riesz decomposition property if for all elements $u, z_1, z_2 \in$ satisfying $u \leq z_1 +$ there exist an elements $u_1, u_2 \in$ such that $u_1 \leq z_1, u_2 \leq$ and $u = u_1 +$

Definition 2.11 (Reflexive Normed Space): A normed space is said to be reflexive, if the canonical mapping $J_X : X \rightarrow X$of into its bidual is surjective.

III. RESULTS

We will begin the section with a result based on the Riesz decomposition property which is due to [3], and use it in proving some of our results that will follow.

Theorem 3.1 [3] For an ordered Banach space X with a closed, generating and normal cone, the following are equivalent:

i. X has the Riesz decomposition property
ii. X' is a Riesz space
iii. X' has the Riesz decomposition property.

Theorem 3.2 [7] Let X be a complete, metrizable topological linear space. Suppose that A, B are closed wedges such that, given $x \in X$, there exist a bounded sequences $\{a_n\} \in A$ and $\{b_n\} \in B$ such that $a_n - b_n \rightarrow x$. Then A, B gives an open decomposition of X.

The above theorem can be simply put as: In a Banach space any closed and generating cone gives an open decomposition.

The following proposition is due to a remark given in [5].

Proposition 3.3 If a Banach space X has a reflexive and generating cone P. Then X is reflexive.

Proof

Let X be a Banach space with reflexive and generating cone P . Since P is a reflexive cone, then if $x_n \in P$ and $x_n \to x$, there exist $\alpha > 0$ such that

$x_n \in \alpha B_X^+ \ for\ each\ n, \Rightarrow x \in \alpha B_X^+ \subseteq P$

Thus, P is closed. Since P is closed and also generating, by Theorem 3.2 P gives an open decomposition. i.e. there exist $\alpha > 0$ so that $\alpha B_X \subseteq B_X^+ - B_X^+$. But the cone P is reflexive, therefore B_X^+ is weakly compact. Thus, αB_X is also weakly compact. Therefore the space X is reflexive.

Theorem 3.4 [3] If the cone P in a locally convex space X is normal, then it dual wedge P' is generating. That is $X' = P' - P'$.

Theorem 3.5 [3] If the cone of an ordered Banach space is closed and generating, then its dual cone is norm normal

Since locally convex spaces are examples of topological vector space that generalizes normed spaces, we can connect the two theorems above, and have the following Corollary.

Corollary 3.6 If the cone of an ordered Banach space is closed, normal and generating, then its dual cone P' is norm normal and generating.

Proposition 3.7 If an ordered Banach space X with a normal, generating and reflexive cone P has a Riesz decomposition property. Then, X is a Riesz space.

Proof 2

Let X be an ordered Banach space with normal, generating and reflexive cone P, from the proof of Proposition 3.3 we know that a reflexive cone in a Banach space is closed. Thus, X is an ordered Banach space with closed, generating and normal cone P and by Theorem 3.1, the dual X' is also a Riesz space, since X has a Riesz decomposition property.

By Corollary 3.6 the dual cone of X' is also a norm normal and generating, because X has a closed, normal and generating cone. Thus X' has a Riesz decomposition property from Theorem 3.1, which means the second dual X'' is also a Riesz space.

Now since the cone P is reflexive, by Proposition 3.3, X is also reflexive. Hence the canonical embedding between X and X'' is isometrically isomorphic. Thus, X is a Riesz space.

Theorem 3.8 Let X be an ordered normed vector space with a reflexive cone P. Then, X is an ordered Banach space.

Proof

Let x_n and y_n be two arbitrary Cauchy sequence in X and P be a reflexive cone. Since X is an ordered normed vector space. Then,

$x_n \leq y_n \Leftrightarrow y_n - x_n \in P$

It suffices to show that $\lim_{n\to\infty} x_n \to x \in X$ and $\lim_{n\to\infty} y_n \to y \in X$. Taking the limit of both side, we have $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n \Rightarrow \lim_{n\to\infty} y_n - \lim_{n\to\infty} x_n \in P$, since P is reflexive

$$\lim_{n\to\infty} y_n - \lim_{n\to\infty} x_n = \lim_{n\to\infty}(y_n - x_n) \in P$$

$$= \lim_{n\to\infty}(y_n^{**} - x_n^{**}) \in P^{**}$$

$$= y - x \in P \Rightarrow x \leq y$$

Thus, $\lim_{n\to\infty} x_n \leq \lim_{n\to\infty} y_n \Rightarrow x \leq y$. Hence $\lim_{n\to\infty} x_n = x$ and $\lim_{n\to\infty} y_n = y$.

Remark 3.9

If X is an ordered Banach space, the cone P in X is not necessarily reflexive. The following theorem gives the necessary and sufficient condition for a cone P of a Banach space to be reflexive.

Theorem 3.10 [5] A closed cone P of a Banach space X is reflexive if and only if P does not contain a closed cone isomorphic to the positive cone of l_1.

Example 3.11 Let $l_1$be the space of real sequences $x = (x_i)_1^\infty$ such that $\sum_{i=1}^\infty |x_i| < \infty$ endowed with a norm

$$\|x\|_1 = \sum_{i=1}^{\infty} |x_i|$$

Let also l_∞ be the space of real bounded sequences x = $(x_i)_1^\infty$ endowed with a supremum norm $\|x\|_\infty = \max_{i\in N} |x_i|$

Let l_1^+ be the positive cone of l_1 defined by $l_1^+ = \{x = (x_i)_1^\infty \in l_1 \mid x_i \geq 0, i = 1,2,\ldots\}$

The above spaces l_1 and l_∞ are all Banach spaces but their positive cones are isomorphic to the positive cone l_1^+ defined above. Thus, by Theorem 3.10 l_1^+ and l_∞^+ are not reflexive.

Definition 3.12 [3] An ordered vector space X is said to be Archimedean or has an Archimedean property whenever it follows from $y \in X, x \in X^+$, and $ny \leq x$ for all $n = 1,2,\ldots$ and $y \leq 0$.

A cone P of a vector space X is said to be Archimedean if the order induced by P on X makes X an Archimedean ordered vector space.

Proposition 3.13 Let X be an ordered vector space with a reflexive cone P. Then, X is an Archimedean space.

Proof

We are to show that $nx \leq y$ for $n \in \mathbb{N}, x \in X$ and $y \in P \Rightarrow x = 0$.

Let $nx \leq y$ for $x \in X$ and $y \in P$, then $x \leq \frac{1}{n}y \Rightarrow \frac{1}{n}y - x \in P$. But P is reflexive, thus $\lim_{n\to\infty} \frac{1}{n}y - x \to 0 - x = -x \in P$, hence $x \in -P$ which implies that $x = 0$.

Below are some examples of an ordered vector space with a Reflexive cone that are Archimedean.

Example 3.14

1. The space $\mathbb{R}^n$ with a positive cone $P = \{(x_1, x_2, \ldots, x_2) \in \mathbb{R}^n \mid x_i \geq 0, i = 1,2,\ldots,n\}$ which has a coordinate-wise ordering is an Archimedean space. The cone P defined above is a Reflexive cone in $\mathbb{R}^n$.
2. Let $1 < p < \infty$. The sequence space $l_p = \{x = (x_1, x_2, \ldots) \mid \sum_{i=1}^{\infty} |x_i|^p < \infty\}$ with a cone $l_p^+ = \{x = (x_1, x_2, \ldots) \mid x_i \geq 0, i = 1,2,\ldots, \sum_{i=1}^{\infty} |x_i|^p < \infty\}$ with the same ordering as above, is an Archimedean space.

We are now ready to give another property of an Archimedean space in terms of reflexive cone which is Archimedean Riesz space. Archimedean Riesz space has some interesting property that Band and Disjoint complement in them are the same, see [10].

Theorem 3.15 If an ordered Banach space with normal, generating and reflexive cone P has a Riesz decomposition property. Then X is an Archimedean Riesz space.

Proof

Let X be an ordered Banach space with normal, generating and reflexive cone P which has a Riesz decomposition property, then X is a Riesz space by Proposition 3.7. Also by Proposition 3.13 the space X is Archimedean. Hence X is an Archimedean Riesz space.

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