A Study on Pseudo 0 - Distributivity And Super 0 – Distributivity In The Subgroup Lattice of 2X2 Matrices Over Z_3, Z_5 AND Z_7

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Abstract: *In this paper we verify the lattice theoretic properties like the pseudo 0 - distributivity and super 0 – distributivity in the subgroup lattice of 2x2 matrices over z_3, z_5 and z_7.*

Keywords: *Matrix group, subgroups, Lattice, pseudo 0 – distributivity, super 0 -distributivity*

1. Introduction:

Let L(G) denotes the lattice of subgroups of G, where G is the group of 2×2 matrices over Zp having determinant value 1 under matrix multiplication modulo p, where p is a prime number. Let $\mathcal{G}$ = $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix}$:a,b,c,d $\in Z_p$, ad-bc≠ 0}. Then $\mathcal{G}$ is a group under matrix multiplication modulo p. Let G = $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathcal{G}$: ad-bc = 1}. Then G is a subgroup of $\mathcal{G}$. We have, o($\mathcal{G}$) = $p(p^2-1)(p-1)$ [6] and o(G) = $p(p^2-1)$.[6]

2. Preliminaries:

Definition 2.1

A partial order on a non-empty set P is a binary relation≤ on P that is reflexive, anti-symmetric and transitive. The pair (P, ≤) is called a **partially ordered set or poset**. A poset. (P, ≤) is totally ordered if every x, y ∈ P are comparable, that is either x ≤ y or y ≤ x. A non-empty subset S of P is a chain in P if S is totally ordered by ≤.

Definition 2.2

Let (P, $\leq$) be a poset and let $S \subseteq P$. An upper bound of S is an element $x \in P$ for which $s \leq x$ for all $s \in S$. The least upper bound of S is called the **supremum or join** of S.A lower bound for S is an element $x \in P$ for which $x \leq s$ for all $s \in S$. The greatest lower bound of S is called the **infimum or meet** of S.

Definition 2.3

Poset (P, $\leq$) is called a **lattice** if every pair x, y elements of P has a supremum and an infimum, which are denoted by $x \vee y$ and $x \wedge y$ respectively.

Definition 2.4

For two elements a and b in P, a is said to ***cover*** b or b is said to be covered by a (in notation, $a \succ b$ or $b \prec a$) if and only if $b < a$ and, for no $x \in P$, $b < x < a$.

Definition 2.5

An element $a \in P$ is called an ***atom***, if $a \succ 0$ and it is a dual atom, if $a \prec 1$.

Definition 2.6

A Lattice L is said to be ***distributive*** if $a \vee (b \wedge c) = [(a \vee b) \wedge (a \vee c)]$ for all $a, b, c \in L$.

Definition 2.7

A Lattice L is said to be ***0-distributive*** if for all $x, y, z \in L$, whenever $x \wedge y = 0$ and $x \wedge z = 0$ then $x \wedge (y \vee z) = 0$.

Definition 2.8

A lattice L is said to be **pseudo 0-distributive** if for all $x, y, z \in L$ with $x \wedge y = 0$, $x \wedge z = 0$ we have $(x \vee y) \wedge z = y \wedge z$.

Definition 2.9

A lattice L is said to be **super 0-distributive** if for all $x, y, z \in L$, $x \wedge y = 0$ implies that $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$.

We give below the diagrams of L(G) when P=3 and 5.

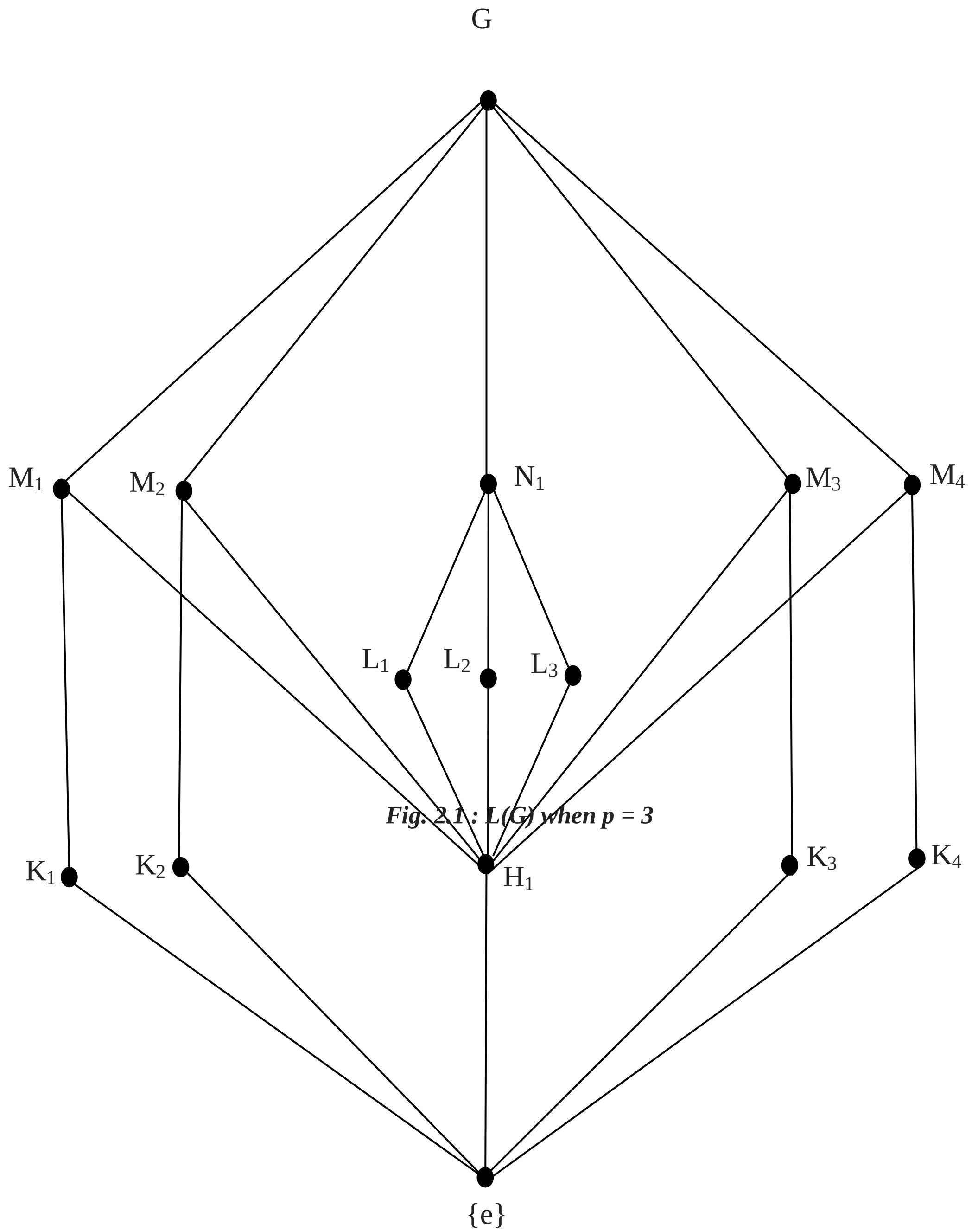

Fig. 2.1 : L(G) when p = 3

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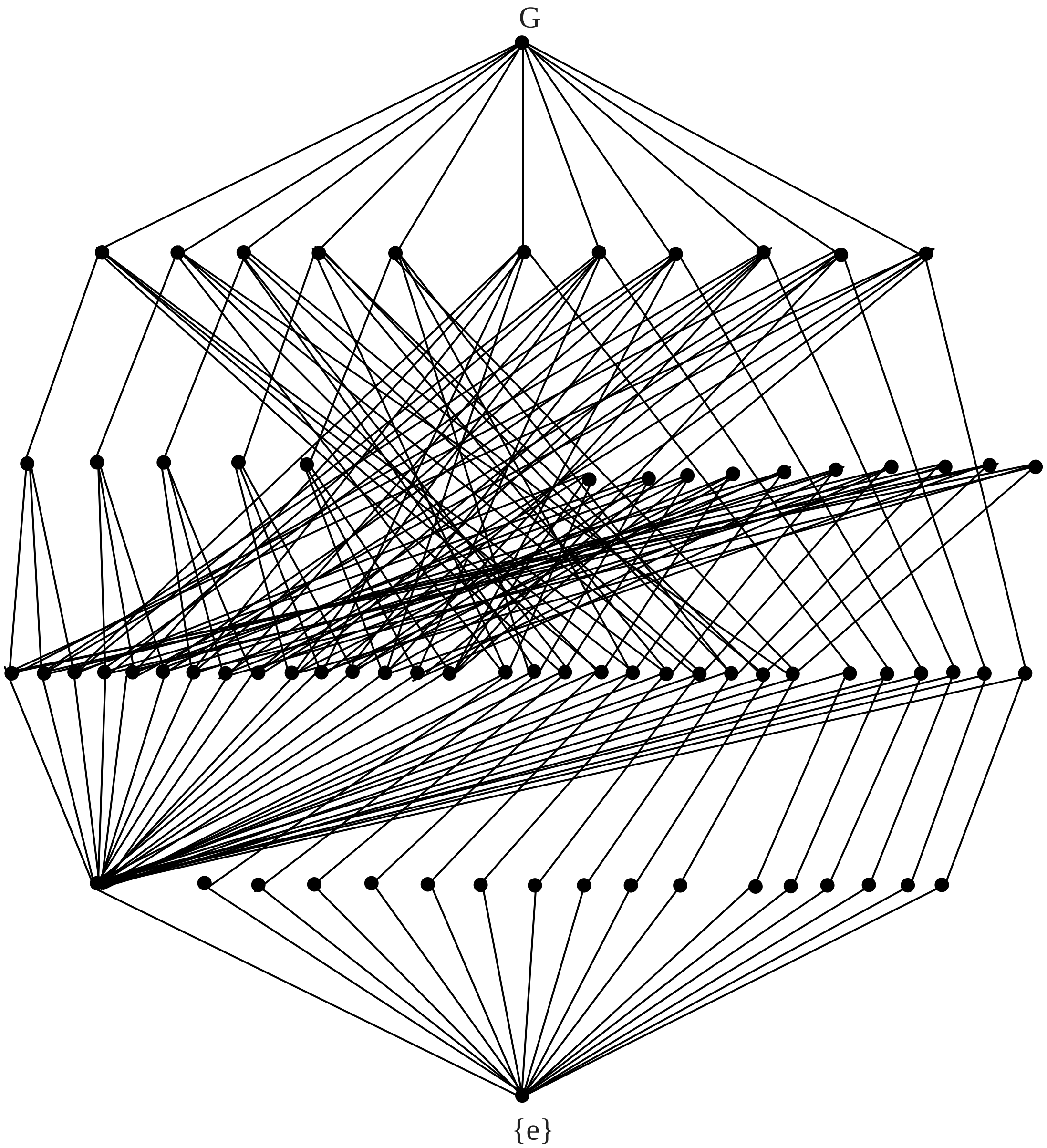

Fig. 2.2 : L(G) when p = 5

Row I : (Left to right): S_1 to S_5 and T_1 to T_6

Row II : (Left to right): P_1 to P_5 and R_1 to R_{10}

Row III: (Left to right): L_1 to L_{15}, N_1 to N_{10} and Q_1 to Q_6

Row IV: (Left to right): H_1, K_1 to K_{10} and M_1 to M_6.

3. Lattice identities in the subgroup lattice of 3x3 matrices over Z_3,Z_5 and Z_7

Lemma: 3.1

L(G) is not Pseudo 0-distributive if P = 3, 5 and 7.

Proof.

When p = 3

We choose three subgroups K_1, K_2 & H_1 in L (G) such that $K_1 \wedge K_2 = \{e\}$ and $K_1 \wedge H_1 = \{e\}$

Now, $(K_1 \vee K_2) \wedge H_1 = G \wedge H_1 = H_1$

But $K_2 \wedge H_1 = \{e\} \neq H_1$.

Therefore, L(G) is not Pseudo 0-distributive when P = 3.

When p = 5

We choose three subgroups K_1, K_2 & M_1 in L (G) such that $K_1 \wedge K_2 = \{e\}$ and $K_1 \wedge M_1 = \{e\}$

Now $(K_1 \vee K_2) \wedge M_1 = G \wedge M_1 = M_1$

But $K_2 \wedge M_1 = \{e\} \neq M_1$. Therefore L(G) is not Pseudo 0-distributive when P = 5.

When p = 7

We choose three subgroups N_1, N_2 & K_2 in L(G) when P = 7, such that $N_1 \wedge N_2 = \{e\}$ and $N_1 \wedge K_2 = \{e\}$

Now, $(N_1 \vee N_2) \wedge K_2 = G \wedge K_2 = K_2$

But $N_2 \wedge K_2 = \{e\} \neq K_2$.

Therefore, L(G) is not Pseudo 0-distributive when p = 7.

Hence L(G) is not Pseudo 0-distributive when p = 3, 5 and 7.

Lemma: 3.2

L(G) is not super 0-distributive if p = 3, 5 and 7.

Proof

When p = 3

We consider three subgroups K_1, K_2 & K_3 in L(G) such that $K_1 \wedge K_2 = \{e\}$

Now, $(K_1 \vee K_2) \wedge K_3 = G \wedge K_3 = K_3$

But, $(K_1 \wedge K_3) \vee (K_2 \wedge K_3) = \{e\} \vee \{e\} = \{e\}$

Therefore, $(K_1 \vee K_2) \wedge K_3 \neq (K_1 \wedge K_3) \vee (K_2 \wedge K_3)$

Therefore, L(G) is not super 0-distributive when p = 3.

When p = 5

Consider three elements K_1, K_2 & M_1 in L (G) such that $K_1 \wedge K_2 = \{e\}$

Now, $(K_1 \vee K_2) \wedge M_1 = G \wedge M_1 = M_1$

But, $(K_1 \wedge M_1) \vee (K_2 \wedge M_1) = \{e\} \vee \{e\} = \{e\}$

Therefore, $(K_1 \vee K_2) \wedge M_1 \neq (K_1 \wedge M_1) \vee (K_2 \wedge M_1)$

Therefore, L(G) is not super 0-distributive when p = 5.

When p = 7

Consider three subgroups N_1, N_2 & K_1 in L (G) when p = 7, such that $N_1 \wedge N_2 = \{e\}$

Now, $(N_1 \vee N_2) \wedge K_1 = G \wedge K_1 = K_1$

But, $(N_1 \wedge K_1) \vee (N_2 \wedge K_1) = \{e\} \vee \{e\} = \{e\}$

Therefore, $(N_1 \vee N_2) \wedge K_1 \neq (N_1 \wedge K_1) \vee (N_2 \wedge K_1)$

Therefore, L(G) is not super 0-distributive when p = 7.

Hence L(G) is not super 0-distributive when p = 3, 5 and 7.

4. CONCLUSION

In this paper we proved that the pseudo 0 - distributivity and super 0 – distributivity in the subgroup lattice of 2x2 matrices over z_3, z_5 and z_7

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