A Study of Some Fractional Functions

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Abstract - *This paper studies the basic properties of some elementary fractional functions such as fractional exponential function, fractional trigonometric functions, and fractional hyperbolic functions. The Mittag-Leffler function plays an important role in this article, and the results obtained in this article are the generalizations of the ones of the classical functions, and are useful to solve the fractional differential problems.*

Keywords - *Fractional exponential function, Fractional trigonometric functions, Fractional hyperbolic functions, Mittag-Leffler function.*

I. INTRODUCTION

Fractional calculus is a field of applied mathematics that deals with derivatives and integrals of any arbitrary real or complex order. It arises from a question proposed by L'Hospital and Leibniz in 1695, the history of fractional derivatives were planted over 300 years ago. Since that time the fractional calculus has drawn the attention of many great mathematicians of their times, such as N. H. Abel, M. Caputo, L. Euler, J. Fourier, A.K. Grunwald, J. Hadamard, G. H. Hardy, O. Heaviside, H. J. Holmgren, P. S. Laplace, G. W. Leibniz, A. V. Letnikov, J. Liouville, B. Riemann, M. Riesz, and H. Weyl. With the great efforts of researchers there have been rapid developments on the theory of fractional calculus and its applications. During this last decades, the fractional calculus have been applied in widespread fields of science and engineering [1-5]. In this paper, we study some basic properties of several fractional functions, for example, fractional exponential function, fractional trigonometric functions and fractional hyperbolic functions which are concerned with the Mittag-Leffler function, and our results are the generalizations of the ones of the traditional elementary functions.

II. BASIC PROPERTIES

In the following, we introduce some fractional functions and their fundamental properties.

Notation 2.1: If α is a real number, then the greatest integer less than or equal to α is denoted by $\lfloor\alpha\rfloor$.

Definition 2.2 ([6])**:** The Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}, \tag{1}$$

where α is a real number, $\alpha \geq 0$, and z is a complex variable.

Definition 2.3 ([7])**:** Let $0 < \alpha \leq 1$, λ be a complex number, and x be a real variable. Then $E_\alpha(\lambda x^\alpha)$ is called α-order fractional exponential function, and the α-order fractional cosine and sine function are defined as follows:

$$cos_\alpha(\lambda x^\alpha) = \sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{2k}x^{2k\alpha}}{\Gamma(2k\alpha+1)}, \tag{2}$$

and

$$sin_\alpha(\lambda x^\alpha) = \sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{2k+1}x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}. \tag{3}$$

Remark 2.4: If $\alpha = 1$, $\lambda = 1$, then $cos_1(x) = cosx$, and $sin_1(x) = sinx$.

Notation 2.5: Let $z = a + ib$ be a complex number, where $i = \sqrt{-1}$, and a, b are real numbers. a is the real part of z, and denoted by $\text{Re}(z)$; b is the imaginary part of z, denoted by $\text{Im}(z)$.

Proposition 2.6 (fractional Euler's formula)[8]**:** *Let* $0 < \alpha \leq 1$, *then*

$$E_\alpha(ix^\alpha) = cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha). \tag{4}$$

Remark 2.7: If $\alpha = 1$, we obtain Euler's formula $e^{ix} = cosx + isinx$.

Proposition 2.8 (fractional DeMoivre's formula)[8]**:** *Let* $0 < \alpha \leq 1$, *and* n *be a positive integer, then*

$$[cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha)]^n = cos_\alpha(nx^\alpha) + isin_\alpha(nx^\alpha). \tag{5}$$

Remark 2.9: The case $\alpha = 1$ is the classical DeMoivre's formula $(cosx + isinx)^n = cosnx + isinnx$.

Proposition 2.10: *If* $0 < \alpha \leq 1$, *and* n *is a positive integer, then*

$$cos_\alpha(nx^\alpha) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!(-1)^k}{(2k)!(n-2k)!} [cos_\alpha(x^\alpha)]^{n-2k} [sin_\alpha(x^\alpha)]^{2k}, \tag{6}$$

and

$$sin_\alpha(nx^\alpha) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{n!(-1)^k}{(2k+1)!(n-2k-1)!} [cos_\alpha(x^\alpha)]^{n-2k-1} [sin_\alpha(x^\alpha)]^{2k+1} \ . \tag{7}$$

Proof $cos_\alpha(nx^\alpha) = \text{Re}[[cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha)]^n]$ (by Eq. (5))

$$= \text{Re}\left[\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} [cos_\alpha(x^\alpha)]^{n-k} [isin_\alpha(x^\alpha)]^k\right]$$

$$= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!(-1)^k}{(2k)!(n-2k)!} [cos_\alpha(x^\alpha)]^{n-2k} [sin_\alpha(x^\alpha)]^{2k} \ .$$

Similarly,

$$sin_\alpha(nx^\alpha) = \text{Im}[[cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha)]^n]$$

$$= \text{Im}\left[\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} [cos_\alpha(x^\alpha)]^{n-k} [isin_\alpha(x^\alpha)]^k\right]$$

$$= \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} \frac{n!(-1)^k}{(2k+1)!(n-2k-1)!} [cos_\alpha(x^\alpha)]^{n-2k-1} [sin_\alpha(x^\alpha)]^{2k+1}.$$

q.e.d.

Proposition 2.11: *Let* $0 < \alpha \le 1$, *then*

$$cos_\alpha((x+y)^\alpha) = cos_\alpha(x^\alpha) \cdot cos_\alpha(y^\alpha) - sin_\alpha(x^\alpha) \cdot sin_\alpha(y^\alpha), \tag{8}$$

and

$$sin_\alpha((x+y)^\alpha) = sin_\alpha(x^\alpha) \cdot cos_\alpha(y^\alpha) + cos_\alpha(x^\alpha) \cdot sin_\alpha(y^\alpha). \tag{9}$$

Proof From [9], we have

$$E_\alpha(\lambda(x+y)^\alpha) = E_\alpha(\lambda x^\alpha) \cdot E_\alpha(\lambda y^\alpha) \ , \tag{10}$$

for any complex number λ.

Let $\lambda = i$, then

$$cos_\alpha((x+y)^\alpha) = \text{Re}[E_\alpha(i(x+y)^\alpha)]$$

$$= \text{Re}[E_\alpha(ix^\alpha) \cdot E_\alpha(iy^\alpha)]$$

$$= \text{Re}\left[\left(cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha)\right) \cdot \left(cos_\alpha(y^\alpha) + isin_\alpha(y^\alpha)\right)\right]$$

$$= cos_\alpha(x^\alpha) \cdot cos_\alpha(y^\alpha) - sin_\alpha(x^\alpha) \cdot sin_\alpha(y^\alpha).$$

And,

$$sin_\alpha((x+y)^\alpha) = \text{Im}[E_\alpha(i(x+y)^\alpha)]$$

$$= \text{Im}\left[\left(cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha)\right) \cdot \left(cos_\alpha(y^\alpha) + isin_\alpha(y^\alpha)\right)\right]$$

$$= sin_\alpha(x^\alpha) \cdot cos_\alpha(y^\alpha) + cos_\alpha(x^\alpha) \cdot sin_\alpha(y^\alpha).$$

q.e.d.

Next, we define the other fractional trigonometric function.

Definition 2.12: Let $0 < \alpha \le 1$, and λ be a complex number, then

$$tan_\alpha(\lambda x^\alpha) = \frac{sin_\alpha(\lambda x^\alpha)}{cos_\alpha(\lambda x^\alpha)} \tag{11}$$

is called α-order fractional tangent function.

$$cot_\alpha(\lambda x^\alpha) = \frac{cos_\alpha(\lambda x^\alpha)}{sin_\alpha(\lambda x^\alpha)} \tag{12}$$

is the α-order fractional cotangent function.

$$sec_\alpha(\lambda x^\alpha) = \frac{1}{cos_\alpha(\lambda x^\alpha)} \tag{13}$$

is the α-order fractional secant function.

$$csc_\alpha(\lambda x^\alpha) = \frac{1}{sin_\alpha(\lambda x^\alpha)} \tag{14}$$

is the α-order fractional cosecant function.

Proposition 2.13: *Let* $0 < \alpha \le 1$, *then*

$$sin_\alpha(-x^\alpha) = -sin_\alpha(x^\alpha), \tag{15}$$

$$cos_\alpha(-x^\alpha) = cos_\alpha(x^\alpha), \tag{16}$$

$$[sin_\alpha(x^\alpha)]^2 + [cos_\alpha(x^\alpha)]^2 = 1, \tag{17}$$

$$1 + [tan_\alpha(x^\alpha)]^2 = [sec_\alpha(x^\alpha)]^2, \tag{18}$$

$$1 + [cot_\alpha(x^\alpha)]^2 = [csc_\alpha(x^\alpha)]^2, \tag{19}$$

$$[cos_\alpha(x^\alpha)]^2 = \frac{1+cos_\alpha(2x^\alpha)}{2}, \tag{20}$$

$$[sin_\alpha(x^\alpha)]^2 = \frac{1-cos_\alpha(2x^\alpha)}{2}, \tag{21}$$

$$tan_\alpha((x+y)^\alpha) = \frac{tan_\alpha(x^\alpha)+tan_\alpha(y^\alpha)}{1-tan_\alpha(x^\alpha)\cdot tan_\alpha(y^\alpha)} \; . \tag{22}$$

Proof By Eq. (2) and Eq. (3), we can easily obtain Eq. (15) and Eq. (16). By [7], we have

$$E_\alpha(\lambda x^\alpha)\cdot E_\alpha(\mu x^\alpha) = E_\alpha((\lambda+\mu)x^\alpha), \tag{23}$$

for any complex numbers λ, μ. Thus,

$$\begin{aligned}[sin_\alpha(x^\alpha)]^2 + [cos_\alpha(x^\alpha)]^2 &= \left[\frac{E_\alpha(ix^\alpha)-E_\alpha(-ix^\alpha)}{2i}\right]^2 + \left[\frac{E_\alpha(ix^\alpha)+E_\alpha(-ix^\alpha)}{2}\right]^2 \\ &= -\frac{E_\alpha(i2x^\alpha)+E_\alpha(-i2x^\alpha)-2}{4} + \frac{E_\alpha(i2x^\alpha)+E_\alpha(-i2x^\alpha)+2}{4} \\ &= 1.\end{aligned}$$

On the other hand,

$$1 + [tan_\alpha(x^\alpha)]^2 = \frac{[sin_\alpha(x^\alpha)]^2+[cos_\alpha(x^\alpha)]^2}{[cos_\alpha(x^\alpha)]^2} = [sec_\alpha(x^\alpha)]^2.$$

$$1 + [cot_\alpha(x^\alpha)]^2 = \frac{[sin_\alpha(x^\alpha)]^2+[cos_\alpha(x^\alpha)]^2}{[sin_\alpha(x^\alpha)]^2} = [csc_\alpha(x^\alpha)]^2 \; .$$

Moreover, by Eq. (9),

$$\begin{aligned}cos_\alpha(2x^\alpha) &= [cos_\alpha(x^\alpha)]^2 - [sin_\alpha(x^\alpha)]^2 \quad \text{(by Eq. (6))} \\ &= [cos_\alpha(x^\alpha)]^2 - (1-[cos_\alpha(x^\alpha)]^2) \quad \text{(by Eq. (17))} \\ &= 2\cdot[cos_\alpha(x^\alpha)]^2 - 1.\end{aligned}$$

And hence,

$$[cos_\alpha(x^\alpha)]^2 = \frac{1+cos_\alpha(2x^\alpha)}{2}.$$

Similarly,

$$\begin{aligned}cos_\alpha(2x^\alpha) &= [cos_\alpha(x^\alpha)]^2 - [sin_\alpha(x^\alpha)]^2 \\ &= (1-[sin_\alpha(x^\alpha)]^2) - [sin_\alpha(x^\alpha)]^2 \\ &= 1 - 2\cdot[sin_\alpha(x^\alpha)]^2.\end{aligned}$$

Therefore,

$$[sin_\alpha(x^\alpha)]^2 = \frac{1-cos_\alpha(2x^\alpha)}{2}.$$

In addition,

$$\begin{aligned}tan_\alpha((x+y)^\alpha) &= \frac{sin_\alpha((x+y)^\alpha)}{cos_\alpha((x+y)^\alpha)} \\ &= \frac{sin_\alpha(x^\alpha)\cdot cos_\alpha(y^\alpha)+cos_\alpha(x^\alpha)\cdot sin_\alpha(y^\alpha)}{cos_\alpha(x^\alpha)\cdot cos_\alpha(y^\alpha)-sin_\alpha(x^\alpha)\cdot sin_\alpha(y^\alpha)} \quad \text{(by Eqs. (8), (9))} \\ &= \frac{tan_\alpha(x^\alpha)+tan_\alpha(y^\alpha)}{1-tan_\alpha(x^\alpha)\cdot tan_\alpha(y^\alpha)} \; .\end{aligned}$$

q.e.d.

In the following, we study the properties of fractional hyperbolic functions.

Definition 2.14: If $0 < \alpha \leq 1$, and λ is a complex number, then

$$cosh_\alpha(\lambda x^\alpha) = \sum_{k=0}^{\infty} \frac{\lambda^{2k} x^{2k\alpha}}{\Gamma(2k\alpha+1)} \tag{24}$$

is called α-order fractional hyperbolic cosine function.

$$sinh_\alpha(\lambda x^\alpha) = \sum_{k=0}^{\infty} \frac{\lambda^{2k+1} x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} \tag{25}$$

is called α-order fractional hyperbolic sine function.

Proposition 2.15: *Assume that* $0 < \alpha \leq 1$, *then*

$$cosh_\alpha(\lambda x^\alpha) = \frac{E_\alpha(\lambda x^\alpha) + E_\alpha(-\lambda x^\alpha)}{2}, \tag{26}$$

$$sinh_\alpha(\lambda x^\alpha) = \frac{E_\alpha(\lambda x^\alpha) - E_\alpha(-\lambda x^\alpha)}{2}. \tag{27}$$

Proof $\frac{E_\alpha(\lambda x^\alpha)+E_\alpha(-\lambda x^\alpha)}{2} = \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{\lambda^k x^{k\alpha}}{\Gamma(k\alpha+1)} + \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k x^{k\alpha}}{\Gamma(k\alpha+1)}\right)$

$$= \sum_{k=0}^{\infty} \frac{\lambda^{2k} x^{2k\alpha}}{\Gamma(2k\alpha+1)}$$

$$= cosh_\alpha(\lambda x^\alpha).$$

$$\frac{E_\alpha(\lambda x^\alpha)-E_\alpha(-\lambda x^\alpha)}{2} = \frac{1}{2}\left(\sum_{k=0}^{\infty} \frac{\lambda^k x^{k\alpha}}{\Gamma(k\alpha+1)} - \sum_{k=0}^{\infty} \frac{(-1)^k \lambda^k x^{k\alpha}}{\Gamma(k\alpha+1)}\right)$$

$$= \sum_{k=0}^{\infty} \frac{\lambda^{2k+1} x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}$$

$$= sinh_\alpha(\lambda x^\alpha).$$ q.e.d.

Corollary 2.16: $E_\alpha(x^\alpha) = cosh_\alpha(x^\alpha) + sinh_\alpha(x^\alpha)$. (28)

$$E_\alpha(-x^\alpha) = cosh_\alpha(x^\alpha) - sinh_\alpha(x^\alpha). \tag{29}$$

Definition 2.17: Suppose that $0 < \alpha \leq 1$, and λ is a complex number,

$$tanh_\alpha(\lambda x^\alpha) = \frac{sinh_\alpha(\lambda x^\alpha)}{cosh_\alpha(\lambda x^\alpha)} \tag{30}$$

is called α-order fractional hyperbolic tangent function.

$$coth_\alpha(\lambda x^\alpha) = \frac{cosh_\alpha(\lambda x^\alpha)}{sinh_\alpha(\lambda x^\alpha)} \tag{31}$$

is α-order fractional hyperbolic cotangent function.

$$sech_\alpha(\lambda x^\alpha) = \frac{1}{cosh_\alpha(\lambda x^\alpha)} \tag{32}$$

is α-order fractional hyperbolic secant function.

$$csch_\alpha(\lambda x^\alpha) = \frac{1}{sinh_\alpha(\lambda x^\alpha)} \tag{33}$$

is α-order fractional hyperbolic cosecant function.

Proposition 2.18: *Let* $0 < \alpha \leq 1$,

$$cosh_\alpha((x+y)^\alpha) = cosh_\alpha(x^\alpha) \cdot cosh_\alpha(y^\alpha) + sinh_\alpha(x^\alpha) \cdot sinh_\alpha(y^\alpha), \tag{34}$$

and

$$sinh_\alpha((x+y)^\alpha) = sinh_\alpha(x^\alpha) \cdot cosh_\alpha(y^\alpha) + cosh_\alpha(x^\alpha) \cdot sinh_\alpha(y^\alpha). \tag{35}$$

Proof Since

$$E_\alpha(\lambda(x+y)^\alpha) = E_\alpha(\lambda x^\alpha) \cdot E_\alpha(\lambda y^\alpha) \ ,$$

for any complex number λ.

Let $\lambda = 1$, then

$$E_\alpha((x+y)^\alpha) = E_\alpha(x^\alpha) \cdot E_\alpha(y^\alpha). \tag{36}$$

If $\lambda = -1$, we have

$$E_\alpha(-(x+y)^\alpha) = E_\alpha(-x^\alpha) \cdot E_\alpha(-y^\alpha). \tag{37}$$

Thus,

$cosh_\alpha((x+y)^\alpha)$

$= \frac{1}{2}[E_\alpha((x+y)^\alpha) + E_\alpha(-(x+y)^\alpha)]$ (by Eq. (26))

$= \frac{1}{2}[E_\alpha(x^\alpha) \cdot E_\alpha(y^\alpha) + E_\alpha(-x^\alpha) \cdot E_\alpha(-y^\alpha)]$ (by Eqs. (36), (37))

$= \frac{1}{2}[(cosh_\alpha(x^\alpha) + sinh_\alpha(x^\alpha)) \cdot (cosh_\alpha(y^\alpha) + sinh_\alpha(y^\alpha)) + (cosh_\alpha(x^\alpha) - sinh_\alpha(x^\alpha)) \cdot (cosh_\alpha(y^\alpha) - sinh_\alpha(y^\alpha))]$

(by Eqs. (28), (29))

$= cosh_\alpha(x^\alpha) \cdot cosh_\alpha(y^\alpha) + sinh_\alpha(x^\alpha) \cdot sinh_\alpha(y^\alpha)$.

Similarly,

$sinh_\alpha((x+y)^\alpha)$

$= \frac{1}{2}[E_\alpha((x+y)^\alpha) - E_\alpha(-(x+y)^\alpha)]$ (by Eq. (27))

$= \frac{1}{2}[E_\alpha(x^\alpha) \cdot E_\alpha(y^\alpha) - E_\alpha(-x^\alpha) \cdot E_\alpha(-y^\alpha)]$

$= \frac{1}{2}[(cosh_\alpha(x^\alpha) + sinh_\alpha(x^\alpha)) \cdot (cosh_\alpha(y^\alpha) + sinh_\alpha(y^\alpha)) - (cosh_\alpha(x^\alpha) - sinh_\alpha(x^\alpha)) \cdot (cosh_\alpha(y^\alpha) - sinh_\alpha(y^\alpha))]$

$= sinh_\alpha(x^\alpha) \cdot cosh_\alpha(y^\alpha) + cosh_\alpha(x^\alpha) \cdot sinh_\alpha(y^\alpha)$. q.e.d.

Proposition 2.19: *Let* $0 < \alpha \le 1$, *then*

$$sinh_\alpha(-x^\alpha) = -sinh_\alpha(x^\alpha), \tag{38}$$

$$cosh_\alpha(-x^\alpha) = cosh_\alpha(x^\alpha), \tag{39}$$

$$[cosh_\alpha(x^\alpha)]^2 - [sinh_\alpha(x^\alpha)]^2 = 1, \tag{40}$$

$$1 - [tanh_\alpha(x^\alpha)]^2 = [sech_\alpha(x^\alpha)]^2, \tag{41}$$

$$[coth_\alpha(x^\alpha)]^2 - 1 = [csch_\alpha(x^\alpha)]^2, \tag{42}$$

$$[cosh_\alpha(x^\alpha)]^2 = \frac{cosh_\alpha(2x^\alpha)+1}{2}, \tag{43}$$

$$[sinh_\alpha(x^\alpha)]^2 = \frac{cosh_\alpha(2x^\alpha)-1}{2}, \tag{44}$$

$$tanh_\alpha((x+y)^\alpha) = \frac{tanh_\alpha(x^\alpha)+tanh_\alpha(y^\alpha)}{1+tanh_\alpha(x^\alpha) \cdot tanh_\alpha(y^\alpha)} . \tag{45}$$

Proof Eqs. (38), (39) are easily obtained by Eqs. (24), (25) respectively. On the other hand,

$$[cosh_\alpha(x^\alpha)]^2 - [sinh_\alpha(x^\alpha)]^2 = \left[\frac{E_\alpha(x^\alpha)+E_\alpha(-x^\alpha)}{2}\right]^2 - \left[\frac{E_\alpha(x^\alpha)-E_\alpha(-x^\alpha)}{2}\right]^2$$

$$= \frac{E_\alpha(2x^\alpha)+E_\alpha(-2x^\alpha)+2}{4} - \frac{E_\alpha(2x^\alpha)+E_\alpha(-2x^\alpha)-2}{4}$$

$$= 1.$$

And,

$$1 - [tanh_\alpha(x^\alpha)]^2 = 1 - \left[\frac{sinh_\alpha(\lambda x^\alpha)}{cosh_\alpha(\lambda x^\alpha)}\right]^2$$

$$= \frac{[cosh_\alpha(x^\alpha)]^2 - [sinh_\alpha(x^\alpha)]^2}{[cosh_\alpha(x^\alpha)]^2}$$

$$= \frac{1}{[cosh_\alpha(x^\alpha)]^2}$$

$$= [sech_\alpha(x^\alpha)]^2 .$$

Moreover,

$$[coth_\alpha(x^\alpha)]^2 - 1 = \left[\frac{cosh_\alpha(x^\alpha)}{sinh_\alpha(x^\alpha)}\right]^2 - 1$$
$$= \frac{[cosh_\alpha(x^\alpha)]^2 - [sinh_\alpha(x^\alpha)]^2}{[sinh_\alpha(x^\alpha)]^2}$$
$$= \frac{1}{[sinh_\alpha(x^\alpha)]^2}$$
$$= [csch_\alpha(x^\alpha)]^2 \ .$$

And,

$$[cosh_\alpha(x^\alpha)]^2 = \left[\frac{E_\alpha(x^\alpha)+E_\alpha(-x^\alpha)}{2}\right]^2$$
$$= \frac{E_\alpha(2x^\alpha)+E_\alpha(-2x^\alpha)+2}{4}$$
$$= \frac{2cosh_\alpha(2x^\alpha)+2}{4}$$
$$= \frac{cosh_\alpha(2x^\alpha)+1}{2} \ .$$

$$[sinh_\alpha(x^\alpha)]^2 = \left[\frac{E_\alpha(x^\alpha)-E_\alpha(-x^\alpha)}{2}\right]^2$$
$$= \frac{E_\alpha(2x^\alpha)+E_\alpha(-2x^\alpha)-2}{4}$$
$$= \frac{2cosh_\alpha(2x^\alpha)-2}{4}$$
$$= \frac{cosh_\alpha(2x^\alpha)-1}{2} \ .$$

$$tanh_\alpha((x+y)^\alpha) = \frac{sinh_\alpha((x+y)^\alpha)}{cosh_\alpha((x+y)^\alpha)}$$
$$= \frac{sinh_\alpha(x^\alpha)\cdot cosh_\alpha(y^\alpha)+cosh_\alpha(x^\alpha)\cdot sinh_\alpha(y^\alpha)}{cosh_\alpha(x^\alpha)\cdot cosh_\alpha(y^\alpha)+sinh_\alpha(x^\alpha)\cdot sinh_\alpha(y^\alpha)} \quad \text{(by Eqs. (34), (35))}$$
$$= \frac{tanh_\alpha(x^\alpha)+tanh_\alpha(y^\alpha)}{1+tanh_\alpha(x^\alpha)\cdot tanh_\alpha(y^\alpha)} \ .$$

q.e.d.

In the following, we find the relationship between fractional trigonometric functions and fractional hyperbolic functions.

Proposition 2.20: *Let* $0 < \alpha \leq 1$, *then*

$$cos_\alpha(ix^\alpha) = cosh_\alpha(x^\alpha), \tag{46}$$
$$sin_\alpha(ix^\alpha) = isinh_\alpha(x^\alpha), \tag{47}$$
$$tan_\alpha(ix^\alpha) = itanh_\alpha(x^\alpha), \tag{48}$$
$$cot_\alpha(ix^\alpha) = -icoth_\alpha(x^\alpha), \tag{49}$$
$$sec_\alpha(ix^\alpha) = sech_\alpha(x^\alpha), \tag{50}$$
$$csc_\alpha(ix^\alpha) = -icsch_\alpha(x^\alpha). \tag{51}$$

Proof

$$cos_\alpha(ix^\alpha) = \sum_{k=0}^{\infty} \frac{x^{2k\alpha}}{\Gamma(2k\alpha+1)} \quad \text{(by Eq. (2))}$$
$$= cosh_\alpha(x^\alpha).$$

$$sin_\alpha(ix^\alpha) = i\cdot\sum_{k=0}^{\infty} \frac{x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)} \quad \text{(by Eq. (3))}$$
$$= isinh_\alpha(x^\alpha).$$

$$tan_\alpha(ix^\alpha) = \frac{sin_\alpha(ix^\alpha)}{cos_\alpha(ix^\alpha)}$$
$$= \frac{isinh_\alpha(x^\alpha)}{cosh_\alpha(x^\alpha)}$$

$$= itanh_{\alpha}(x^{\alpha}).$$

$$cot_{\alpha}(ix^{\alpha}) = \frac{1}{tan_{\alpha}(ix^{\alpha})}$$

$$= \frac{1}{itanh_{\alpha}(x^{\alpha})}$$

$$= -icoth_{\alpha}(x^{\alpha}).$$

$$sec_{\alpha}(ix^{\alpha}) = \frac{1}{cos_{\alpha}(ix^{\alpha})}$$

$$= \frac{1}{cosh_{\alpha}(x^{\alpha})}$$

$$= sech_{\alpha}(x^{\alpha}).$$

$$csc_{\alpha}(ix^{\alpha}) = \frac{1}{sin_{\alpha}(ix^{\alpha})}$$

$$= \frac{1}{isinh_{\alpha}(x^{\alpha})}$$

$$= -icsch_{\alpha}(x^{\alpha}).$$

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III. CONCLUSIONS

The fractional functions studied in this paper are closely related to Mittag-Leffler function and are the generalizations of classical elementary functions such as exponential function, trigonometric functions, and hyperbolic functions. The fundamental properties of these fractional functions are the same as the ones of these classical functions. In the future, we will study the fractional differential problems of these fractional functions.

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