Lie Symmetry Solutions of Coupled Lotka-Volterra Competition-Diffusion Model

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Abstract
In this paper we consider the coupled Lotka-Volterra competition-diffusion interaction model. We employ the concept of Lie group theory in constructing the Lie generator, developed the k^{th} order prolongation of the generator of the coupled system. The invariant solution and the new symmetry solutions of the coupled Lotka-Volterra competition-diffusion system are obtained when the diffusive coefficients are equal. The group transformations of solutions of the system are also presented.

Keywords — *Coupled Lotka-Volterra competition-diffusion interaction model, Lie symmetry, infinitesimal transformation, Prolongation, Invariant solution, Transformation of solution.*

I. Introduction

Lotka-Volterra competition equations are generally accepted in the field of population biology. The equations are physiological models that describe competing interaction of multispecies. In this paper we consider two competing species whose interactions exhibit coupled density oscillation with diffusion. The differential equation that models diffusion accounts for heterogeneity of both the populations and the environmental resources involved [1]. Reference [2] explains that the spatial pattern formation even in the absence of environmental heterogeneity is another phenomenon associated with diffusion models. The *Lotka-Volterra* system can be modified by taking into account diffusion of the two species in one-dimensional space to obtain a general coupled Lotka-Volterra competition-diffusion interaction system of partial differential equations given as [3], [4];

$$u_t - \gamma_1 u_{xx} = u\left(\kappa_1 - \alpha_1 u - \beta_1 v\right) \tag{1a}$$

$$v_t - \gamma_2 v_{xx} = v\left(\kappa_2 - \alpha_2 v - \beta_2 u\right) \tag{1b}$$

where $\alpha_1 > 0$ and $\alpha_2 > 0$ model inter-specific competition, the terms involving partial derivatives u_t and v_t model the population density change of the competing species with respect to time $,t$ while u_{xx} and v_{xx} model the effect of transportation in the habitat, β_1 and β_2 being the strength of the interaction for the two species while γ_1 and γ_2 are diffusion coefficients. The interaction terms β_1 and β_2 represents logistic growth with competition. The model under consideration in this study is the modified coupled Lotka-Volterra competition-diffusion system [5], [6]

$$U_T = \gamma_1 U_{XX} + \kappa_1 U\left(1 - \Im_1 U - \wp_1 V\right) \tag{2a}$$

$$V_T = \gamma_2 V_{XX} + \kappa_2 V\left(1 - \Im_2 U - \wp_2 V\right) \tag{2b}$$

where $U(T, X)$ and $V(T, X)$ denote the population densities at position X and time T of two different species that compete for the same resources ; κ_i are net birth rates $\frac{1}{\wp_i}$ are carrying capacities, $\Im_i$ are competition coefficients and γ_i are diffusion coefficients.

We non dimensionalise the system to obtain

$$u_t = u_{xx} + u\left(1 - u - \beta_1 v\right) \tag{3a}$$

$$v_t = dv_{xx} + \alpha v\left(1 - \beta_2 u - v\right) \tag{3b}$$

By setting

$$x = X\left(\frac{\kappa_1}{\gamma_1}\right)^{\frac{1}{2}} \quad \beta_1 = \frac{\wp_1}{\wp_2}, \beta_2 = \frac{\Im_2}{\Im_1}, t = \kappa_1 T, u = \Im_1 U, v = \Im_2 V, \alpha = \frac{\kappa_1}{\kappa_2}, d = \frac{\gamma_2}{\gamma_1}$$

Assuming the first population outcompete the second then we have $\beta_1 < 0$ $\beta_2 > 0$.

Reference [1] used the (G'/G)-expansion method in getting travelling wave solutions of system of the Lotka-Volterra competition equations with diffusion terms which they classified into three sub-classes as trigonometric, hyperbolic and rational. Through numerical simulations, they further explored exact solutions. To construct the explicit solutions Li-Chang used Ansätz method hence obtaining new exact travelling solutions of the coupled *Lotka-Volterra* systems of two competing species by constructing explicit solutions. Using travelling wave solutions, further determination on which species would survive the completion was conducted. The result showed that the system has four equilibriums and the asymptotic behavior of the solution $u(x,0), v(x,0) > 0$ can be classified into four cases [7]. Reference [8] investigated the competition diffusion system of partial differential equation assuming that the coefficients diffusivity of the system are not equal hence proving the existence of a wave front solution connecting two nonzero rest points for systems of two equations of the kind (3). The travelling wave solution of two species coupled completion-diffusion systems with the *Lotka-Volterra* type interaction was studied and the magnitude of the diffusion coefficients of the former species over the latter explained as small enough [9]. Reference [3] made an investigation on the existence of travelling wave solutions of the form $u = u(\xi)$, $v = v(\xi)$. The solutions obtained were positive and monotone about ξ in the interval of negative infinity to positive infinity and satisfy the boundary condition

$\left(u(-\infty) = u_-, u(+\infty) = u_+\right)$ and $\left(v(-\infty) = v_-, v(+\infty) = v_+\right)$ with $(u_-, v_-), (u_+, v_+)$ being the rest points of the system and $\xi = x + ct$. The value of wave speed, c for the coupled Lotka-Volterra competition-diffusion system was also estimated. The same system for N- equations (species) we as for the system with 3-equations (one predator-two prey model) was investigated establishing the existence of a positive steady-state solution as well as the existence and stability of various semi trivial steady –state solutions in terms of the natural growth rates of the three species [10], [11].

Kan-On proved the linearized stability of the travelling waves for the system [12]. The existence of the travelling waves of the Lotka-Volterra competition-diffusion equations was investigated with an assumption that the first population outcompete the second such that $\alpha_2 > \beta_1, \beta_2 > \alpha_1$ and the boundary conditions $u = 1, v = 0$ at $z \to -\infty, u = 0, v = 1$ at $z \to +\infty$ under certain restrictions on the values of the parameter. It was also shown that in general case, the system of differential equations cannot be analytically solved but some analytical results can be obtained only in special case where $\frac{\gamma_2}{\gamma_1} = \kappa_1 = \kappa_2 = 1, \beta_1 + \beta_2 = 2$ [13].

Reference [14] did an investigation on the existence of travelling wave fronts connecting the equilibrium states (1, 0) and (0, 1) for the bi-stable case thus $\beta_1, \beta_2 > 1$. The comparison principle was applied to couple Lotka-Volterra competition-diffusion equations on R in proving the existence of an entire solution which behaves as two monotone waves propagating from both sides of x-axis, such that the entire solution is defined for all space and time variable. It was also shown that the global dynamics for this entire solution exhibits the extinction of the inferior species by the superior species invading from both sides of the x-axis [15]. Guo and Wu presented the entire solutions for a two- component competition-diffusion system while Wang and Li did investigate the entire solutions of coupled Lotka-Volterra competition-diffusion system with nonlocal delays [16], [17]. Rodrigo and Mimera applied certain ansatz to the coupled Lotka-Volterra competition-diffusion equations obtaining exact travelling wave and standing wave solutions under certain parameter restrictions and for some given correlations between diffusivity coefficients γ_1 and γ_2 such that $\gamma_1 \neq \gamma_2$. Among the obtained results solutions for the case when $\gamma_1 = \gamma_2$ such that $\frac{\gamma_2}{\gamma_1} = 1$ are missing [18]. Nikolay and Anastasia did an investigation on the painlev'e property and found the exact travelling wave solutions of the coupled Lotka-Volterra competition-diffusion equations for the case $\gamma_1 = \gamma_2$. They also obtained the periodic solutions which they expressed in terms of the Weierstrass elliptic function [5].

In this study, we constructed the Lie generator of the system, develop the k^{th} order prolongation of the generator for the system and use them in obtaining the invariant solutions and the group transformations of solutions of the system with the restrictions that the diffusivity coefficients γ_1 and γ_2 are equal and their ratio is one.

II. Symmetry Solutions of Lotka-Volterra Competition-Diffusion Interaction Model

Assuming the connectivity of the transformation group **G** and since the system has two independent variables x, t and two dependent variables u, v, the required symmetry groups of transformation takes the form

$$t^* = A(x,t,u,v;\lambda),\ x^* = B(x,t,u,v;\lambda),\ u^* = C(x,t,u,v;\lambda),\ v^* = D(x,t,u,v;\lambda),$$

With the corresponding infinitesimals as

$$\xi(x,t,u,v) = \left(\frac{\partial A(x,t,u,v;\lambda)}{\partial \lambda} |_{\lambda=0} \right),\ \tau(x,t,u,v) = \left(\frac{\partial B(x,t,u,v;\lambda)}{\partial \lambda} |_{\lambda=0} \right),$$
$$\phi(x,t,u,v) = \left(\frac{\partial C(x,t,u,v;\lambda)}{\partial \lambda} |_{\lambda=0} \right),\quad \varphi(x,t,u,v) = \left(\frac{\partial D(x,t,u,v;\lambda)}{\partial \lambda} |_{\lambda=0} \right)$$

the infinitesimal generator given as

$$U = \xi(t,x,u,v)\frac{\partial}{\partial x} + \tau(t,x,u,v)\frac{\partial}{\partial t} + \phi(t,x,u,v)\frac{\partial}{\partial u} + \varphi(t,x,u,v)\frac{\partial}{\partial v} \tag{4}$$

The prolongation of the vector field U defined by equation (4) [19],[20], [21], [22] is given by

$$\Pr^{(k)}U=\sum_{\gamma=1}^{m}\xi_{\gamma}\left(x,v\right)\frac{\partial}{\partial x_{\gamma}}+\sum_{\beta=1}^{n}\phi_{\beta}\left(x,v\right)\frac{\partial}{\partial v^{\beta}}+\sum_{\beta=1}^{n}\sum_{J}\phi_{\beta}^{J}\left(x,v^{(k)}\right)\frac{\partial}{\partial v_{J}^{\beta}} \tag{5}$$

Such that the second summation $\sum_{J}$ extends to all multi-indices $J=\left(j_1,j_2,\ldots,j_l\right)$ with $1\le j_l\le m, 1\le l\le k$, the k^{th} prolongation coefficients ϕ_{β}^{J} are given by

$$\phi_{\beta}^{J}\left(x,v^{(k)}\right)=D_J\left(\phi_{\beta}-\sum_{\gamma=1}^{m}\xi_{\gamma}v_{\gamma}^{\beta}\right)+\sum_{\gamma=1}^{m}\xi_{\gamma}v_{J,\gamma}^{\beta} \tag{6}$$

Where, $v_{\gamma}^{\beta}=\frac{\partial v^{\beta}}{\partial x_{\gamma}}$, $v_{J,\gamma}^{\beta}=\frac{\partial v_{J}^{\beta}}{\partial x_{\gamma}}$ and D_J is the total derivative that treat the dependent variables u,v and their derivatives as functions of independent variables and is defined as

$$D_J=\frac{\partial}{\partial x_{\gamma}}+v_{\gamma}^{\beta}\frac{\partial}{\partial v^{\beta}}+v_{J,\gamma}^{\beta}\frac{\partial}{\partial v_{J}^{\beta}}+\ldots;\gamma=1,2,\ldots,q \tag{7}$$

Since the Lotka-Volterra competition-diffusion system is a second order partial differential equation, we obtain the first and second prolongation of its infinitesimal generator as follows;

$$\Pr^{(k)}U=\xi\left(t,x,u,v\right)\frac{\partial}{\partial x}+\tau\left(t,x,u,v\right)\frac{\partial}{\partial t}+\phi\left(t,x,u,v\right)\frac{\partial}{\partial u}+\varphi\left(t,x,u,v\right)\frac{\partial}{\partial v}$$
$$+\sum_{\beta=1}^{n}\sum_{J}\left[D_J\left(\phi_{\beta}-\sum_{\gamma=1}^{m}\xi_{\gamma}v_{\gamma}^{\beta}\right)+\sum_{\gamma=1}^{m}\xi_{\gamma}v_{J,\gamma}^{\beta}\right]\frac{\partial}{\partial v_{J}^{\beta}}$$

$$\Pr^{(k)}U=\xi\left(t,x,u,v\right)\frac{\partial}{\partial x}+\tau\left(t,x,u,v\right)\frac{\partial}{\partial t}+\phi\left(t,x,u,v\right)\frac{\partial}{\partial u}+\varphi\left(t,x,u,v\right)\frac{\partial}{\partial v}$$
$$+\sum_{\beta=1}^{n}\sum_{J}\left[D_J\left(\phi_{\beta}-\sum_{\gamma=1}^{m}\xi_{\gamma}\frac{\partial v^{\beta}}{\partial x_{\gamma}}\right)+\sum_{\gamma=1}^{m}\xi_{\gamma}\frac{\partial v_{J}^{\beta}}{\partial x_{\gamma}}\right]\frac{\partial}{\partial v_{J}^{\beta}}$$

$$\Pr^{(k)}U=\xi\left(t,x,u,v\right)\frac{\partial}{\partial x}+\tau\left(t,x,u,v\right)\frac{\partial}{\partial t}+\phi\left(t,x,u,v\right)\frac{\partial}{\partial u}+\varphi\left(t,x,u,v\right)\frac{\partial}{\partial v}$$
$$+\sum_{\beta=1}^{n}\sum_{J}\left[\left(\frac{\partial}{\partial x_{\gamma}}+v_{\gamma}^{\beta}\frac{\partial}{\partial v^{\beta}}+v_{J,\gamma}^{\beta}\frac{\partial}{\partial v_{J}^{\beta}}+\ldots\right)\left(\phi_{\beta}-\sum_{\gamma=1}^{m}\xi_{\gamma}\frac{\partial v^{\beta}}{\partial x_{\gamma}}\right)+\sum_{\gamma=1}^{m}\xi_{\gamma}\frac{\partial v_{J}^{\beta}}{\partial x_{\gamma}}\right]\frac{\partial}{\partial v_{J}^{\beta}} \tag{8}$$

To obtain the first prolongations we set

$\alpha=2;n=\beta=2;m=\gamma=2;k=1$

$$\Pr^{(1)}U=\xi\left(t,x,u,v\right)\frac{\partial}{\partial x}+\tau\left(t,x,u,v\right)\frac{\partial}{\partial t}+\phi\left(t,x,u,v\right)\frac{\partial}{\partial u}+\varphi\left(t,x,u,v\right)\frac{\partial}{\partial v}$$
$$+\sum_{\beta=1}^{2}\sum_{J}\left[\left(\frac{\partial}{\partial x_{\gamma}}+v_{\gamma}^{\beta}\frac{\partial}{\partial v^{\beta}}+v_{J,\gamma}^{\beta}\frac{\partial}{\partial v_{J}^{\beta}}+....\right)\left(\phi_{\beta}-\sum_{\gamma=1}^{2}\xi_{\gamma}\frac{\partial v^{\beta}}{\partial x_{\gamma}}\right)+\sum_{\gamma=1}^{2}\xi_{\gamma}\frac{\partial v_{J}^{\beta}}{\partial x_{\gamma}}\right]\frac{\partial}{\partial v_{J}^{\beta}}$$

$$\Pr^{(1)}U=\xi\left(t,x,u,v\right)\frac{\partial}{\partial x}+\tau\left(t,x,u,v\right)\frac{\partial}{\partial t}+\phi\left(t,x,u,v\right)\frac{\partial}{\partial u}+\varphi\left(t,x,u,v\right)\frac{\partial}{\partial v}$$
$$+\phi^{x}\left(t,x,u,v\right)\frac{\partial}{\partial u_{x}}+\varphi^{x}\left(t,x,u,v\right)\frac{\partial}{\partial v_{x}}+\phi^{t}\left(t,x,u,v\right)\frac{\partial}{\partial u_{t}}+\varphi^{t}\left(t,x,u,v\right)\frac{\partial}{\partial v_{t}} \tag{9}$$

while the second prolongations is obtained as follows by setting

$\alpha=2; n=\beta=2; m=\gamma=2; k=2$

$$\Pr^{(2)}U=\xi\left(t,x,u,v\right)\frac{\partial}{\partial x}+\tau\left(t,x,u,v\right)\frac{\partial}{\partial t}\phi\left(t,x,u,v\right)\frac{\partial}{\partial u}+\varphi\left(t,x,u,v\right)\frac{\partial}{\partial v}$$
$$+\phi^{x}\left(t,x,u,v\right)\frac{\partial}{\partial u_{x}}+\varphi^{x}\left(t,x,u,v\right)\frac{\partial}{\partial v_{x}}+\phi^{t}\left(t,x,u,v\right)\frac{\partial}{\partial u_{t}}+\varphi^{t}\left(t,x,u,v\right)\frac{\partial}{\partial v_{t}}$$
$$+\phi^{xx}\left(t,x,u,v\right)\frac{\partial}{\partial u_{xx}}+\varphi^{xx}\left(t,x,u,v\right)\frac{\partial}{\partial v_{xx}}+\phi^{xt}\left(t,x,u,v\right)\frac{\partial}{\partial u_{xt}}+\varphi^{xt}\left(t,x,u,v\right)\frac{\partial}{\partial v_{xt}}$$
$$+\phi^{tt}\left(t,x,u,v\right)\frac{\partial}{\partial u_{tt}}+\varphi^{tt}\left(t,x,u,v\right)\frac{\partial}{\partial v_{tt}} \tag{10}$$

These generators always satisfy the necessary and sufficient criterion of infinitesimal invariance

$$\left[\sum_{\gamma=1}^{m}\xi_{\gamma}\left(x,v\right)\frac{\partial}{\partial x_{\gamma}}+\sum_{\beta=1}^{n}\phi_{\beta}\left(x,v\right)\frac{\partial}{\partial v^{\beta}}+\sum_{\beta=1}^{n}\sum_{J}\phi_{\beta}^{J}\left(x,v^{(k)}\right)\frac{\partial}{\partial v_{J}^{\beta}}\right]\left[\Delta^{\gamma}\left(x,v^{(k)}\right)\right]|_{\Delta^{\gamma}\left(x,v^{(k)}\right)=0}=0$$

where $|_{\Delta^{\gamma}\left(x,v^{(k)}\right)=0}$ means evaluated on the surface $\Delta^{\gamma}\left(x,v^{(k)}\right)=0$ and $\Pr^{(k)}U$ is the k^{th} prolongation of the vector field U.

We use the obtained second prolongations in solving the system as

$$\Pr^{(2)}U\left(u_{t}-u_{xx}-u+u^{2}+\beta_{1}uv\right)=0,$$

$$\Pr^{(2)}U\left(v_{t}-v_{xx}-\alpha v+\alpha v^{2}+\alpha\beta_{2}uv\right)=0$$

for simplicity $\xi\left(t,x,u,v\right)\frac{\partial}{\partial x}=\xi\frac{\partial}{\partial x},\phi^{x}\left(t,x,u,v\right)\frac{\partial}{\partial u_{x}}=\phi^{x}\frac{\partial}{\partial u_{x}},etc$ we obtain

$$\begin{pmatrix} \xi\frac{\partial}{\partial x}+\tau\frac{\partial}{\partial t}+\phi\frac{\partial}{\partial u}+\varphi\frac{\partial}{\partial v}+\phi^{x}\frac{\partial}{\partial u_x}+\varphi^{x}\frac{\partial}{\partial v_x}+\phi^{t}\frac{\partial}{\partial u_t}+\varphi^{t}\frac{\partial}{\partial v_t} \\ +\phi^{xx}\frac{\partial}{\partial u_{xx}}+\varphi^{xx}\frac{\partial}{\partial v_{xx}}+\phi^{xt}\frac{\partial}{\partial u_{xt}}+\varphi^{xt}\frac{\partial}{\partial v_{xt}}+\phi^{tt}\frac{\partial}{\partial u_{tt}}+\varphi^{tt}\frac{\partial}{\partial v_{tt}} \end{pmatrix}\left(u_t-u_{xx}-u+u^2+\beta_1 uv\right)=0$$

$$\begin{pmatrix} \xi\frac{\partial}{\partial x}+\tau\frac{\partial}{\partial t}+\phi\frac{\partial}{\partial u}+\varphi\frac{\partial}{\partial v}+\phi^{x}\frac{\partial}{\partial u_x}+\varphi^{x}\frac{\partial}{\partial v_x}+\phi^{t}\frac{\partial}{\partial u_t}+\varphi^{t}\frac{\partial}{\partial v_t} \\ +\phi^{xx}\frac{\partial}{\partial u_{xx}}+\varphi^{xx}\frac{\partial}{\partial v_{xx}}+\phi^{xt}\frac{\partial}{\partial u_{xt}}+\varphi^{xt}\frac{\partial}{\partial v_{xt}}+\phi^{tt}\frac{\partial}{\partial u_{tt}}+\varphi^{tt}\frac{\partial}{\partial v_{tt}} \end{pmatrix}\left(v_t-v_{xx}-\alpha v+\alpha v^2+\alpha\beta_2 uv\right)=0$$

On expansion we get

$$\begin{aligned}
&\xi\frac{\partial}{\partial x}(u_t)+\tau\frac{\partial}{\partial t}(u_t)+\phi\frac{\partial}{\partial u}(u_t)+\varphi\frac{\partial}{\partial v}(u_t)+\phi^{x}\frac{\partial}{\partial u_x}(u_t)+\varphi^{x}\frac{\partial}{\partial v_x}(u_t)+\phi^{t}\frac{\partial}{\partial u_t}(u_t)\\
&+\varphi^{t}\frac{\partial}{\partial v_t}(u_t)+\phi^{xx}\frac{\partial}{\partial u_{xx}}(u_t)+\varphi^{xx}\frac{\partial}{\partial v_{xx}}(u_t)+\phi^{xt}\frac{\partial}{\partial u_{xt}}(u_t)+\varphi^{xt}\frac{\partial}{\partial v_{xt}}(u_t)+\phi^{tt}\frac{\partial}{\partial u_{tt}}(u_t)\\
&+\varphi^{tt}\frac{\partial}{\partial v_{tt}}(u_t)-\xi\frac{\partial}{\partial x}(u_{xx})-\tau\frac{\partial}{\partial t}(u_{xx})-\phi\frac{\partial}{\partial u}(u_{xx})-\varphi\frac{\partial}{\partial v}(u_{xx})-\phi^{x}\frac{\partial}{\partial u_x}(u_{xx})-\varphi^{x}\frac{\partial}{\partial v_x}(u_{xx})\\
&-\phi^{t}\frac{\partial}{\partial u_t}(u_{xx})-\varphi^{t}\frac{\partial}{\partial v_t}(u_{xx})-\phi^{xx}\frac{\partial}{\partial u_{xx}}(u_{xx})-\varphi^{xx}\frac{\partial}{\partial v_{xx}}(u_{xx})-\phi^{xt}\frac{\partial}{\partial u_{xt}}(u_{xx})-\varphi^{xt}\frac{\partial}{\partial v_{xt}}(u_{xx})\\
&-\phi^{tt}\frac{\partial}{\partial u_{tt}}(u_{xx})-\varphi^{tt}\frac{\partial}{\partial v_{tt}}(u_{xx})-\xi\frac{\partial}{\partial x}(u)-\tau\frac{\partial}{\partial t}(u)-\phi\frac{\partial}{\partial u}(u)-\varphi\frac{\partial}{\partial v}(u)-\phi^{x}\frac{\partial}{\partial u_x}(u)\\
&-\varphi^{x}\frac{\partial}{\partial v_x}(u)-\phi^{t}\frac{\partial}{\partial u_t}(u)-\varphi^{t}\frac{\partial}{\partial v_t}(u)-\phi^{xx}\frac{\partial}{\partial u_{xx}}(u)-\varphi^{xx}\frac{\partial}{\partial v_{xx}}(u)-\phi^{xt}\frac{\partial}{\partial u_{xt}}(u)-\varphi^{xt}\frac{\partial}{\partial v_{xt}}(u)\\
&-\phi^{tt}\frac{\partial}{\partial u_{tt}}(u)-\varphi^{tt}\frac{\partial}{\partial v_{tt}}(u)+\xi\frac{\partial}{\partial x}(u^2)+\tau\frac{\partial}{\partial t}(u^2)+\phi\frac{\partial}{\partial u}(u^2)+\varphi\frac{\partial}{\partial v}(u^2)+\phi^{x}\frac{\partial}{\partial u_x}(u^2)\\
&+\varphi^{x}\frac{\partial}{\partial v_x}(u^2)+\phi^{t}\frac{\partial}{\partial u_t}(u^2)+\varphi^{t}\frac{\partial}{\partial v_t}(u^2)+\phi^{xx}\frac{\partial}{\partial u_{xx}}(u^2)+\varphi^{xx}\frac{\partial}{\partial v_{xx}}(u^2)+\phi^{xt}\frac{\partial}{\partial u_{xt}}(u^2)\\
&+\varphi^{xt}\frac{\partial}{\partial v_{xt}}(u^2)+\phi^{tt}\frac{\partial}{\partial u_{tt}}(u^2)+\varphi^{tt}\frac{\partial}{\partial v_{tt}}(u^2)+\xi\frac{\partial}{\partial x}(\beta_1 uv)+\tau\frac{\partial}{\partial t}(\beta_1 uv)+\phi\frac{\partial}{\partial u}(\beta_1 uv)\\
&+\varphi\frac{\partial}{\partial v}(\beta_1 uv)+\phi^{x}\frac{\partial}{\partial u_x}(\beta_1 uv)+\varphi^{x}\frac{\partial}{\partial v_x}(\beta_1 uv)+\phi^{t}\frac{\partial}{\partial u_t}(\beta_1 uv)+\varphi^{t}\frac{\partial}{\partial v_t}(\beta_1 uv)\\
&+\phi^{xx}\frac{\partial}{\partial u_{xx}}(\beta_1 uv)+\varphi^{xx}\frac{\partial}{\partial v_{xx}}(\beta_1 uv)+\phi^{xt}\frac{\partial}{\partial u_{xt}}(\beta_1 uv)+\varphi^{xt}\frac{\partial}{\partial v_{xt}}(\beta_1 uv)+\phi^{tt}\frac{\partial}{\partial u_{tt}}(\beta_1 uv)\\
&+\varphi^{tt}\frac{\partial}{\partial v_{tt}}(\beta_1 uv)=0
\end{aligned}$$

(11a)

$$\xi\frac{\partial}{\partial x}(v_t)+\tau\frac{\partial}{\partial t}(v_t)+\phi\frac{\partial}{\partial u}(v_t)+\varphi\frac{\partial}{\partial v}(v_t)+\phi^x\frac{\partial}{\partial u_x}(v_t)+\varphi^x\frac{\partial}{\partial v_x}(v_t)+\phi^t\frac{\partial}{\partial u_t}(v_t)$$
$$+\varphi^t\frac{\partial}{\partial v_t}(v_t)+\phi^{xx}\frac{\partial}{\partial u_{xx}}(v_t)+\varphi^{xx}\frac{\partial}{\partial v_{xx}}(v_t)+\phi^{xt}\frac{\partial}{\partial u_{xt}}(v_t)+\varphi^{xt}\frac{\partial}{\partial v_{xt}}(v_t)+\phi^{tt}\frac{\partial}{\partial u_{tt}}(v_t)$$
$$+\varphi^{tt}\frac{\partial}{\partial v_{tt}}(v_t)-\xi\frac{\partial}{\partial x}(v_{xx})-\tau\frac{\partial}{\partial t}(v_{xx})-\phi\frac{\partial}{\partial u}(v_{xx})-\varphi\frac{\partial}{\partial v}(v_{xx})-\phi^x\frac{\partial}{\partial u_x}(v_{xx})-\varphi^x\frac{\partial}{\partial v_x}(v_{xx})$$
$$-\phi^t\frac{\partial}{\partial u_t}(v_{xx})-\varphi^t\frac{\partial}{\partial v_t}(v_{xx})-\phi^{xx}\frac{\partial}{\partial u_{xx}}(v_{xx})-\varphi^{xx}\frac{\partial}{\partial v_{xx}}(v_{xx})-\phi^{xt}\frac{\partial}{\partial u_{xt}}(v_{xx})-\varphi^{xt}\frac{\partial}{\partial v_{xt}}(v_{xx})$$
$$-\phi^{tt}\frac{\partial}{\partial u_{tt}}(v_{xx})-\varphi^{tt}\frac{\partial}{\partial v_{tt}}(v_{xx})-\xi\frac{\partial}{\partial x}(\alpha v)-\tau\frac{\partial}{\partial t}(\alpha v)-\phi\frac{\partial}{\partial u}(\alpha v)-\varphi\frac{\partial}{\partial v}(\alpha v)-\phi^x\frac{\partial}{\partial u_x}(\alpha v)$$
$$-\varphi^x\frac{\partial}{\partial v_x}(\alpha v)-\phi^t\frac{\partial}{\partial u_t}(\alpha v)-\varphi^t\frac{\partial}{\partial v_t}(\alpha v)-\phi^{xx}\frac{\partial}{\partial u_{xx}}(\alpha v)-\varphi^{xx}\frac{\partial}{\partial v_{xx}}(\alpha v)-\phi^{xt}\frac{\partial}{\partial u_{xt}}(\alpha v)$$
$$-\varphi^{xt}\frac{\partial}{\partial v_{xt}}(\alpha v)-\phi^{tt}\frac{\partial}{\partial u_{tt}}(\alpha v)-\varphi^{tt}\frac{\partial}{\partial v_{tt}}(\alpha v)+\xi\frac{\partial}{\partial x}(\alpha v^2)+\tau\frac{\partial}{\partial t}(\alpha v^2)+\phi\frac{\partial}{\partial u}(\alpha v^2)$$
$$+\varphi\frac{\partial}{\partial v}(\alpha v^2)+\phi^x\frac{\partial}{\partial u_x}(\alpha v^2)+\varphi^x\frac{\partial}{\partial v_x}(\alpha v^2)+\phi^t\frac{\partial}{\partial u_t}(\alpha v^2)+\varphi^t\frac{\partial}{\partial v_t}(\alpha v^2)+\phi^{xx}\frac{\partial}{\partial u_{xx}}(\alpha v^2)$$
$$+\varphi^{xx}\frac{\partial}{\partial v_{xx}}(\alpha v^2)+\phi^{xt}\frac{\partial}{\partial u_{xt}}(\alpha v^2)+\varphi^{xt}\frac{\partial}{\partial v_{xt}}(\alpha v^2)+\phi^{tt}\frac{\partial}{\partial u_{tt}}(\alpha v^2)+\varphi^{tt}\frac{\partial}{\partial v_{tt}}(\alpha v^2)$$
$$+\xi\frac{\partial}{\partial x}(\alpha\beta_2 uv)+\tau\frac{\partial}{\partial t}(\alpha\beta_2 uv)+\phi\frac{\partial}{\partial u}(\alpha\beta_2 uv)+\varphi\frac{\partial}{\partial v}(\alpha\beta_2 uv)+\phi^x\frac{\partial}{\partial u_x}(\alpha\beta_2 uv)$$
$$+\varphi^x\frac{\partial}{\partial v_x}(\alpha\beta_2 uv)+\phi^t\frac{\partial}{\partial u_t}(\alpha\beta_2 uv)+\varphi^t\frac{\partial}{\partial v_t}(\alpha\beta_2 uv)+\phi^{xx}\frac{\partial}{\partial u_{xx}}(\alpha\beta_2 uv)+\varphi^{xx}\frac{\partial}{\partial v_{xx}}(\alpha\beta_2 uv)$$
$$+\phi^{xt}\frac{\partial}{\partial u_{xt}}(\alpha\beta_2 uv)+\varphi^{xt}\frac{\partial}{\partial v_{xt}}(\alpha\beta_2 uv)+\phi^{tt}\frac{\partial}{\partial u_{tt}}(\alpha\beta_2 uv)+\varphi^{tt}\frac{\partial}{\partial v_{tt}}(\alpha\beta_2 uv)=0 \qquad \textbf{(11b)}$$

We now differentiate partially with respect to the partial variables $u_t,\ v_t,\ u_x,\ v_x,\ u_{xx},\ v_{xx},\ u_{tt},\ v_{tt},\ u_{xt},\ v_{xt}$ and $t,\ x,\ u,\ v$ as algebraic variables and on substitution for $\phi^t,\ \varphi^t,\ \phi^{xx}$ and φ^{xx} obtained from equations (6) and (7) as in equations (12)-(21) below

$$\phi^t=\phi_t-\xi_t u_x+(\phi_u-\tau_t)u_t-\xi_u u_x u_t-\tau_u u_t^{\,2} \qquad (12)$$

$$\varphi^t=\varphi_t-\xi_t v_x+(\varphi_v-\tau_t)v_t-\xi_v v_x v_t-\tau_v v_t^{\,2} \qquad (13)$$

$$\phi^x=\phi_x-\tau_x u_t+(\phi_u-\xi_x)u_x-\xi_u u_x^{\,2}-\tau_u u_t u_x \qquad (14)$$

$$\varphi^x=\varphi_x-\tau_x v_t+(\varphi_v-\xi_x)v_x-\xi_v v_x^{\,2}-\tau_v v_t v_x \qquad (15)$$

$$\phi^{xx} = D^2{}_x\phi - u_x D^2{}_x\xi - u_t D^2{}_x\tau - 2u_{xx}D_x\xi - 2u_{xt}D_x\tau$$

$$D^2{}_x(\varphi) = \varphi_{xx} + 2v_x\varphi_{vx} + v_{xx}\varphi_v + v^2{}_x\varphi_{vv}$$

$$D^2{}_x(\xi) = \xi_{xx} + 2u_x\xi_{ux} + u_{xx}\xi_u + u^2{}_x\xi_{uu}$$

$$D^2{}_x(\tau) = \tau_{xx} + 2u_x\tau_{ux} + u_{xx}\tau_u + u^2{}_x\tau_{uu}$$

$$\begin{aligned}\phi^{xx} &= \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \tau_{xx}u_t + (\phi_{uu} - 2\xi_{xu})u_x{}^2 - 2\tau_{xu}u_xu_t - \xi_{uu}u_x{}^3\\ &-\tau_{uu}u_x{}^2u_t + (\phi_u - 2\xi_x)u_{xx} - 2\tau_xu_{xt} - 3\xi_uu_xu_{xx} - \tau_uu_tu_{xx} - 2\tau_uu_xu_{xt}\end{aligned} \tag{16}$$

$$\begin{aligned}\varphi^{xx} &= \varphi_{xx} + (2\varphi_{xv} - \xi_{xx})v_x - \tau_{xx}v_t + (\varphi_{vv} - 2\xi_{xv})v_x{}^2 - 2\tau_{xv}v_xv_t - \xi_{vv}v_x{}^3\\ &-\tau_{vv}v_x{}^2v_t + (\varphi_v - 2\xi_x)v_{xx} - 2\tau_xv_{xt} - 3\xi_vv_xv_{xx} - \tau_vv_tv_{xx} - 2\tau_vv_xv_{xt}\end{aligned} \tag{17}$$

$$\begin{aligned}\phi^{tt} &= \phi_{tt} + (2\phi_{ut} - \tau_{uu})u_t - \xi_{tt}u_x + (\phi_{uu} - 2\tau_{ut})u_t{}^2 - 2\xi_{ut}u_xu_t - \tau_{uu}u_t{}^3\\ &-\xi_{uu}u_xu^2{}_t + (\phi_u - 2\tau_t)u_{tt} - 2\xi_tu_{xt} - 3\tau_uu_tu_{tt} - \xi_uu_xu_{tt} - 2\xi_uu_tu_{xt}\end{aligned} \tag{18}$$

$$\begin{aligned}\varphi^{tt} &= \varphi_{tt} + (2\varphi_{vt} - \tau_{vv})v_t - \xi_{tt}v_x + (\varphi_{vv} - 2\tau_{vt})v_t{}^2 - 2\xi_{vt}v_xv_t - \tau_{vv}v_t{}^3\\ &-\xi_{vv}v_xv^2{}_t + (\varphi_v - 2\tau_t)v_{tt} - 2\xi_tv_{xt} - 3\tau_vv_tv_{tt} - \xi_vv_xv_{tt} - 2\xi_vv_tv_{xt}\end{aligned} \tag{19}$$

$$\begin{aligned}\varphi^{tx} &= \varphi_{xt} + (\varphi_{xv} - \tau_{tx})v_t - \xi_tv_{xx} + (\varphi_{tv} - \xi_{tx})v_x - 2\xi_vv_xv_{xt} - 2\tau_{vt}v_{tx} - \xi_{vv}v_x^2v_t + (\varphi_v - 2\tau_t - \xi_x)v_{xt}\\ &-\xi_{tv}v_x^2 - \tau_vv_{tt} - \xi_vv_xv_{tt} - \xi_vv_tv_{xx} - \tau_{xv}v_t^2 - \tau_{vv}v_t^2v_x + (\varphi_{vv} - \tau_{tv} - \xi_{xv})v_tv_x\end{aligned} \tag{20}$$

$$\begin{aligned}\varphi^{tx} &= \varphi_{xt} + (\varphi_{xv} - \tau_{tx})v_t - \xi_tv_{xx} + (\varphi_{tv} - \xi_{tx})v_x - 2\xi_vv_xv_{xt} - 2\tau_{vt}v_{tx} - \xi_{vv}v_x^2v_t + (\varphi_v - 2\tau_t - \xi_x)v_{xt}\\ &-\xi_{tv}v_x^2 - \tau_vv_{tt} - \xi_vv_xv_{tt} - \xi_vv_tv_{xx} - \tau_{xv}v_t^2 - \tau_{vv}v_t^2v_x + (\varphi_{vv} - \tau_{tv} - \xi_{xv})v_tv_x\end{aligned} \tag{21}$$

we get

$$\begin{aligned}&\phi_t - \xi_tu_x + (\phi_u - \tau_t)u_t - \xi_uu_xu_t - \tau_uu_t{}^2 - \phi_{xx} - (2\phi_{xu} - \xi_{xx})u_x + \tau_{xx}u_t - (\phi_{uu} - 2\xi_{xu})u_x{}^2\\ &+2\tau_{xu}u_xu_t + \xi_{uu}u_x{}^3 + \tau_{uu}u_x{}^2u_t - (\phi_u - 2\xi_x)u_{xx} + 2\tau_xu_{xt} + 3\xi_uu_xu_{xx} + \tau_uu_tu_{xx} + 2\tau_uu_xu_{xt}\\ &-\phi + 2u\phi + \beta_1v\phi + \beta_1u\varphi = 0\end{aligned} \tag{22a}$$

$$\varphi_t-\xi_t v_x+(\varphi_v-\tau_t)v_t-\xi_v v_x v_t-\tau_v v_t^2-\varphi_{xx}-(2\varphi_{xv}-\xi_{xx})v_x+\tau_{xx}v_t-(\varphi_{vv}-2\xi_{xv})v_x^2$$
$$+2\tau_{xv}v_x v_t+\xi_{vv}v_x^3+\tau_{vv}v_x^2 v_t-(\varphi_v-2\xi_x)v_{xx}+2\tau_x v_{xt}+3\xi_v v_x v_{xx}+\tau_u v_t v_{xx}+2\tau_v v_x v_{xt}$$
$$-\alpha\varphi+2\alpha v\varphi+\beta_2\alpha u\varphi+\beta_2\alpha v\phi=0 \tag{22b}$$

On replacing u_t by $u_{xx}+u-u^2-\beta_1 uv$ and v_t by $v_{xx}+\alpha v-\alpha v^2-\alpha\beta_2 uv$ whenever they occurs, Since ξ, τ, ϕ, φ are functions of x,t,u,v, then equating the coefficients of various monomials in the first, second and the other order partial derivatives of u,v and their powers, we obtain the determining equations for the symmetry group of the coupled Lokta-Volterra competition-diffusion bellow;

$$2\tau_x=0 \quad 2\tau_u=0 \quad 2\tau_v=0 \quad \xi_u=0 \quad \xi_v=0 \quad \tau_{uu}=0 \quad \tau_{uv}=0$$
$$\tau_{vv}=0 \quad \xi_{uu}=0 \quad \xi_{vv}=0 \quad \xi_{uv}=0 \quad \phi_{vv}=0 \quad \varphi_{uu}=0$$
$$\phi_u-\phi_{xx}+\phi=0 \quad \varphi_v-\varphi_{xx}+\varphi=0 \quad -\xi_t-(2\phi_{xu}-\xi_{xx})=0$$
$$-\xi_t-(2\varphi_{xv}-\xi_{xx})=0 \quad \phi_u-\tau_t+\tau_{xx}-(\phi_u-2\xi_x)=0 \quad -(\varphi_{vv}-2\xi_{xv})=0$$
$$\varphi_v-\tau_t+\tau_{xx}-(\varphi_v-2\xi_x)=0 \quad \phi_u-\tau_t+\tau_{xx}+2\phi+\beta_1\varphi-\alpha\beta_2\varphi=0$$
$$\alpha\varphi_v-\alpha\tau_t+\alpha\tau_{xx}+2\varphi-\beta_1\phi+\alpha\beta_2\phi=0 \quad -\phi_u+\tau_t-\tau_{xx}-\tau_u=0$$
$$-\alpha\phi_u+\alpha\tau_t-\alpha\tau_{xx}-\alpha^2\tau_u=0 \quad -\beta_1\phi_u+\beta_1\tau_t-\beta_1\tau_{xx}=0 \quad -(\phi_{uu}-2\xi_{xu})=0$$
$$-\alpha\beta_2\varphi_v+\alpha\beta_2\tau_t-\alpha\beta_2\tau_{xx}=0 \quad -\xi_u+2\tau_{xu}+3\xi_u=0 \quad -\xi_v+2\tau_{xv}+3\xi_v=0$$
$$-\xi_u+2\tau_{xu}=0 \quad -\alpha\xi_v+2\alpha\tau_{xv}=0 \quad -2\tau_u+\tau_u=0 \quad -2\alpha\tau_v+\alpha\tau_v=0 \tag{23}$$

After lengthy and straightforward calculations, we obtain and express τ,ξ,ϕ,φ the infinitesimal transformations in the standard basis to get the spinning set of the Lie algebra of the infinitesimal symmetries U_i of coupled Lotka-Volterra competition-diffusion equations as;

$$U_1=\frac{\partial}{\partial t},\ U_2=\frac{\partial}{\partial x},\ U_3=u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v},\ U_4=2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x},\ U_5=2t\frac{\partial}{\partial t}-xu\frac{\partial}{\partial u}-xv\frac{\partial}{\partial v}$$
$$U_6=4t^2\frac{\partial}{\partial t}+4tx\frac{\partial}{\partial x}-(2ut+x^2u)\frac{\partial}{\partial u}-(2vt+x^2v)\frac{\partial}{\partial v} \tag{24}$$

The one-parameter group G_i admitted by the infinitesimal generators $U_1,U_2,U_3,U_4,U_5,$ and U_6 are determined by solving the corresponding Lie equations which gives the groups as;

$$\frac{\partial}{\partial t};G_1=\mathrm{X}_1(x,t,u,v;\lambda)\to\mathrm{X}_1(x,t+\lambda,u,v)$$

$$\frac{\partial}{\partial x}; G_2 = \mathrm{X}_2\left(x,t,u,v;\lambda\right) \to \mathrm{X}_2\left(x+\lambda,t,u,v\right)$$

$$u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}; G_3 = \mathrm{X}_3\left(x,t,u,v;\lambda\right) \to \mathrm{X}_3\left(x,t+\lambda,e^{\lambda}u,e^{\lambda}v\right)$$

$$2t\frac{\partial}{\partial t}+x\frac{\partial}{\partial x}; G_4 = \mathrm{X}_4\left(x,t,u,v;\lambda\right) \to \mathrm{X}_4\left(e^{\lambda}x,e^{2\lambda}t,u,v\right)$$

$$2t\frac{\partial}{\partial x}-xu\frac{\partial}{\partial u}-xv\frac{\partial}{\partial v}; G_5 = \mathrm{X}_5\left(x,t,u,v;\lambda\right) \to \mathrm{X}_5\left(x+2\lambda t,t,ue^{-\left(\lambda x+\lambda^2 t\right)},ve^{-\left(\lambda x+\lambda^2 t\right)}\right)$$

$$4t^2\frac{\partial}{\partial t}+4t\frac{\partial}{\partial x}-\left(2ut+x^2u\right)\frac{\partial}{\partial u}-\left(2vt+x^2v\right)\frac{\partial}{\partial v}; G_6 = \mathrm{X}_6\left(x,t,u,v;\lambda\right)$$
$$\to \mathrm{X}_6\left(\frac{x}{1-4\lambda t},\frac{t}{1-4\lambda t},u\sqrt{1-4\lambda t}e^{\left(\frac{-\lambda x^2}{1-4\lambda t}\right)},v\sqrt{1-4\lambda t}e^{\left(\frac{-\lambda x^2}{1-4\lambda t}\right)}\right)$$

We note that the groups U_5 and U_6 are non-trivial while U_1, U_2, U_3 and U_4 are translations and scaling (trivial groups).

III. Invariant Solutions

Suppose a group G of transformations T map a solution into itself the resulting solution is known as group invariant solution [23]. For the infinitesimal generator (4) of the coupled Lotka-Volterra competition-diffusion equation (3a,b), we obtain the invariant solution generated by U under the one –parameter group such that the general solution is obtained by integrating the corresponding characteristic system of ordinary differential equations

$$\frac{dx}{\xi\left(x,t,u,v\right)}=\frac{dt}{\xi\left(x,t,u,v\right)}=\frac{du}{\phi\left(x,t,u,v\right)}=\frac{dv}{\varphi\left(x,t,u,v\right)} \tag{25}$$

to obtain the independent invariants $\Omega_1 = k\left(x,t\right),\ \Omega_2 = \mu\left(x,t,u\right),\ \Omega_3 = \omega\left(x,t,v\right)$

We then designate two of the invariants as a function of the other as $\mu = \Psi\left(\Omega_1\right)$ and $\omega = \Theta\left(\Omega_1\right)$ and solve with respect to u and v. the obtained results are tabulated bellow;

Table I

Invariant Solution of Coupled Lotka-Volterra Competition-Diffusion Equations

Generator U_i	Invariants solutions
$U_1=\frac{\partial}{\partial t}$	$u=\Psi(x),\ v=\Theta(x)$
$U_2=\frac{\partial}{\partial x}$	$u=\Psi(t),\ v=\Theta(t)$
$U_3=u\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}$	$u=\Psi(x,t),\ v=\Theta(x,t)$
$U_4=x\frac{\partial}{\partial x}+2t\frac{\partial}{\partial t}$	$u=\Psi\left(\frac{x}{t}\right),\ v=\Theta\left(\frac{x}{t}\right)$
$U_5=2t\frac{\partial}{\partial x}-xu\frac{\partial}{\partial u}-xv\frac{\partial}{\partial v}$	$u=\Psi(x,t)e^{\frac{x^2}{4t}},\ v=\Theta(x,t)e^{\frac{x^2}{4t}}$
$U_6=4t^2\frac{\partial}{\partial t}+4tx\frac{\partial}{\partial x}-\left(2tu+x^2u\right)\frac{\partial}{\partial u}-\left(2tv+x^2v\right)\frac{\partial}{\partial v}$	$u=\Psi\left(xe^{-\frac{1}{t}}\right)(2t+x^2)^{-1}$ $v=\Theta\left(xe^{-\frac{1}{t}}\right)(2xt+x^2)^{-1}$

Table II

New Symmetry Solutions of Coupled Lotka-Volterra Competition-Diffusion Equations under U_6

Generator U_i	New Symmetry solutions
$U_6 = 4t^2\frac{\partial}{\partial t} + 4tx\frac{\partial}{\partial x} - \left(2tu + x^2u\right)\frac{\partial}{\partial u} - \left(2tv + x^2v\right)\frac{\partial}{\partial v}$	$\hat{u}_{1,6} = \Psi(x)\sqrt{1+4\lambda t}e^{\left(\frac{-\lambda x^2}{1+4\lambda t}\right)}$, $\hat{v}_{1,6} = \Theta(x)\sqrt{1+4\lambda t}e^{\left(\frac{-\lambda x^2}{1+4\lambda t}\right)}$
	$\hat{u}_{2,6} = \Psi(t)\sqrt{1+4\lambda t}e^{\left(\frac{-\lambda x^2}{1+4\lambda t}\right)}$, $\hat{v}_{2,6} = \Theta(t)\sqrt{1+4\lambda t}e^{\left(\frac{-\lambda x^2}{1+4\lambda t}\right)}$
	$\hat{u}_{3,6} = \Psi(x,t)\sqrt{1+4\lambda t}e^{\left(\frac{-\lambda x^2}{1+4\lambda t}\right)}$, $\hat{v}_{3,6} = \Theta(x,t)\sqrt{1+4\lambda t}e^{\left(\frac{-\lambda x^2}{1+4\lambda t}\right)}$
	$\hat{u}_{4,6} = \Psi\left(\frac{x}{t}\right)\sqrt{1+4\lambda t}e^{\left(\frac{-\lambda x^2}{1+4\lambda t}\right)}$, $\hat{v}_{4,6} = \Theta\left(\frac{x}{t}\right)\sqrt{1+4\lambda t}e^{\left(\frac{-\lambda x^2}{1+4\lambda t}\right)}$
	$\hat{u}_{5,6} = \Psi(x,t)e^{\frac{x^2}{4t}}\sqrt{1+4\lambda t}e^{\left(\frac{-\lambda x^2}{1+4\lambda t}\right)}$, $\hat{v}_{5,6} = \Theta(x,t)e^{\frac{x^2}{4t}}\sqrt{1+4\lambda t}e^{\left(\frac{-\lambda x^2}{1+4\lambda t}\right)}$
	$\hat{u}_{6,6} = \Psi\left(xe^{-\frac{1}{t}}\right)(2t+x^2)^{-1}\sqrt{1+4\lambda t}e^{\left(\frac{-\lambda x^2}{1+4\lambda t}\right)}$, $\hat{v}_{6,6} = \Theta\left(xe^{-\frac{1}{t}}\right)(2t+x^2)^{-1}\sqrt{1+4\lambda t}e^{\left(\frac{-\lambda x^2}{1+4\lambda t}\right)}$

IV. Group Transformations of Solutions

Since symmetry groups transform solutions of any equation into new solutions of the same equation, then; we let

$$\bar{x} = g(x,t,u,v), \quad \bar{t} = h(x,t,u,v), \quad \overline{u_n} = f_n(x,t,u,v) \tag{26}$$

for n dependent variables be a symmetry transformation group of coupled partial differential equation and suppose $u_n = \zeta_n(x,t)$ solve the equation, we can write the solutions in the new variables as $\overline{u_n} = \zeta_n(\bar{x},\bar{t})$. Suppose (15) are group transformations of coupled partial differential equation with $\overline{u_n}$ of the form $\overline{u_n} = \overline{\psi_n}(x,t,u_n,\lambda)$ for some explicit functions ψ_n, then by applying the inverse mapping, the new solutions takes the form

$$\overline{\overline{u_n}} = \overline{\psi_n}\left(\psi_n\left(\Upsilon_\lambda^{-1}(\bar{x}),\Upsilon_\lambda^{-1}(\bar{t})\right),\Upsilon_\lambda^{-1}(\bar{x}),\Upsilon_\lambda^{-1}(\bar{t}),\lambda^{-1}\right) \tag{27}$$

The symmetry inversion theory requires that for each symmetry group G_i admitted by the infinitesimal generators U_i and suppose $u=\psi(x,t)$ and $v=\Theta(x,t)$ are solutions of the coupled Lotka-Volterra competition-diffusion equation (3a,b), then the functions $\hat{u}_i$ and $\hat{v}_i$ are also solutions since symmetry transformations changes known solutions into new solutions [22] and we obtain

$$\hat{u}_1=\psi(x,t-\lambda),\ \ \hat{v}_1=\Theta(x,t-\lambda)$$

$$\hat{u}_2=\psi(x-\lambda,t),\ \ \hat{v}_2=\Theta(x-\lambda,t)$$

$$\hat{u}_3=e^{\lambda}\psi(x,t),\ \ \hat{v}_3=e^{\lambda}\Theta(x,t)$$

$$\hat{u}_4=\psi\left(e^{-\lambda}x,e^{-2\lambda}t\right),\ \ \hat{v}_4=\Theta\left(e^{-\lambda}x,e^{-2\lambda}t\right)$$

$$\hat{u}_5=e^{\left(-\lambda x+\lambda^2 t\right)}\psi(x-2\lambda t,t),\ \ \hat{v}_5=e^{\left(-\lambda x+\lambda^2 t\right)}\Theta(x-2\lambda t,t)$$

$$\hat{u}_6=\frac{1}{\sqrt{1+4\lambda t}}e^{\frac{-\lambda x^2}{(1+4\lambda t)}}\psi\left(\frac{x}{1+4\lambda t},\frac{t}{1+4\lambda t}\right),\ \ \hat{v}_6=\frac{1}{\sqrt{1+4\lambda t}}e^{\frac{-\lambda x^2}{(1+4\lambda t)}}\Theta\left(\frac{x}{1+4\lambda t},\frac{t}{1+4\lambda t}\right)$$

V. Conclusion

In this work, the coupled Lotka-Volterra competition-diffusion interaction system of equations which models two competing species whose interactions exhibit coupled density oscillation with diffusion are studied. We first non-dimensionalise (2a, b) to (3a, b) reducing the number of constants. We then constructed the Lie generator (4) of the system, develop the k^{th} order prolongation (8) of the generator for the system and use them in obtaining the invariant solutions (Table I), new symmetry solutions (Table II) and the group transformations of solutions (section 4) of the system with the restrictions that the diffusivity coefficients γ_1 and γ_2 are equal and their ratio is one. We recommend that a study on this topic be done using non-classical Lie symmetry groups and potential symmetries.

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