Algorithm for solving Proportion Equations

Dr. J. Suresh Kumar

Assistant Professor and Research Supervisor,
Post Graduate and Research Department of Mathematics,
N.S.S. Hindu College, Changanacherry, Kottayam Dist., Kerala, India-686102

Abstract:

Proportion Equation is an equality of two or more ratios. A Proportion Equation containingn ratios contains 2nterms. It is a novel Mathematical model of many Physical, Biological or Economics Problems. We consider Proportion Equations involving one or more ratios and in which exactly one of the terms is an indeterminate. Solving the Proportion Equation means to find the value of the indeterminate term knowing the values of all the other terms. Proportion Equations are very much important in the Practical Real-life Applications like Differential or Difference Equations. In this paper, we discuss the algorithm, based on the ancient Indian Mathematical works of Bhaskara[2], but in the modern perspective, for solving the Proportion Equations with any number of terms and in which exactly one of the terms is unknown. We also discuss its Methodology, Mathematical correctness, Algorithmic efficiency, Relevance and the Real-world Applications.

I. Introduction

The Ancient Indian Mathematics is a fertile area of research to bring out the significant Mathematical ideas and techniques. Ancient Indian Mathematics isa science and an art rather than a mere mental exercise. The Ancient Indian Mathematics gives a more intuitive appeal and pedagogical simplicity to the Mathematical methods and stress on the high speed mental computation with due relevance and applications to the real life problems, unlike the so-called Pure Mathematics. Suresh Kumar [1] initiated an approach to study the Ancient Indian Mathematicalmethods in the modern perspective and to discuss their methodology, Mathematical Correctness, Relevance and Real world applications.

A Proportion Equation is an equality of ratios. The proportion $A/B = C/D$ is commonly denoted as $A \propto B = C \propto D$(read as "A is to B" in the same way as "C is to D") or mathematically denoted as$A{:}B = C{:}D$,each side of the proportion contains two numbers each. In this Paper, we consider the Proportion Equationsinvolving one or more ratios and in which exactly one of the terms is an indeterminate variable. Solving the Proportion Equation means to find the value of the indeterminateterm knowing the values of all the other terms.

A Proportion Equation with 2, 3 or 4 ratios will have 3, 5 or 7 terms respectively and it is called as a 3-Rasika, 5-Rasika or 7-Rasika respectively in the Ancient Indian Mathematics [2] and separate methodswere suggested for each of them. In this Paper, we consider the problem in general and discuss a single algorithm for solving it. We also present a mathematical proof that this algorithm works for all cases and produces the desired result, which is an aspect usually missing in Ancient Indian Mathematics. However, the relevance and applications of the algorithm is well illustrated through real life problemsin Ancient Indian Mathematics, a few of which will be discussed here to illustrate the algorithm and its real life applications.

II. Main Results

In this section, we consider Proportion Equations, involving one or more ratios and in which exactly one of the terms is unknown. Solving the Proportion Equation means to find the value of the indeterminate term knowing the values of all the other terms. Consider a proportion Equation withonly two ratios: $A{:}B = C{:}D$. We call A as cause-1, C as cause-2, B as effect-1 and D as effect-2.Sometimes, the cause may have several factors in it. If the cause has only two factors in it, it will correspond to a Proportion with three terms on both sides of the proportion Equation and denoted by$A{:}B{:}C = D{:}E{:}F$. In terms of ratios, it means $(A/D)(B/E) = (C/F)$. When the cause has three factors in it, it will correspond to a Proportion Equation with four terms on both sides of the proportion and denoted by $A{:}B{:}C{:}D = E{:}F{:}G{:}H$. In terms of ratios, it means$(A/E)(B/F)(C.G) = (D/H)$.

In general, we are interested in the problem of finding the effect or some factors of the cause knowing the values of all other terms. In this Paper, we consider proportion Equationsin which the cause may have any numbers of factors in it. We present an algorithm in the modern perspective based on the works of Bhaskara [2]. A general Proportion Equation contains n factors for the cause and will have $(n+1)$terms on both of its sides. The problem is to find the effect A or Bor some factors $A_i$of the cause knowing the values of all other terms.

A. The Algorithm for Proportion Equations

The general Proportion Equation is: $A_1{:}A_2{:}\ldots{:}A_n{:}A = B_1{:}B_2{:}\ldots{:}B_n{:}B$, where some A_i or A is indeterminate, The problem is to find the effect A or B or some factors $A_i$of the cause knowing the values of all other terms. The algorithm is described as below:

Step-1: Write the terms of the general proportion Equation, $A_1{:}A_2{:}\ldots{:}A_n{:}A = B_1{:}B_2{:}\ldots{:}B_n{:}B$where some A_i or A is indeterminate, into two columns as follows:

<table>
<tr><td rowspan="4">Cause-1</td><td>A_1</td><td>B_1</td><td rowspan="4">Cause-2</td></tr>
<tr><td>A_2</td><td>B_2</td></tr>
<tr><td colspan="2">……………</td></tr>
<tr><td>A_n</td><td>B_n</td></tr>
<tr><td>Effect-1</td><td>$\boldsymbol{A}$</td><td>B</td><td>Effect-2</td></tr>
</table>

Step-2: Interchange the effect terms, $A$and B.

<table>
<tr><td rowspan="4">Cause-1</td><td>A_1</td><td>B_1</td><td rowspan="4">Cause-2</td></tr>
<tr><td>A_2</td><td>B_2</td></tr>
<tr><td colspan="2">……………</td></tr>
<tr><td>A_n</td><td>B_n</td></tr>
<tr><td>Effect-1</td><td>B</td><td>$\boldsymbol{A}$</td><td>Effect-2</td></tr>
</table>

Step-3: If the cause produces an inverse effect (that is, the magnitude of the effect term increase or decrease according as the magnitude of the cause term decrease or increase), then replace the effectterms, A, Brespectively by their reciprocals.$1/A$, $1/B$.

Step-4: Compute products of the known terms in each column. Then indeterminate term is given by:

$$indeterminate\ term = \frac{Product\ of\ the\ terms\ in\ the\ column\ without\ indeterminate\ term}{Product\ of\ the\ known\ terms\ in\ the\ column\ with\ indeterminate\ term}$$

B. The Mathematical Correctness:

Theorem.2.1. The above algorithm works for all proportions and produces the desired result.

Proof. We prove the theorem by Mathematical induction on the number of ratios, say n, present on either side of the proportion Equation.

If $n = 1$, the proportion is $A{:}B = C{:}D$. Writing column wise, we get

A	C
B	D

Interchanging the effect terms, we get

A	C
D	B

Computing the products of terms in each column, $AD = BC$, which is same as $A/B = C/D$ and which is the same as the proportion Equation, $A{:}B = C{:}D$. So the result is true when$n = 1$.

If $n = 2$, the proportion Equation is $A{:}B{:}C = D{:}E{:}F$. Writing column wise, we get

A	D
B	E
C	F

Interchanging the effect terms, we get

A	D
B	E
F	C

Computing the products of the terms in each column, we get $ABF = DEC$, which is equivalent to $(A/D)(B/E) = (C/F)$ and which is the same as the proportionEquation, $A{:}B{:}C = D{:}E{:}F$. Hence the result is true for $n = 2$.

Assume that the result is true for all proportion Equations with n ratios on either side of the proportion. Consider a proportion with $(n+1)$ ratios on any side of the proportion Equation, $A_1{:}A_2{:}\ldots{:}A_n{:}A = B_1{:}B_2{:}\ldots{:}B_n{:}B$ where $A_1{:}A_2{:}\ldots{:}A_n$are the factors of the cause-1 and $B_1{:}B_2{:}\ldots{:}B_n$,are the factors of the cause-2, A is the effect-1 and B isthe effect-2.Now, consider the given proportion Equation with $(n+1)$ ratios on any side of the proportion Equation, $A_1{:}A_2{:}\ldots{:}A_n{:}A = B_1{:}B_2{:}\ldots{:}B_n{:}B$.Writing into two columns, we get

A_1	B_1
A_2	B_2
A_3	B_3
$\ldots.$	$\ldots\ldots$
A_n	B_n
A	B

Interchanging the effect terms, we get

A_1	B_1
A_2	B_2
A_3	B_3
$\ldots.$	$\ldots.$
A_n	B_n
B	A

The products of the terms in each column gives$A_1\, A_2\, A_3 \ldots . A_n\, B = B_1 B_2\, B_3 \ldots . B_n\, A$, which further implies that $A_1(\,A_2\, A_3 \ldots . A_n\, B) = B_1(B_2\, B_3 \ldots . B_n\, A)$. That is, we have the following proportion Equation:$(A_1/B_1) = (\,A_2\, A_3 \ldots . A_n\, B/B_2\, B_3 \ldots . B_n\, A)$…………………(1)

By induction hypothesis, the result is true forthe proportion Equation,$A_2{:}\ldots{:}A_n{:}A = B_2{:}\ldots{:}B_n{:}B$. Thus, $A_2 A_3 \ldots . A_n B = B_2 B_3 \ldots . B_n A$implies theproportionEquation $(A_2/B_2)(A_3/B_3)(A_4/B_4)\ldots.(A_n/B_n) = (A/B)$.

Hence(1) implies$(A_1/B_1)(A_2/B_2)(A_3/B_3)(A_4/B_4)\ldots.(A_n/B_n) = (A/B)$.So the result is true for $n+1$. So by Mathematical Induction, the theorem follows.

Remark: The efficiency of this algorithm is evident, as one can even mentally solve problems by this method. Now, we turn to the relevance and real-world applications of this problem and its algorithm.

C. The Relevance and Real-world Applications:

The Proportion Equations occurs and play a significant role in many phenomena in the Physical sciences, Biological sciences and Economics. We discuss some practical real-life problems in which Proportion Equations occur. The problems involving proportion Equations are very significant in real life situations as well as in the physical or social sciences. The problem of solving Proportion Equations was a significant aspect of everyday life in the ancient India and many ancient Indian Mathematicians like Bhaskara studied this problem. We now discuss some real life problems originally posed by Bhaskara [2].

Problem.1. If 2½ units of rice is worth for 3/7 coins, then how much shall we get for 9 coins?

Solution: Let the required answer be x. Then we have the Proportion Equation,2½ :3/7 = x: 9.

Writing the terms of the Proportion Equation column wise,

2½	x
3/7	9

Interchanging the effect terms,

2½	x
9	3/7

Hence, the unknown term is given by,

$x = \frac{(2\frac{1}{2})\times 9}{\frac{3}{7}} = \frac{\left(\frac{5}{2}\right)\times 9}{\frac{3}{7}} = \frac{5\times 9\times 7}{2\times 3} = \frac{315}{6} = \frac{105}{2} = 52\frac{1}{2}$. Thus the answer is $52\frac{1}{2}$ units of rice.

Problem.2. If we get 32 coins for a cow of 16 years old, then how much shall we get fora cow of 20 years old?

Solution: Let the required answer be x. Since the age of a cow is inversely proportional to its cost in the market, the cost is directly proportional to the reciprocal of the age. Then we have the Proportion Equation, $1/16 : 32 = 1/20 : x$

Writing the terms of the Proportion Equation column wise,

$1/16$	32
x	$1/20$

Interchanging the effect terms, we get

$1/16$	$1/20$
x	32

Hence, the unknown term is given by,

$x = \frac{32(1/20)}{(1/16)} = \frac{32\times 16}{20} = \frac{128}{5}$. Thus the answer is $\frac{128}{5}$ coins.

Problem.3. If 5000 rupees is the interest for a loan of 100000rupees for one month, then in how many months shall we get an interest of 9600rupees for a loan of 16000rupees?

Solution: Let the required answer be x. Since there are three quantities on either side of the Proportion Equation, namely the Principal, the Months and the Interest, the Proportion Equation has six terms.Thus, we have the Proportion Equation, $1{,}00{,}000 : 1 : 5000 = 16000 : x : 9600$.

Writing the terms of the Proportion Equation column wise,

100000	16000
1	x
5000	9600

Interchanging the effect terms, we get

100000	16000
1	x
9600	5000

Hence, the unknown term is given by,

$x = \frac{(100000 \times 1 \times 9600)}{(16000 \times 5000)} = 12$. Thus the answer is 12 months.

Problem.4. If 8 silk Sarees of 8 feet length and 3 feet breadth cost 100000rupees, how much will a Sareeof 3½ feet length and ½ feet breadth cost?

Solution: Let the required answer be x. Here, there are three factors, namely the length, the breadth and the number of Sarees for the "cause" and the "effect" is the cost of theSarees. So, the Proportion Equation has eight terms.Thus, we have the Proportion Equation, 8: 3: 8: 100000=3½ :½ : 1; x

Writing the terms of the Proportion Equation column wise,

8	3½
3	½
8	1
100000	x

Interchanging the effect terms, we get

8	3½
3	½
8	1
x	100000

Hence, the unknown term is given by,

$x = \frac{(3.5 \times 0.5 \times 1 \times 100000)}{(8 \times 3 \times 8)} = \frac{175 \times 1000}{192} = 910$. Thus the answer is 910 rupees.

Problem.4. If 30 wooden planks, each of length 14ft, breadth 16ft and width 12ft cost 100000 rupees, how much will 14 planks each of dimension 4ft less worth?

Solution: Let the required answer be x. Here, there are four factors, namely the Length, the Breadth, the Width and the Number of planksthat form the "cause" and the "effect" is the Cost of the planks. Thus, we have the Proportion Equation, 14: 16: 12: 30: 100000 = 10:12: 8: 14 : x.

Writing the terms of the Proportion Equation column wise,

14	10
16	12
12	8
30	14
100000	x

Interchanging the effect terms, we get

14	10
16	12
12	8
30	14
x	100000

Hence, Unknown term is given by,

$x = \frac{(10\times12\times8\times14\times100000\,)}{(14\times16\times12\times30)} = \frac{50000}{3} = 16670$. Thus the answer is 16670 coins.

References:

[1] J. Suresh Kumar, "Algorithms in Quantitative Aptitude: An ancient Indian approach based on Bhaskara's works", Project Reports ubmitted to UGC-ASC, Orientation Course for College or University Teachers held from 27-07-2012 to 23-08-2012, UGC-ASC, University of Kerala, Trivandrum, Kerala, 2012.

[2] Bhaskara, "Sidddhantha Siromani Part-II: Lilavathy" (Sanskrit) (AD 1114-1185)