G- Frame Operator in C* Algebra

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Abstract: - The g-frame operator for g-frames in C* algebra is introduced. The results on g-frame operators are proved. Frame identities are shown. Result on direct sum of g-frame operators on direct sum of Hilbert Spaces is presented.

1. Introduction

Frames are generalization of bases .D. Han and D. R. Larson have developed a number of basic concepts of operator theoretic approach to frame theory in C* algebra. Peter G Casazza presented a tutorial on frame theory and he suggested the major directions of research in frame theory. Radu V. Balan and Peter G. Casazza have analyzed decomposition of a normalized tight frame and obtained identities for frames. A. Najati and A. Rahimi have developed the generalized frame theory and introduced methods for generating g-frames of a C* algebra.

1.1 Banach Algebra:-A Banach Algebra is a complex Banach space A together with an associative and distributive multiplication such that

$\lambda(ab)= \lambda(a)\lambda(b)$

$\|ab\| \leq \|a\|\|b\|$ ∀a,b∈ A and $\lambda \in C$

Forany x,$x^1, y, y^1 \in A$wehave $\|xy - x^1y^1\| \leq$

$\|x\|\|y - y^1\|$+ $\|x - x^1\|$

The algebra A is said to be commutative ifab=ba$\forall a, b \in A$

1.2 Definition (Involution of an algebra):- Let A be a Banach algebra .An involution on A ~~is~~ amap*:A A suchthat

1. a^{**}=a

2.$(\lambda a + \mu b)^*$=$\bar{\lambda}$a*+$\bar{\mu}$b*

3.$(ab)^*$=b*a*

1.3 Definition :- (C* algebra) If A is a Banach algebra with involutionand also $\|aa*\|$= $\|a\|^2$then A is called a c*algebra.

Example:- C(X) let X be a compact space and C(X) is a Banach spaceof all complex valued functions on X with norm$\|f\|$= $sup_{x \in X}|f(x)|$ Multiplication on C(X) is defined as pointwiseief.g(x)=f(x)g(x) And involution by complex conjugationf*(x)=$\overline{f(x)}$

2. G- frame and g-frame operator

Throughoutthispaper{A_jj∈J}willdenote asequenceofC*algebras

LetL (A,A_j) be a collection of bounded linear operators fromAto A_j and { Δ_j∈L (A , A_j) ; j ∈J} we obtain some characterization of g-frame operator. They are the generalizations of results of frameoperator.

2.1 Definition: - A sequence of operators $\{\Delta_j\}_{j\in J}$is said to be g-frame for C* algebra A with respect to sequence of C* algebras { A_j, j ∈J}if there exists twoconstants0 $\leq$ A$<\infty$ B for any vector f ∈H,

A $\| \bar{f} \|^2 \leq \sum_{j\in J} \|\Delta_j \bar{f}\|^2 \leq$ B$\|\bar{f}\|^2$ where $\bar{f}$(x) = f* (x)

The above inequality is called a g-frame inequality. The numbers A, B are called the lower frame bound and upper frame bound respectively.

2.2 Definition: - A g-frame for $\{\Delta_j\}_{j\in J}$is said to be g-tight frame if A=B, then wehave

A$\|\bar{f}\|^2 = \sum_{j\in J} \|\Delta_j f\|^2$forallf*∈A

2.3 Definition: - A g-frame $\{\Delta_j\}_{j\in J}$for A is said to be a g-normalized tight frame for A if A = B = 1. Thenwehave $\|\bar{f}\|^2 = \sum_{j\in J}\|\Delta_j \bar{f}\|^2$forall f∈ H

2.4 Definition: - Let $\{\Delta_j\}_{j\in J}$be a g-frame for c* algebra.G-frame operator:AS^g → A is definedas

$S^g f = \sum_{j\in J} \Delta_j^* \Delta_j f^*$ for all f* ∈A

Byusing above definitions we have the followingtheorems.

Theorem:- If $S^g$is a g-frame operator, then wehave

1) $< S^g$ f,f $>$ = $\sum_{j\in J} \|\Delta_j f\|^2$for all f ∈A

2) S^g is a positiveoperator

3) S^g is a self-adjointoperator.

Proof:- $S^g$is a g-frame operatormeans

$$S^g f = \sum_{j\in J} \Delta_j^* \Delta_j f^* \text{forall } f^* \in A$$

1) $< S^g f,f > = < \sum_{j\in J} \Delta_j^* \Delta_j f^* f, f >$

$$= \sum < \Delta_j^* \Delta_j f, \Delta_j^* \Delta_j f >$$

$$= \sum \| \Delta_j f\|^2 [\because \quad < x, x > = \|x^2\|]$$

(2) Clearly $S^g$is a positive operator bydefinition

(3) It is left to thereader

2.6 Theorem: Suppose $\{\Delta_j\}_{j\in J}$is a g- frame if and only ifAI ≤ S^g ≤BIand $\{\Delta_j\}_{j\in J}$is a g-normalized tight frame if andonlyif S^g =I where I is an identity operator onA.

Proof: Since $\{\Delta_j\}_{j\in J}$is a g-frame so wehave

A$\|\bar{f}^2\| \leq \|\Delta_j \bar{f}\|^2 \leq$B$\|\bar{f}^2\|$forall $\bar{f} \in$A

Consider$\langle AI\bar{f},\bar{f}\rangle$=A$\langle \bar{f},\bar{f}\rangle$=A $\|\bar{f}^2\| \leq \sum_{j\in J}[\![\Delta_j \bar{f}]\!]^2 \leq B\|f\|^2$

=B$\langle \bar{f},\bar{f}\rangle$= $\langle BI\bar{f},\bar{f}\rangle$

Conversely supposeAI$\leq S^g \leq BI$

$\Rightarrow \langle AI\bar{f},\bar{f}\rangle \leq \langle S^g\bar{f},\bar{f}\rangle \leq \langle IB\bar{f},\bar{f}\rangle$for all$\bar{f} \in$A

$\Rightarrow$ A$\|\bar{f}^2\| \leq \sum_{j\in J}\|\Delta_j \bar{f}\|^2 \leq B\|\bar{f}\|^2$

Which implies $\{\Delta_j\}_{j\in J}$is a g-frame forA.

Suppose $\{\Delta_j\}_{j\in J}$is a g-normalized tight frame forA

$\Leftrightarrow \sum_{j\in J}\|\Delta_j \bar{f}\|^2$=$\|\bar{f}\|^2$forall $\bar{f} \in$A

If and only if $\langle s^g f,f\rangle$= $\langle If,f\rangle$

If and only ifS^g=I

Note: We can easily see thatframeoperator S^g is invertible and$S^{g^{-1}}$is a positiveoperator.

2.7 Theorem: A sequence of operators $\{\Delta_j\}_{j\in J}$where $\bar{\Delta}_j$= $\Delta_j S^{g^{-1}}$

is a G-frame for C* algebra with frame bounds 1/B and1/A

Proof: Consider$\sum_{j\in J}\|\Delta^*{}_j f\|^2$= $\sum_{j\in J}\|\bar{\Delta}_j f\|^2$

$$= \sum_{j\in J}\left\|\Delta_j S^{g^{-1}} f\right\|^2$$

$$= \sum_{j\in J}\left\langle \Delta_j^* \Delta_j S^{g^{-1}} f, S^{g^{-1}} f\right\rangle$$

$$= \left\langle \sum_{j\in J} \Delta_j^* \Delta_j\, S^{g^{-1}} f, S^{g^{-1}} f\right\rangle$$

$$= \langle S^g S^{g^{-1}} f, S^{g^{-1}} f\rangle$$

$$= \langle f, S^{g^{-1}} f\rangle$$

$$\leq \frac{1}{A}\|f\|^2$$

$$\Rightarrow \sum_{j\in J}\left\|\bar{\Delta}_j f\right\|^2 \leq \frac{1}{A}\|f\|^2$$

$$\|f\|^2 = \langle \sum_{j\in J} \bar{\Delta}_j^*\, \Delta_j f, f\rangle = \sum_{j\in J}\langle \Delta_j^* \Delta_j f, f\rangle$$

$$= \sum_{j\in J}\left\langle \Delta_j f\, \bar{\Delta}_j f\right\rangle$$

$$\leq \left[\sum_{j\in J}\left\|\Delta_j f\right\|^2\right]^{\frac{1}{2}} \left[\sum_{j\in J}\left\|\bar{\Delta}_j f\right\|^2\right]^{\frac{1}{2}}$$

$$\leq \sqrt{B}\,\|f\|\left[\sum_{j\in J}\left\|\bar{\Delta}_j f\right\|^2\right]^{1/2}$$

$\Rightarrow \|f\|^2 \leq \sqrt{B}\|f\|\left[\sum_{j\in J}\left\|\bar{\Delta}_j f\right\|^2\right]^{1/2}$

$\Rightarrow \frac{1}{B}\|f\|^2 \leq \sum_{j\in J}\left\|\bar{\Delta}_j f\right\|^2$

Hence $\Rightarrow \frac{1}{B}\|f\|^2\|f\|^2 \leq \sum_{j\in J}\left\|\bar{\Delta}_j f\right\|^2 \leq \frac{1}{A}$

Which shows that the sequence of operators $\{\bar{\Delta}_j\}_{j\in J}$is a g-frame for the C* algebra A with framebounds $\frac{1}{B}$ and $\frac{1}{A}$

2.8 Theorem: - Let $\{\Delta_j\}_{j\in J}$be a g-frame for C* algebra A with respectto $\{A_j$,j∈J} and V∈B(H)bean invertible operator. Then$\{\Delta_j V\}_{j\in J}$ is a G-frame for A with respect to to$\{A_j$,j∈J}and its g-frame operator isV^*S^g V.

Proof: - Since V∈B(H),∀f∈H, we have Vf ∈H given that $\{\Delta_j\}_{j\in J}$isa G-frame for H, so for all$V$f∈H we have ‖Vf‖^2≤$\sum_{j\in J}\|\Delta_j Vf\|^2$ ≤B ‖Vf‖^2

Since V is invertible operator, therefore we have

‖Vf‖^2 ≤‖V‖^2‖f‖^{2}and‖V^{-1}‖$^{-2}$‖f‖^2‖Vf‖^2 ≤

By above inequalities, the equation become A‖ V^{-1}‖$^{-2}$‖f‖^2$\sum_{j\in J}$≤‖$\Delta_j Vf$‖^2 ≤B‖V‖^2‖f‖^2,f∈H ∀

$\Rightarrow \{\Delta_j V\}_{j\in J}$ is a g-frame for A.

For each f∈A.We have S^gV f= $\sum_{j\in J} \Delta_j^*\Delta_j$

$\Longrightarrow$ V^*S^gVfVf =$\sum_{j\in J} V^* \Delta_j^*\Delta_j$Vf

$\Longrightarrow$ V^*S^gV is a g-frame operator for the frame $\{\Delta_j V\}_{j\in J}$

Frame Identities forg-frames

2.9 Proposition: - If$T_1$and $T_2$are two operators in a C* algebraA SatisfyingT_1+ T_2= I, Then T_1- T_2= T_1^2- T_2^2

Proof:-ConsiderT_1-T_2=T_1–(I -T_1)=T_1^2–(I- 2T_1+T_1^2)

= T_1^2– (I - T_1)^2

= T_1^2- T_2^2

Theorem: - Let $\{\Delta_j\}_{j\in J}$be a g-normalize d tight frame for A for ICJ,then

$$\sum_{j\in J} \|\Delta_j f\|^2 = \|\sum_{j\in J} \Delta_j^*\Delta_j\|^2 S_I^g S_{I^c}^g \Leftrightarrow =0$$

Proof:-Consider$\sum_{j\in J} \|\Delta_j f\|^2 = \|\sum_{j\in J} \Delta_j^*\Delta_j\|^2 \sum_{j\in J} \|\Delta_j f\|^2$

- $\| \sum_{j\in J} \Delta_j^*\Delta_j\|^2$=0

$\Leftrightarrow \sum_{j\in J} \| \Delta_j f\|^2$- $< \sum_{j\in J} \Delta_j^* \Delta_j f ,\sum_{j\in J} \Delta_j^*\Delta_j f >$=0

$\Leftrightarrow < S_I^g f$,f $>$ - $<$ $S_I^g f, S_I^g f >$=0

$\Leftrightarrow < (S_I^g$- $S_I^g)^2$f, f $>$ =0

$\Leftrightarrow$ S_I^g(I - S_I^g) f =0

$\Leftrightarrow$ $S_I^g = S_{I^c}^g$ for allf∈H

2.10 Theorem: - Let $\{\Delta_j\}_{j\in J}$be a g-normalized tight frame for H, for ICJ with respect to[H_j,j∈J].Then for ICJ and for all f∈H.

$\sum_{j\in J} ||\Delta_j f||^2 + \|S_{I^c}^g f\|^2 = \sum_{j\in J} ||\Delta_j f||^2 + \|S_I^g f\ ||^2$

Proof: - Science $\{\Delta_j\}_{j\in J}$is a g-normalized tight frame for H,

Therefore S^g = I and $S_I^g + S_{I^c}^g$= I, for all f∈H.

Consider $\sum_{j\in J} ||\Delta_j f\,||^2 + \| S_{I^c}^g f||^2 = \quad < S_I^g f, f > + < S_{I^c}^g f, S_{I^c}^g f >$

$= < (S_I^g + S_{I^c}^g)^2 f, f > = < (S_I^g + (I - S_I^g)^2 f, f >$

$= < (I - S_I^g + S_I^g)^2 f, f >$

$= < (S_{I^c}^g + S_I^g)^2 f, f >$

$= < \quad f, f \quad + \quad f, S_I^g\ f >$

$= \sum_{j\in I^c} ||\Delta_j\ f\,||^2 \ + ||S_I^g f\,||^2$

References

[1] M.R.Abdollahpour ,M.H. Faroughi and A.Rahimi ,"PG-Frames in Banach spaces" Methods of Functional analysis and topology Vol.13NO.3(2007),pp.201-210

[2] R.BalanP.G.Casazza, D.Edidin and G.Kutynoik " Decomopostion of frames and anew frame identity " Wavelet XI (San Diego),CA(2005) pp.379-388 SPIE Proc 5914 ,SPIE BellinginamWA.

[3] P.G.Casazza,The art of Frame theory Taiwanese Journal of Mathematics Vol.4,No.2(2000) pp129-201

[4] D.Han and D.R.Larson "Frames Bases and Group Representations" Memories Ams Nov 7 (2000) ProvidenceRI

[5] A.Najati and A.Rahimi "Generalized frames in Hilbert spaces" Bulletin of the Iranian Mathematical society Vol.35,No.1(2009)pp.97-109.

[6] A.Najati and M.H.Faroughi "P- frames of subspaces of separable Hilbert spacesSouth east AsainBulletion of Mathematics31(2007)pp.713-726.

[7] G. Upender Reddy and N.GopalReddy ,Some results of Frame operator in Hilbert space, Journal of Mathematical education ,Volume XLV,No.3 September2011.