# Fibonacci Sequence with Golden Ratio and Its Application 

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#### Abstract

In this paper the Fibonacci numbers are an interesting sequence of integers and related to the Golden Ratio and many of the natural things such as branching trees, the arrangement of leaves on a stem (phyllotaxis), the birth rates of rabbits and other natural phenomena, that shows up in many places in mathematics. The Fibonacci numbers play very important role in coding theory.


Keywords - Fibonacci numbers, Golden Ratio, Natural phenomena, Coding

## I. INTRODUCTION

In recent mathematical trained, there are many useful concepts in the several aspects with applications. The concept of Fibonacci number is one of them. These concepts were discovered by mathematician Leonardo Pisano Fibonacci, found in nature such as Marry gold portals, Sun flowers, some tree branches, the hypothetical birth rates of rabbits and also in other natural phenomena. That has been shown in many places in mathematics in the $13^{\text {th }}$ century and it is known as the Fibonacci sequence. The $\mathrm{n}^{\text {th }}$ Fibonacci number is denoted by $\mathrm{F}_{\mathrm{n}}$ and is defined for $n=1$ by $F_{1}=F_{2}=1$ and for $n \geq 2$ by $F_{n}=F_{n-1}+F_{n-2}$.

These sequences become one of the most intriguing concepts in the entire of mathematics with remarkable characteristics and its useful applications to various mathematical fields such as Number Theory, Discrete mathematics, geometry and its clear presentation of the elegant nature of God.

## II. FIBONACCI NUMBERS

The numbers $1,1,2,3,5,8,13,21,34,55, \ldots$ are called Fibonacci numbers(e.g. [2, 6]). Any Fibonacci number except first two terms is the sum of two immediately preceding Fibonacci numbers. Consequently the following recursive definition of nth Fibonacci number $F_{n}$ such that

$$
\begin{array}{cl}
F_{1}=F_{2}=1 & \text { (Initial condition ) } \\
F_{n}=F_{n-1}+F_{n-2}, \mathrm{n} \geq 3 & \text { (Recurrence relation) }
\end{array}
$$

Where $n$ is any positive number.
The Fibonacci numbers contain even index and odd index. The numbers $F_{2}, F_{4}, F_{6} \ldots F_{2 n}$ are called even index Fibonacci numbers \& $F_{1}, F_{3}, F_{5}, \ldots, F_{2 n-1}$ are called odd index Fibonacci numbers. Where $F_{1}=1, F_{2}=1$, $F_{3}=2, F_{4}=3, F_{5}=5, F_{6}=8, F_{7}=13, \quad F_{8}=21, F_{9}=34, F_{10}=55, F_{11}=89, F_{11}=144, \ldots$

Thus $1,3,8,21,55,144, \ldots$ are even Fibonaccis, and $1,2,5,13,34,89, \ldots$ are odd Fibonaccis, since their indexes are even and odd but the numbers themselves are not always even and odd respectively.

## A. Characteristics of Fibonacci numbers and hypothetical situation

Fibonacci also introduced a unique sequence of numbers with interesting characteristics. This sequence eventually become one of the most famous sequences in the field of mathematics. In $19^{\text {th }}$ century when a number theorist named Edouard Lucas examined a problem in Fibonacci Liber Abaci and linked Fibonacci's name to the sequence that problem involves. Fibonacci introduced the sequence as the following hypothetical situation:

A man put one pair of rabbits in a certain place entirely surrounded by a wall. How many pairs of rabbits can be produced from that pair in a year. If the nature of these rabbits is such that every month each pair bears a new pair which the second month on becomes producing? This particular problem began by Fibonacci to examine and discovered a sequence involving the numbers of pairs of rabbits.

The problem begins with a pair of baby rabbits. Once the first month has concluded, that initial pair of baby rabbits has reached adulthood and is now capable of reproducing (e.g. [1]). Assuming that the average gestation period for a rabbit is one month, the initial pair of rabbits will give birth to a second pair of rabbits at the beginning of the third month. At this point in time in the problem, there currently exist a pair of adult rabbits and
a pair of baby rabbits. Fibonacci assumes in his problem that once a pair of rabbits has reached adulthood, they reproduce another pair of rabbits each month afterward (e.g. [1]). The current pair of baby rabbits in the problem is able to reproduce by the beginning of the fourth month, and they give birth to a pair of baby rabbits each month thereafter. In order to maintain uniformity in his problem, Fibonacci also assumes that none of the rabbits die (e.g. [2]). After each month of the problem, Fibonacci counted the number of pairs of rabbits; and his conclusions led him to a sequence of numbers with the number of pairs of rabbits as the terms of the sequence and the corresponding month numbers as the subscripts for those terms. Fibonacci's rabbit problem is illustrated in Figure 1 with each rabbit image representing a pair of rabbits (e.g. [1]). The smaller rabbit images represent rabbits that have been newly birthed while the larger rabbit images represent adult rabbits that are at least one month old.


Fig. 1.1: Fibonacci's hypothetical rabbit problem.

## B. The Fibonacci Problem

Suppose there are two new born rabbits one male other female. The number of rabbits produces in a year if

- Each pair takes one month to become mature
- Each pair produces a mixed pair every month from the second month and
- No rabbits die.

For convenience, let us suppose that the original pair of rabbits was born in January one. They take a month to become mature so there is still only one pair in February one. On March one, they are two months old and produce a new mixed pair, a total of two pair. Continuing like this, there will be three pairs on April one, five pairs on May one and so on.
We draw a table as below:

| No. of Pairs | Jan. | Feb. | March | April | May | June | July |
| :--- | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| Adults | 0 | 1 | 1 | 2 | 3 | 5 | 8 |
| Babies | 1 | 0 | 1 | 1 | 2 | 3 | 5 |
| Total | 1 | 1 | 2 | 3 | 5 | 8 | 13 |

## C. Fibonacci sequence in nature:

Above mentioned the rabbit experiment is not only famous for Fibonacci numbers in nature, but the Fibonacci number is also famous for beautiful flowers (Internet access, 12 ). On the head of sunflower and the seeds are packed in a definite way. So that they follows the pattern of the Fibonacci sequence. The petals of flowers and other plants may also be related to the Fibonacci sequence in the way that they create new petals (Internet access, 10).

## D. The number or arrangement of Petals on flowers

The number or arrangement of petals on a flower that still has all of its petals intact and has not lost any, for many flowers is a Fibonacci number (Internet access, 8)
1 petal: white cally lily
3 petals : lily, iris
5 petals: buttercup, wild rose, larkspur, columbine
8 petals: delphiniums
13 petals :ragwort, corn marigold, cineraria.
21 petals: aster, black- eyed susan, chicory.


## E. Golden Ratio

The number $\frac{1+\sqrt{5}}{2}$ is known as the Golden Ratio (e.g. [5], [8]). It has also been called the Golden Section (in an 1835 book by Martin Ohm). It is thought to reflect the ideal proportions of nature and to even possess some mystical powers. It is an irrational number, and is denoted by the symbol $\phi$ :

$$
\phi=1.6180339887 \ldots
$$

It does have some very special, though not so mysterious, properties.
For example, its square,

$$
\phi^{2}=2.6180339887 \ldots
$$

is obtainable by adding 1 to $\phi$. Its reciprocal,

$$
1 / \phi=0.6180339887 \ldots
$$

is the same as subtracting 1 from $\phi$. These properties are not mysterious at all, if we recall that $\phi$ is a solution of equation.
In terms of $\phi$, the general solution can be written as :

$$
F_{n}=a \phi^{n}+b\left(-\frac{1}{\phi}\right)^{n}
$$

Since $\phi>1$, the second term diminishes in importance as $n$ increases, so that for $n>1$,

$$
F_{n} \approx a \phi^{n}
$$

Therefore the ratio of successive terms in the Fibonacci sequence approaches the Golden Ratio:

$$
\begin{equation*}
\frac{F_{n+1}}{F_{n}} \rightarrow \frac{a \phi^{n+1}}{a \phi^{n}}=\phi=1.6180339887 \ldots, \text { as } n \rightarrow \infty \tag{1}
\end{equation*}
$$

This ratio converging to the Golden Ratio is independent of $a$ and $b$, as long as $a$ is not zero, it is satisfied by all solutions to the difference equation, including the Lucas sequence, which is the sequence of numbers starting with $F_{0}=2$ and $F_{1}=1: 2,1,3,4,7,11,18,29, \ldots$.
For our later use, we also list the result.

$$
\begin{equation*}
\frac{F_{n+2}}{F_{n}} \rightarrow \frac{a \phi^{n+2}}{a \phi^{n}}=\phi^{2} \tag{2}
\end{equation*}
$$

As we know that an irrational number is a number that cannot be expressed as the ratio $\mathrm{m} / \mathrm{n}$ of two integers, $m$ and $n$. Mathematicians sometimes are interested in the rational approximation of an irrational number; that is, finding two integers, $m$ and $n$, whose ratio, $m / n$, gives a good approximation of the irrational number with an error.

Example 1:, the irrational number $\pi=3.14159265 \ldots$ can be approximated by the ratio $22 / 7=3.142857 \ldots$, with error 0.00126 . This is the best rational approximation if $n$ is to be less than 10 . When we make $m$ and $n$ larger, the error goes down rapidly.

Example 2: The number $355 / 113$ is a rational approximation of $\pi$ (with $n$ less than 200) with an error of 0.000000266 . We measure the degree of irrationality of an irrational number by how slowly the error of its best rational approximation approaches zero when we allow $m$ and $n$ to get bigger and bigger. In this sense $\pi$ is "not too irrational."

From equation (1) we see that the value of $\phi$ can thus be approximated by the rational ratio: $8 / 5=1.6$, or $13 / 8=1.625$, or $21 / 13=1.615385 \ldots$, or $34 / 21=1.619048 \ldots$, or $55 / 34=1.617647 \ldots$, or $89 / 55=1.618182$ $\ldots$, or $144 / 89=1.617978 \ldots$. Hence the ratios of successive terms in the Fibonacci sequence will eventually converge to the Golden Ratio. One therefore can use the ratio of successive Fibonacci numbers as the rational approximation to the Golden Ratio. Such rational ratios, however, converge to the Golden Ratio extremely slowly. Thus we might say that the Golden Ratio is the most irrational of the irrational numbers.

More importantly, the Golden Ratio has its own geometrical significance, first recognized by the Greek mathematicians Pythagoras ( $560-480$ BC), and Euclid ( $365-325$ BC). The Golden Ratio is the only positive number that, when 1 is subtracted from it, equals its reciprocal. Euclid in fact defined it, without using the name Golden Ratio, when he studied the division of a line into what he called the "extreme and mean ratio":

A straight line is said to have been cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the lesser.


Fig. 1.2: A straight line cut into extreme and mean ratios.


Fig. 1.3 : The Great Pyramid at Giza and the "Egyptian Triangle.

From figure 1.2, consider the straight line $a b c$, divide into two segments $a b$ and $b c$, in such a way that the "extreme ratio" $\overline{a b c} / \overline{a b}$ is equal to its "mean ratio" $\overline{b c} / \overline{a b c}$.

Without loss of generality, let the length of small segment $\overline{b c}$ be 1 , and $\overline{a b}$ be $x$, so the whole line $\overline{a b c}$ is $1+x$. The line is said to be cut in extreme and mean ratio when $\quad(1+x) / x=1 / x$; this is the same as $x^{2}=x+1$, which is equation is the only positive root of that equation. Many authors reported that the ancient Egyptians possessed the knowledge of the Golden Ratio even earlier and incorporated it in the geometry of the Great Pyramid of Khufu at Giza, to 2480 BC. Midhat Gazale, who was the president of AT\&T-France, wrote in his popular 1999 book, Gnomon: From Pharaohs to Fractals:

It was reported that the Greek historian Herodotus learned from the Egyptian priests that the square of the Great Pyramid's height is equal to the area of its triangular lateral side. Referring to Figure 1.3, we consider the upright right triangle formed by the height of the pyramid (from its base to its apex), the slanted height of the triangle on its lateral side (the length from the base to the apex of the pyramid along the slanted lateral triangle), and a horizontal line joining these two lines inside the base.

We see that if the above statement is true, then the ratio of the hypotenuse to the base of that triangle is equal to the Golden Ratio. However, as pointed out by Mario Livio in his wonderful 2002 book, The Golden Ratio, Gazale was repeating an earlier misinterpretation by the English author John Taylor in his 1859 book, The Great Pyramid: Why Was It Built and Who Built It, in which Taylor was trying to argue that the construction of the Great Pyramid was through divine intervention. "Its base is square, each side is eight plethra long and its height the same." One plethronwas 100 Greek feet, approximately 101 English feet. Nevertheless, there is no denying that the physical dimensions of the Great Pyramid as it stands now do give a ratio of hypotenuse to base rather close to the Golden Ratio. The base of the pyramid is approximately a square with sides measuring 756 feet each, and its height is 481 feet. So the base of the upright right triangle is $756 / 2=378$ feet, while the hypotenuse is, by the Pythagorean Theorem, 612 feet. Their ratio is then $612 / 378=1.62$, which is in fact quite close to the Golden Ratio. The debate continues. All we can say is that, casting aside the claims of some religious cults, there is no historical or archeological evidence that the ancient Egyptians knew about the Golden Ratio.

## F. Fibonacci number use in coding

At present the Fibonacci sequence and Golden ratio use in different field of science, high energy Physics, Cryptography and Coding. Many researchers are taking interest to use the both numbers. Ref. [7] developed a paper classical encryption technique for securing data. Ref. [1] showed that communication may be secured by the use of Fibonacci numbers. For the application of Fibonacci in Cryptography, we consider an example. Let us suppose that the original Message 'CODE" to be Encrypted. It is sent through an unsecured channel. Security key is chosen based on the Fibonacci number. Any one character may be chosen as a first security key to generate cipher texted and then Fibonacci sequence can be used. The Fibonacci sequence used for Encryption data by References [2-4].

## III. CONCLUSION

The Fibonacci numbers are an interesting sequence of numbering system who is appear in nature everywhere, from leaf arrangement in plans to the pattern of the florets of a flower, the bracts of a pinecone and the Golden Ratio, continued fractions, many of the natural things such as branching trees, the birth rates of rabbits and other natural phenomena, that shows up in many places in mathematics. The Fibonacci numbers play very important role in coding theory.

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