4-Roman Coloring of graphs

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Abstract:

Motivated from the Roman Military defense strategy, Suresh Kumar [12] introduced a new type of graph coloring, namely, Roman Coloring. A Roman coloring of a graph G is an assignment of four colors, {0, 1, 2, 3}, to the vertices of G such that every vertex with the color, 0 must be adjacent to some vertex of degree 2 or 3. In this paper, we extend the concept of Roman coloring to 4-colorings of graphs. We introduce and study the 4-Roman coloring of graphs and the related parameter, 4-Roman chromatic number.

Keywords: *Graph, Roman Coloring, 4-Roman coloring, 4-Roman Chromatic Number.*

I. INTRODUCTION

The majority of graph theory research on graph coloring focuses on vertex coloring that satisfies some specified property for the induced edge coloring [5]. The coloring is also played an important role in combinatorial optimization problems and critical (Optimal) graphs were crucial in the Chromatic number Theory [7, 8, 9, 10, 11]. Jason Robert Lewis [1] suggested several new graph parameters in his Doctoral Thesis. Several studies were made in applying such parameters to Roman defense strategy [2, 3, 4, 5, 6]. The basic idea behind all these works was that for a given city, if the streets are considered as the edges of a graph and the meeting points of the streets, called the junctions, as the vertices of the graph, then we can color each vertex by the number of soldiers deployed at that junction and require that every street (edge) should be guarded by at least one soldier using a strategy that if any street have no soldier, then there must be an adjacent junction with two soldiers so that one among them may be deployed to the former junction in case of emergency. Motivated from this Roman military defense strategy, Suresh Kumar [12] defined a new type of graph coloring, Roman Coloring and the related parameter, Roman Chromatic number. However, this is not a proper coloring. The proper Roman Coloring was introduced and studied by Suresh Kumar and Preethi K Pillai [13].

In this paper, we introduce and study the 4-Roman coloring of graphs. The basic idea and the requirement behind this work is still in line with traditional Roman military defense strategy that every street (edge) should be guarded by at least one soldier using a strategy that if any street has no soldiers, then there must be an adjacent junction with two and three soldiers so that each junction can have at least two soldiers, in case of emergency. For the terms and definitions not explicitly here, refer Harary [14].

II. MAIN RESULTS

In this paper, we introduce the concept of 4-Roman coloring of graphs and the related parameter, 4-Roman Chromatic Number.

Definition.2.1. 4- Roman coloring of a graph G is an assignment of four colors, namely {0, 1, 2, 3}, to the vertices of G such that (1) Every vertex with the color, 0 must be adjacent to two vertices, one of them is of color 2 and the other is of color 3 and (2) Every vertex with color 1 is adjacent to a vertex with color 2 or 3.

Weight of a 4-Roman coloring is defined as the sum of all the vertex colors. The 4- Roman Coloring number of a graph G is defined as the minimum weight of a Roman coloring on G and is denoted by $R_4(G)$. A 4-Roman coloring of G with the minimal weight is called a minimal 4-Roman coloring of G.

Prposition.2.2. If any 4-Roman coloring of G, a vertex of degree less than 2, then it cannot be colored with 0.
Proof. Let v be a vertex of degree less than 2. If this vertex, v is colored with 0, then it must be adjacent to two

vertices so that its degree is at least 2, which is a contradiction. Hence the result follows.

Prposition.2.3. In any minimal 4-Roman coloring of G, an isolated vertex must have the color 2.
Proof. From the definition of the 4-Roman coloring, it follows that 0 or 1 cannot be assigned to an isolated vertex. So it can be colored by 2 or 3. Since the coloring is a minimal 4-Roman coloring of G, it must be colored by 2, since the minimum of these colors is 2.

Proposition. 2.4. If there exist two vertices that are adjacent to all other vertices in G, then $R_4(G) = 5$

Proof. If two vertices u and v are adjacent to all other vertices in G, then assign color 2 and 3 to that adjacent vertices u and v and gave color 0 to all the remaining vertices so that $R_4(G) = 5$

Theorem 2.5. $R_4(P_n) = 4\left\lfloor\frac{n}{3}\right\rfloor, 4\left\lfloor\frac{n}{3}\right\rfloor + 1, 4\left\lfloor\frac{n}{3}\right\rfloor + 3$ according as $n = 3k, 3k+1, 3k+2$ respectively for some positive integer, k.
Proof. P_2 has a minimal 4-Roman coloring by assigning the colors 1 and 2, by Proposition 2.2. and $R_4(P_2) = 3$. For $n \geq 3$, let $V(P_n) = \{v_1, v_2, \dots v_n\}$. Then we cannot color $v_1$with the color 0, thus there are three cases.
Case.1: Color of v_1 is 1
Then the minimum possible color for the vertex v_2 is 2 and the minimum possible color for the vertex v_3 is 1, since if we color it by 0, next vertex should be colored by 3 whereas if it has the color 1, which is minimal so that the minimum possible sum of the colors these 3 vertices is 4. The resulting coloring is: (1, 2, 1).
Now, the same color pattern may be repeated for the next block of 3 vertices and so on.

Case.2: Color of v_1 is 2.
Then there are 2 cases for coloring the vertex v_2.If we color v_2 is 0, then minimum possible value to color v_3 is 2. If we color v_2 by 1, then minimum possible value to color v_3 is 1. Hence the minimum possible sum of the 3 vertices is 4. The resulting coloring is: (2, 0, 2) or (2, 1, 1)
Now, the same color pattern may be repeated for the next block of four vertices and so on.

Case.3: Color of v_1 is 3.
Then, there are two cases for coloring v_2. If we color v_2 by 0, then minimum possible color for v_3 is 2, If we color v_2 by 1, the minimum possible color for v_3 is 1, so that minimum possible sum of colors for these three vertices is 5. Then the minimum possible sum of these 3 vertices is 5. The resulting coloring is: (3, 0, 2) or (3, 1, 1).

Hence a minimal 4-Roman coloring can be obtained by case.1. That is, by assigning the colors (1, 2, 1) to each block of three consecutive vertices of P_n each. Then $R_4(P_n) = 4\left\lfloor\frac{n}{3}\right\rfloor, 4\left\lfloor\frac{n}{3}\right\rfloor + 1, 4\left\lfloor\frac{n}{3}\right\rfloor + 3$ according as $n = 3k, 3k+1, 3k+2$ respectively for some positive integer, k.

Theorem 2.6. For n ≥ 3, $R_4(C_n) = 4\left\lfloor\frac{n}{3}\right\rfloor, 4\left\lfloor\frac{n}{3}\right\rfloor + 1, 4\left\lfloor\frac{n}{3}\right\rfloor + 3$ according as $n = 3k, 3k+1, 3k+2$ respectively for some positive integer, k.
Proof. If n=4, the four vertices of C_n can be colored by (0, 2, 0, 3) in the order so that this is a minimal 4-Roman coloring and $R_4(C_4) = 5$. For $n \geq 5$, let $V(C_n) = \{v_1, v_2, \dots v_n\}$.
Then there are three cases.

Case.1: Color of v_1 is 1
Then the minimum possible color for the vertex v_2 is 2 and the minimum possible color for the vertex v_3 is 1, since if we color it by 0, next vertex should be colored by 3 whereas if it has the color 1, which is minimal so that the minimum possible sum of the colors these 3 vertices is 4. The resulting coloring is: (1, 2, 1).
Now, the same color pattern may be repeated for the next block of 3 vertices and so on.

Case.2: Color of v_1 is 2.
Then there are 2 cases for coloring the vertex v_2.If we color v_2 is 0, then minimum possible value to color v_3 is 2. If we color v_2 by 1, then minimum possible value to color v_3 is 1. Hence the minimum possible sum of the 3 vertices is 4. The resulting coloring is: (2, 0, 2) or (2, 1, 1)
Now, the same color pattern may be repeated for the next block of four vertices and so on.
Case.3: Color of v_1 is 3.
Then, there are two cases for coloring v_2. If we color v_2 by 0, then minimum possible color for v_3 is 2, If we color v_2 by 1, the minimum possible color for v_3 is 1, so that minimum possible sum of colors for these three vertices is 5. Then the minimum possible sum of these 3 vertices is 5. The resulting coloring is: (3, 0, 2) or (3, 1, 1).

Hence a minimal 4-Roman coloring can be obtained by case.1. That is, by assigning the colors (1, 2, 1) to each block of three consecutive vertices of C_n each. Then $R_4(C_n) = 4\left\lfloor\frac{n}{3}\right\rfloor, 4\left\lfloor\frac{n}{3}\right\rfloor + 1, 4\left\lfloor\frac{n}{3}\right\rfloor + 3$ according as $n = 3k, 3k+1, 3k+2$ respectively for some positive integer, k.

Theorem 2.7. $R_4(K_n) = \begin{cases} 5 & if\ n \geq 4 \\ n+1 & if\ n \leq 3 \end{cases}$

Proof. If v is an isolated vertex, we shall assign the color 2 to it so that it is a minimal 4-Roman coloring and $R_4(K_n) = 2 = n + 1$.So K_1 can be colored with 2. K_2 has a minimal 4-Roman coloring by assigning the colors 1 and 2, by Proposition 2.2. and $R_4(K_2) = 3 = n + 1$. K_3 has a minimal 4-Roman coloring by assigning the colors 1,2,1 and $R_4(K_3) = 4 = n + 1$. If n ≥ 4, then two vertices of K_n can be colored with 2, 3 and all other vertices are colored by 0, so that this is a minimal 4-Roman coloring and $R_4(K_n) = 5$.

Theorem 2.8. $R_4(K_{1,n}) = n + 2$

Proof. There are n vertices of degree, 1 in the star graph. These vertices cannot be colored by the color, 0 by Proposition.2.2. Thus the minimum possible color to these vertices is 1. The minimum possible color for the center of the star graph is 2. Thus, $R_4(K_{1,n}) = n + 2$.

Theorem 2.9. For a complete bipartite graph, $K_{m,n}$, $2 \leq m \leq n$

$$R_4(K_{1,n}) = \begin{cases} 5 & if\ m = 2 \\ 8 & if\ m = 3 \\ 9 & if\ m = 4 \\ 10 & if\ m \geq 5 \end{cases}$$

Proof. Let X, Y be the partitions of the graph $K_{m,n}$ with $|X| = m, |Y| = n$. We consider the four cases.

Case-1: m=2. Then we can assign the color 2, 3 to two vertices in X and the color, 0 to all the vertices in Y. Clearly, this is a minimal 4-Roman coloring and $R_4(K_{1,n}) = 5$.

Case-2: m=3. Then we can assign the color 1, 2, 3 to three vertices in X and the color, 2 to one vertex in Y and the color, 0 to all the other vertices in Y. Clearly, this is a minimal 4-Roman coloring and $R_4(K_{1,n}) = 8$.

Case-3: m=4. Then we can assign the color 1, 1, 2, 3 to four vertices in X and the color, 2 to one vertex in Y and the color, 0 to all the other vertices in Y. Clearly, this is a minimal 4-Roman coloring and $R_4(K_{1,n}) = 8$.

Case-4: m≥ 5. Then we can assign the color 2, 3 to two vertices in X and the color, 2, 3 to two vertices in Y and the color, 0 to all the other vertices in Y. Clearly, this is a minimal 4-Roman coloring and $R_4(K_{1,n}) = 10$. Hence the theorem follows.

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