Strong Independent Functions

M. Kavitha[1,*],

[1]Department of Mathematics, KPR Institute of Technology, Coimbatore - 641407, India.

Abstract

A subset S of the vertex set V of a graph G is said to be independent, if no two vertices of S are adjacent. Independent functions and maximal independent functions have been defined and studied already. In this chapter, strong independent functions, maximal strong independent functions and basic maximal strong independent functions are defined and a study of these is made.

Keywords: Strong Independent Function, Maximal Strong Independent Function, Universal Maximal Strong Independent Function

I. Introduction

A function $f : V(G) \to [0,1]$ is called a strong independent function, if the value of the closed strong neighbourhood of any vertex under f is 1, if the vertex gets positive value under f. A function f is called a maximal strong independent function, if it is a strong independent function and the value of the function on the closed strong neighbourhood of a vertex which gets zero value under f is greater than or equal to one. A detailed study of these functions is made in the following.

Definition 1.1: Let $G = (V, E)$ be a simple graph. A function $f : V(G) \to [0,1]$ is called an independent function if for every vertex v with $f(v) > 0$, $\sum\limits_{u \in N[v]} f(u) = 1$.

Definition 1.2: Let $G = (V, E)$ be a simple graph. An independent function $f : V(G) \to [0,1]$ is called a maximal independent function if for any $v \in V$ with $f(v) = 0$,
$\sum\limits_{u \in N[v]} f(u) \geq 1$.

Definition 1.3: Let $G = (V, E)$ be a simple graph. A function $f : V(G) \to [0,1]$ is called a maximal independent function if f is an independent function and for any independent function g, $f \leq g \Rightarrow f = g$.

II. Maximal Strong Independent Functions

Definition 2.1: A function $f : V(G) \to [0,1]$ is called a **Strong Independent Function (SIF)** if for any $u \in V(G)$, $f(u) > 0 \Rightarrow f(N_s[u]) = 1$, where $N_s[u] = \{x \in N[u] \ : \ deg\ x \geq deg\ u\}$.

Definition 2.2: A function $f : V(G) \to [0,1]$ is called a **Maximal Strong Independent Function (MSIF)** if for any $u \in V(G)$,
$f(u) > 0 \Rightarrow f(N_s[u]) = 1$ and
$f(u) = 0 \Rightarrow f(N_s[u]) \geq 1$.

Definition 2.3: $P_f = \{v \in V(G) \ : \ f(v) > 0\}$ and
$B_f^s = \{v \in V(G) \ : \ f(N_s[v]) = 1\}$.

Theorem 2.4: A function $f : V(G) \to [0,1]$ is a strong independent function if and only if $P_f \subseteq B_f^s$.

Proof:
Let $f : V(G) \to [0,1]$ be a strong independent function.

Let $u \in P_f$. Then $f(u) > 0$.
Since f is a strong independent function, $f(N_s[u]) = 1$.
Therefore, $u \in B_f^s$.
Conversely, let $P_f \subseteq B_f^s$.
Let $f(u) > 0$. Then $u \in P_f$.
Therefore, $u \in B_f^s$.
Therefore, $f(N_s[u]) = 1$.
Therefore, f is a strong independent function.

Theorem 2.5: Every MSIF is a minimal strong dominating function.

Proof:

Let $f : V(G) \to [0,1]$ be a MSIF.
Let $u \in V(G)$. Then $f(N_s[u]) \geq 1$.
Therefore, f is a strong dominating function.
Let $g : V(G) \to [0,1]$ be a strong dominating function such that $g \leq f$.
Suppose there exists $u \in V(G)$ such that $g(u) < f(u)$.
Therefore, $f(u) > 0$.
Therefore, $f(N_s[u]) \geq 1$.
Since $g \leq f$, $g(N_s[u]) < 1$.
Therefore, g is not a strong dominating function, a contradiction
Therefore, $g = f$.
Therefore, f is a minimal strong dominating function.

Remark 2.6: If f is a maximal strong independent function of G, then B_f^s is a strong dominating set of G.

Proof:

Let $u \in V(G) - B_f^s$.
Since $P_f \subseteq B_f^s$, $u \notin P_f$.
Therefore, $f(u) = 0$.
Therefore, $f(N_s[u]) \geq 1$.
Therefore, there exists $v \in N_s(u)$ such that $f(v) > 0$.
Therefore, $v \in P_f$. Therefore, $v \in B_f^s$.
Therefore, B_f^s strongly dominates u.
Therefore, B_f^s is a strong dominating set of G.

Remark 2.7: The convex combination of two strong independent functions need not be a strong independent function.

For example,
let $G = P_3$.
Let $V(P_3) = \{u_1, u_2, u_3\}$ where u_1 and u_3 are pendant vertices.
Let $f_1,\ f_2,\ f_3\ :\ V(G) \to [0,1]$ be defined as follows:
$f_1(u_1) = 0,\ f_1(u_2) = 1,\ f_1(u_3) = 0$
$f_2(u_1) = f_2(u_3) = 1,\ f_2(u_2) = 0$
$f_3(u_1) = f_3(u_2) = f_3(u_3) = \dfrac{1}{2}$.
$P_{f_1} = \{u_2\}$,
$B_{f_1} = \{u_1, u_2, u_3\}$,
$P_{f_2} = \{u_1, u_3\}$,

$B_{f_2} = \{u_1, u_3\}$,
$P_{f_3} = \{u_1, u_2, u_3\}$,
$B_{f_3} = \{u_1, u_3\}$.
Since $P_{f_1} \subseteq B_{f_1}$, $P_{f_2} \subseteq B_{f_2}$, f_1 and f_2 are strong independent functions.
Here, f_3 is not a strong independent function, since $P_{f_3} \nsubseteq B_{f_3}$.
Clearly, $f_3 = \frac{1}{2}f_1 + \frac{1}{2}f_2$.
f_3 is a convex combination of strong independent functions f_1 and f_2
and f_3 is not a strong independent function.

Theorem 2.8: Let f_1 and f_2 be two strong independent functions of G.
Let $0 < \lambda < 1$.
Then $h_\lambda = \lambda f_1 + (1-\lambda)f_2$ is a strong independent function if and only if $P_{f_1} \cup P_{f_2} \subseteq B^s_{f_1} \cap B^s_{f_2}$.

Proof:

Let f_1 and f_2 be strong independent functions of G. Let $0 < \lambda < 1$.
Let $h_\lambda = \lambda f_1 + (1-\lambda)f_2$.
Suppose h_λ is a strong independent function.
Then $P_{h_\lambda} \subseteq B^s_{h_\lambda}$.
Let $u \in P_{f_1} \cup P_{f_2}$.
Therefore, $u \in P_{f_1}$ or $u \in P_{f_2}$.
If $u \in P_{f_1}$, then $u \in B^s_{f_1}$ (since f_1 is a strong independent function of G).
Suppose $u \in P_{f_2}$.
Then $u \in B^s_{f_2}$ (since f_2 is a strong independent function of G).
Therefore, $u \in B^s_{f_1} \cap B^s_{f_2}$.
Suppose $u \notin P_{f_2}$.
Therefore, $f_2(u) = 0$.

$$P_{h_\lambda}(u) = \lambda f_1(u) + (1-\lambda)f_2(u)$$
$$= \lambda f_1(u)$$

Since $P_{h_\lambda}(u) \subseteq B^s_{h_\lambda}(u)$, $h_\lambda(N[u]) = 1$
That is, $(\lambda f_1 + (1-\lambda)f_2)(N[u]) = 1$

$$\lambda f_1(N[u]) + (1-\lambda)f_2(N[u]) = 1$$
$$\lambda + (1-\lambda)f_2(N[u]) = 1$$

Therefore, $(1-\lambda)f_2(N[u]) = 1 - \lambda$

$$f_2(N[u]) = 1$$

Therefore, $u \in B^s_{f_2}$.
Hence $P_{f_1} \cup P_{f_2} \subseteq B^s_{f_1} \cap B^s_{f_2}$.
Conversely, let $P_{f_1} \cup P_{f_2} \subseteq B^s_{f_1} \cap B^s_{f_2}$.
Let $u \in P_{h_\lambda}$. Therefore $h_\lambda(u) > 0$.
If $u \in P_{f_1}$ and $u \in P_{f_2}$, then $u \in B^s_{f_1} \cap B^s_{f_2}$.
$f_1(N[u]) = 1$, $f_2(N[u]) = 1$.
Therefore, $\lambda f_1(N[u]) + (1-\lambda)f_2(N[u])$

$$= \lambda + 1 - \lambda = 1.$$

Therefore, $u \in B^s_{h_\lambda}$.
Since $h_\lambda(u) > 0$, $\lambda f_1(u) + (1-\lambda)f_2(u) > 0$.
At least one of $f_1(u)$, $f_2(u)$ is > 0.

Suppose $u \in P_{f_1}$ and $u \notin P_{f_2}$.
(similar proof for $u \notin P_{f_1}$ and $u \in P_{f_2}$).
Then $f_1(N[u]) = 1$.
$\lambda f_1(N[u]) + (1-\lambda)f_2(N[u]) = \lambda + (1-\lambda)f_2(N[u]) \quad \longrightarrow \quad (i)$
Since $u \in P_{f_1} \cup P_{f_2}$, $u \in B^s_{f_1} \cap B^s_{f_2}$.
Therefore, $u \in B^s_{f_2}$. $f_2(N[u]) = 1$.
Therefore, (i) gives $h_\lambda(N[u]) = \lambda + 1 - \lambda = 1$.
Therefore, $u \in B^s_{h_\lambda}$.
That is, $P_{h_\lambda} \subseteq B^s_{h_\lambda}$.
Hence h_λ is a strong independent function.

Remark 2.9: Let f and g be two strong independent functions. If $\lambda f + (1-\lambda)g$ is a strong independent function for some λ, $0 < \lambda < 1$, then any convex combination of f and g is strong independent.

For, since $\lambda f + (1-\lambda)g$ is a strong independent function, $P_f \cup P_g \subseteq B^s_f \cap B^s_g$. Since this is independent of λ, any convex combination of f and g is also strong independent.

Remark 2.10: If f and g are strong independent functions, then either no convex combination of f and g is strong independent or every convex combination of f and g is strong independent.

Theorem 2.11: Let f and g be two maximal strong independent functions. Then either all convex combinations of f and g are maximal strong independent functions or no one of them is a maximal strong independent function.

Proof:
Let f and g be two maximal strong independent functions.
Let $0 < \lambda < 1$.
Let $h_\lambda = \lambda f + (1-\lambda)g$. Then h_λ is strong independent if and only if $P_f \cup P_g \subseteq B^s_f \cap B^s_g$.
Let $u \in V(G)$. Suppose $h_\lambda(u) = 0$.
Therefore, $\lambda f(u) + (1-\lambda)g(u) = 0$.
Therefore, $f(u) = 0$ if and only if $g(u) = 0$.
Since at least one of $f(u)$ or $g(u)$ is zero, we get that both $f(u)$ and $g(u)$ are equal to zero.
Therefore, $f(N[u]) \geq 1$ and $g(N[u]) \geq 1$ (since f and g are maximal strong independent functions).
Therefore, $\lambda f(N[u]) + (1-\lambda)g(N[u]) \geq \lambda + (1-\lambda)$
$= 1$.
Therefore, $h_\lambda(N[u]) \geq 1$.
Therefore, h_λ is a maximal strong independent function if and only if
$P_f \cup P_g \subseteq B^s_f \cap B^s_g$.
Hence the theorem.

Remark 2.12: Let f and g be two MSIF^s. If $h_\lambda = \lambda f + (1-\lambda)g$ is strong independent, then h_λ is a MSIF.

Definition 2.13: Let f be a MSIF. f is said to be **Universal Maximal Strong Independent Function (UMSIF)**, if the convex combinaiton of f with any other maximal strong independent function is a MSIF.

Remark 2.14: A MSIF f of a graph G is universal if and only if $P_f \cup P_g \subseteq B^s_f \cap B^s_g$, for any MSIF g.
For example,
Let $G = K_3$.
Let $f : V(G) \to [0, 1]$ be defined by
$f(v_1) = 1$, $f(v_2) = f(v_3) = 0$, where $V(K_3) = \{v_1, v_2, v_3\}$.
$f(N[v_i]) = 1$, $\forall\ i$, $1 \leq i \leq 3$
Therefore, f is a MSIF.

Let g be any MSIF on K_3.
$v_i \in P_f \cup P_g$ if and only if $f(v_i) > 0$ (or) $g(v_i) > 0$.
$f(N[v_i]) = 1, \forall\ i,\ 1 \leq i \leq 3$.
Therefore, $v_i \in B^s_f$.
Suppose $g(v_i) = 0$.
If $g(v_j) = 1$, for $j \neq i$, then $g(N[v_j]) = 2$, a contradiction.
Therefore, $g(v_j) = 1$, for exactly one j, $j \neq i$, $1 \leq j \leq 3$.
Therefore, $g(N[v_i]) = 1$.
Therefore, $v_i \in B_g$. Therefore, $v_i \in B^s_f \cap B^s_g$.
Therefore, $P_f \cup P_g \subseteq B^s_f \cap B^s_g$.
Hence f is a UMSIF.

Definition 2.15: A function $f : V(G) \to [0,1]$ is **positive** if $f(u) > 0$ for at least one $u \in V(G)$.

Theorem 2.16: Any SIF on K_3 which is positive is a UMSIF.

Proof:

Let f be a SIF on K_3 whose vertex set is $\{v_1, v_2, v_3\}$.
Suppose f is positive on K_3.
Let $f(v_1) = \alpha > 0$, $f(v_2) = \beta$ and $f(v_3) = \gamma$.
Then $\beta + \gamma + \alpha = 1$.
Therefore, $f(N[v_i]) = 1, \forall\ i,\ 1 \leq i \leq 3$.
Therefore, f is a MSIF. Let g be a MSIF.
Suppose $h_\lambda = \lambda f + (1-\lambda)g$ is strong independent.
Then $P_f \cup P_g \subseteq B^s_f \cap B^s_g$.
Let $h_\lambda(v_1) = 1$. Therefore, $\lambda f(v_1) + (1-\lambda)g(v_1) = 0$.
Therefore, $f(v_1) = 0$ and $g(v_1) = 0$.
Therefore, $f(N[v_1]) \geq 1$ and $g(N[v_1]) \geq 1$.
Therefore, $h_\lambda(N[v_1]) \geq 1$.
Therefore, h_λ is a MSIF.
Let $0 < \lambda < 1$. Let $u \in P_f \cup P_g$.
Clearly, $f(N[u]) = 1$ and $g(N[u]) = 1, \forall\ u \in V(K_3)$.
Therefore, $u \in B^s_f \cap B^s_g$. Therefore $B_f \cap B_g = V(K_3)$.
Therefore, $P_f \cup P_g \subseteq B^s_f \cap B^s_g$.
Therefore, for any λ, $0 < \lambda < 1$, h_λ is SIF.
Hence f is a UMSIF.

Remark 2.17: Any SIF on K_n which is positive is a UMSIF.

Observation 2.18: If f is a UMSIF, then $B^s_f = V(G)$.

Proof:

Let $u \in V(G)$.
Then $\{u\}$ is strong independent and hence is contained in a maximal strong independent set, say D of G.
Therefore, χ_D is a maximum strong independent function of G.
Since f is a UMSIF, $P_f \cup P_{\chi_D} \subseteq B^s_f \cap B^s_{\chi_D}$.
Since $\chi_D(u) = 1$, $u \in P_{\chi_D}$.
Therefore, $u \in P_f \cup P_{\chi_D}$.

Therefore, $u \in B^s_f \cap B^s_{\chi_D}$.
Therefore, $u \in B^s_f$.
Therefore, $V(G) \subseteq B^s_f$.
But $B^s_f \subseteq V(G)$.
Hence $B^s_f = V(G)$.

Observation 2.19: If there exist two MSIF^s f and g such that $B^s_f \cap B^s_g = \phi$, then the graph G has no UMSIF.

Proof:

Suppose G has a UMSIF h.
Therefore, any convex combination of h and f is a MSIF.
Therefore, $P_f \cup P_h \subseteq B^s_f \cap B^s_h$.
Similarly, $P_h \cup P_g \subseteq B^s_h \cap B^s_g$.
Therefore, $P_h \subseteq B^s_f \cap B^s_g$.
$B^s_f \cap B^s_g = \phi$, by hypothesis.
Therefore, $P_h = \phi$.
Since h is a MSIF, $P_h \neq \phi$, a contradiction.
Hence G has no UMSIF.

Observation 2.20: Let G be a regular bipartite graph with $\delta(G) \geq 2$. Then G has no UMSIF.

Proof:

Let G be a regular bipartite graph with $\delta(G) \geq 2$.
Let V_1 and V_2 be the bipartitions of $V(G)$.
Define f and g by

$$\begin{aligned} f(v) &= 1, \quad \text{if } v \in V_1 \text{ and} \\ f(v) &= 0, \quad \text{if } v \in V_2 \\ g(v) &= 1, \quad \text{if } v \in V_2 \text{ and} \\ g(v) &= 0, \quad \text{if } v \in V_1. \end{aligned}$$

Then f and g are MSIF^s with $B^s_f \cap B^s_g = \phi$.
Therefore, by the above observation, G has no UMSIF.

References

[1] Acharya, B.D., *The Strong Domination Number of a Graph and Related Concepts*, J.Math. Phys. Sci., 14,471-475(1980).

[2] Allan R.B., and Laskar.R.C., *On Domination and Independent Domination Number of a Graph*, Dicrete Math., 23, 73-76(1978).

[3] Arumugam.S. and Regikumar.K., *Basic Minimal Dominating Functions*, Utilitas Mathematica, 77, pp. 235-247(2008).

[4] Arumugam.S. and Regikumar.K., *Fractional Independence and Fractional Domination Chain in Graphs*, AKCE J.Graphs Combin., 4, No.2, pp.161-169(2007).

[5] Arumugam.S. and Sithara Jerry.A., *A Note On Independent Domination in Graphs*, Bulletin of the Allahabad Mathematical Society, Volume 23, Part 1.(2008).

[6] Arumugam.S. and Sithara Jerry.A., *Fractional Edge Domination in Graphs*, Appl. Anal. Discrete Math.,(2009).

[7] Berge.C., *Theory of Graphs and its Applications*, Dunod, Paris(1958).

[8] Berge.C., *Graphs and Hypergraphs*, North-Holland, Amsterdam(1973).

[9] Cockayne E.J., Dawes R.M. and Hedetniemi.S.T., *Total domination in graphs*, Networks, 10, 211-219(1980).

[10] Cockayne E.J., Favaron O., Payan C. and Thomasaon A.G., *Contributions to the theory of domination, independence and irredundance in graphs*, Discrete Math., 33, 249-258(1981).

[11] Cockayne E.J., MacGillivray G. and Mynhardt C.M., *Convexity of minimal dominating functions and universal in graphs*, Bull. Inst. Combin. Appl., 5, 37-38(1992).

[12] Cockayne E.J. and Mynhardt C.M., *Convexity of minimal dominating functions of trees: A survey*, Quaestiones Math., 16, 301-317(1993).

[13] Cockayne E.J., MacGillivray G. and Mynhardt C.M., *Convexity of minimal dominating functions of trees-II*, Discrete Math., 125, 137-146(1994).

[14] Cockayne E.J. and Mynhardt C.M., *A characterization of universal minimal total dominating functions in trees*, Discrete Math., 141, 75-84(1995).
[15] Cockayne.E.J., Fricke.G., Hedetniemi.S.T. and Mynhardt.C.M., *Properties of Minimal Dominating Functions of Graphs*, Ars Combin., 41,107-115(1995).
[16] Cockayne E.J., MacGillivray G. and Mynhardt C.M., *Convexity of minimal dominating functions of trees*, Utilitas Mathematica, 48, 129-144(1995).
[17] Cockayne.E.J. and Hedetniemi.S.T., *Towards a Theory of Domination in Graphs*, Networks, 7,247-261(1977).
[18] Cockayne.E.J. and Mynhardt.C.M., *Minimality and Convexity of dominating and related functions in Graphs*, A unifying theory, Utilitas Mathematica, 51,145-163(1997).
[19] Cockayne.E.J. and Mynhardt.C.M. and Yu.B., *Universal Minimal Total Dominating Functions in Graphs*, Networks, 24, 83-90(1994).
[20] Grinstead D. and Slater P.J., *Fractional Domination and Fractional Packing in Graphs*, Congr., Numer., 71, 153-172(1990).
[21] Harary.F., *Graph Theory*, Addison-Wesley, Reading, Mass(1969).
[22] Hedetniemi.S.T. and Laskar.R.L., *Bibliography on Domination in Graphs and Some Basic Definitions of Domination Parameters*, Discrete Math., 86,pp 257-277(1990).
[23] Ore.O, *Theory of Graphs*, AMS(1962).
[24] Regikumar K., *Dominating Functions* - Ph.D thesis, Manonmaniam Sundaranar University(2004).
[25] Sampathkumar.E and Pushpa Latha.L., *Strong Weak Domination and Domination Balance in a Graph*, Discrete Math., 161,235-242(1996).
[26] Scheinerman E.R. and Ullman D.H., *Fractional Graph Theory: A rational approach to the theory of graphs*, John Wiley and Sons Inc.,(1997).
[27] Terasa W. Haynes, Stephen T. Hedetneimi, Peter J. Slater, *Fundamentals of Domination in Graphs*, Marcel Dekker Inc.,(1998).
[28] Terasa W. Haynes, Stephen T. Hedetneimi, Peter J. Slater, *Domination in Graphs: Advanced Topics* Marcel Dekker Inc., New York(1998).
[29] Yu.B., *Convexity of Minimal Total Dominating Functions in Graphs*, J.Graph Theory, 24,313-321(1997).