Unique Metro Domination of Cube of Paths

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Abstract — *A dominating set D of G which is also a resolving set of G is called a metro dominating set. A metro dominating set D of a graph G(V,E) is a unique metro dominating set (in short an UMD-set) if |N(v)∩D|=1 for each vertex $v \in V - D$ and the minimum cardinality of an UMD-set of G is the unique metro domination number of G denoted by $\gamma_{\mu\beta}(G)$. In this paper, we determine unique metro domination number of ${P_n}^3$graphs.*

Keywords — *Domination, metric dimension, metro domination, unique metro domination.*

I. INTRODUCTION

All the graphs considered in this paper are simple, connected and undirected. The length of a shortest path between two vertices u and v in a graph G is called the distance between u and v and is denoted by $d(u,v)$. For a vertex v of a graph, $N(v)$ denote the set of all vertices adjacent to v and is called open neighborhood of v. Similarly, the closed neighborhood of v is defined as $N[v] = N(v) \cap \{v\}$. Let $G(V,E)$ be a graph. For each ordered subset $S = \{v_1, v_2, v_3, \dots, v_k\}$ of V, each vertex $v \in V$ can be associated with a vector of distances denoted by $\Gamma(v/S) = (d(v_1,v), d(v_2,v), \dots d(v_k,v))$. The set S is said to be a resolving set of G, if $\Gamma(v/S) \neq \Gamma(u/S)$, for every $u,v \in V - S$. A resolving set of minimum cardinality is a *metric basis* and cardinality of a metric basis is the *metric dimension* of G. The k-tuple, $\Gamma(v/S)$ associated to the vertex $v \in V$ with respect to a metric basis S, is referred as a code generated by S for that vertex v. If $\Gamma(v/S) = (c_1, c_2, \dots, c_k)$, then $c_1, c_2, c_3, \dots, c_k$ are called components of the code of v generated by S and in particular c_i, $1 \leq i \leq k$, is called i^{th}-component of the code of v generated by S.

A dominating set D of a graph $G(V,E)$ is the subset of V having the property that for each vertex $v \in V - D$, there exists a vertex $u \in D$ such that uv is in E. A dominating set D of G which is also a resolving set of G is called a *metro dominating set*. A metro dominating set D of a graph $G(V,E)$ is a *unique metro dominating set* (in short an $UMD-set$) if $|N(v) \cap D| = 1$ for each vertex $v \in V - D$ and the minimum of cardinalities of UMD-sets of G is the *unique metro domination number* of G denoted by $\gamma_{\mu\beta}(G)$.

Consider $P_n, n \geq 4$. Join v_i to v_{i+2} and v_{i+3} for $1 \leq i \leq n-3$. The resulting graph is denoted by P_n^3.

Lemma 1: For any positive integer n, $\gamma_{\mu\beta}(P_n^3) \geq \left\lceil \frac{n}{7} \right\rceil$.

Proof: A vertex v_i dominates seven vertices $v_i, v_{i-1}, v_{i-2}, v_{i-3}, v_{i+1}, v_{i+2}, v_{i+3}$. Therefore, if D is a minimal dominating set then $|D| \geq \frac{n}{7}$. Hence we have $\gamma(P_n^3) \geq \lceil \frac{n}{7} \rceil$.

End vertex v_1 of P_n^3 can dominate only 4 vertices v_1, v_2, v_3 and v_4. As we have to minimize $|D|$, we include v_4 in D , which dominates $v_1, v_2, v_3, v_4, v_5, v_6$ and v_7.

Lemma 2: For $n = 7k$, $\gamma(P_n^3) = \left\lceil \frac{n}{7} \right\rceil$.

Proof: When $k = 1, v_4$ dominates all vertices of $P_7^3 = 1$. Hence $\gamma(P_n^3) = 1$.

Let $n = 7k$. Then $D = \{v_4, v_{11}, v_{18}, \dots, v_{7k-3}\}$ and $|D| = k$. When $n = 7(k+1)$, take $D' = D \cup \{v_{7k+4}\}$. Observe that $|D'| = k+1$ and D'dominates all vertices.

From Lemma1, we have $\gamma\left(P_{7(k+1)}^3\right) \geq \left\lceil \frac{7(k+1)}{7} \right\rceil = k+1$, and $|D'| = k+1$. Therefore we conclude that $\gamma(P_{7(k+1)}^3) = k+1$. Thus by induction $\gamma(P_n^3) = k = \lceil \frac{n}{7} \rceil$.

Lemma 3: If $n = 7k$, then $\gamma_{\mu\beta}(P_n^3) = k = \left\lceil \frac{n}{7} \right\rceil$.

Proof: In P_n^3, consider any v_j and v_{j+7} , $j \geq 4$ in D. Vertex v_j dominates $v_{j-3}, v_{j-2}, v_{j-1}, v_{j+1}, v_{j+2}, v_{j+3}$. Vertex v_{j+7} dominates $v_{j+4}, v_{j+5}, v_{j+6}, v_{j+8}, v_{j+9}$ and v_{j+10}. These vertices are uniquely dominated by v_j and v_{j+7}. The vertices v_1, v_2 and v_3 are uniquely dominated by v_4. The vertex v_{7k}, v_{7k-1} and v_{7k-2} are uniquely dominated by v_{7k-3}.

If $j > i + 14$, then $d(v_i, v_j) = \left\lceil\frac{j-i}{3}\right\rceil$, $d(v_{i+7}, v_j) = \left\lceil\frac{j-i-7}{3}\right\rceil$ and $d(v_{i+14}, v_j) = \left\lceil\frac{j-i-14}{3}\right\rceil$. Hence if $j = 3k$ and $j > i + 14$ then $d(v_i, v_{j-1}) = d(v_i, v_j) = d(v_i, v_{j+1})$, whereas $d(v_{i+7}, v_j) = d(v_{i+7}, v_{j+1})$=$d(v_{i+7}, v_{j+2})$ and $d(v_{i+14}, v_{j+1}) = d(v_{i+14}, v_{j+2}) = d(v_{i+7}, v_{j+3})$. Hence codes generated by v_i, v_{i+7}, v_{i+14} to v_j, $j > i + 14$ are all distinct and therefore $\{v_i, v_{i+7}, v_{i+14}\}$ resolves them. Now take $j = 3k$, $i < j < i + 7$.
Observe that $d(v_i, v_{j-1}) = d(v_i, v_j) = d(v_i, v_{j+1})$, $d(v_{i+7}, v_{j-1}) = d(v_{i+7}, v_j) = d(v_{i+7}, v_{j+1})$ and $d(v_{i+14}, v_j) = d(v_{i+14}, v_{j+1}) = d(v_{i+14}, v_{j+2})$. Hence codes generated by $\{v_i, v_{i+7}, v_{i+14}\}$ to v_j and v_{j+1} is the same. Observe that if $r < i + 21$ then $d(v_{i+21}, v_r) = \left\lceil\frac{i+21-r}{3}\right\rceil$. Hence $d(v_{i+21}, v_j) \neq d(v_{i+21}, v_{j+1})$. Therefore $\{v_i, v_{i+7}, v_{i+14}, v_{i+21}\}$ resolves v_j, $i < j < i + 7$. Similarly we observe that codes generated by $\{v_i, v_{i+7}, v_{i+14}\}$ to v_j and v_{j+1} where $i + 7 < j = 3k < i + 14$ are same. But $d(v_{i+21}, v_j) \neq d(v_{i+21}, v_{j+1})$. Hence $\{v_i, v_{i+7}, v_{i+14}, v_{i+21}\}$ resolves all vertices $v_j, j > i$. When $i = 4$, the codes generated by $\{v_4, v_{11}, v_{18}, v_{25}\}$ to v_1, v_2, v_3 are (1,4,6,8),(1,3,6,8),(1,3,5,8) and hence $\{v_4, v_{11}, v_{18}, v_{25}\}$ resolves all vertices of P_n^3. Therefore to resolve all vertices of P_n^3 we take $n \geq 22$. We observe that D=$\{v_4, v_{11}, v_{18}, \ldots, v_{7k-3}, v_{7k+4}\}$ is a UMD set. Therefore $\gamma_{\mu\beta}(P_n^3) = k = \left\lceil\frac{n}{7}\right\rceil$.
When $n = 7k + 1, 7k + 2, 7k + 3$ and $7k + 4$, $D = \{v_1, v_8, v_{15}, \ldots, v_{7k-6}, v_{7k+1}\}$ is a UMD set.
When $n = 7k + 5, 7k + 6$ we have $D = \{v_4, v_{11}, v_{18}, \ldots, v_{7k-3}, v_{7k+4}\}$ is a UMD set.
Therefore $\gamma_{\mu\beta}(P_n^3) = k + 1$. In all these cases $|D| = k + 1 = \left\lceil\frac{n}{7}\right\rceil$. Thus we obtain that $\gamma_{\mu\beta}(P_n^3) = \left\lceil\frac{n}{7}\right\rceil, \forall n \geq 22$.
If $n < 22$, then we observe that $\gamma_{\mu\beta}(P_n^3) = n$. Hence we have

II. CONCLUSION

Theorem 1. $\gamma_{\mu\beta}(P_n^3) = \begin{cases} \left\lceil\frac{n}{7}\right\rceil, & for\ n \geq 22 \\ n, & for\ n < 22 \end{cases}$.

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