History, Development And Application of Pseudocontractive Mapping With Fixed Point Theory

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ABSTRACT: *In this Paper, Brief history, origin and relevant role of Pseudocontractive mapping and Fixed point theory in Mathematics with the help of some definitions, theorems and types of this theory is presented. This article is to make accessible material that might be of interest to students and research scholars. Some important results from beginning up to now are incorporated in this paper. The theory of fixed point with Pseudo contractive mapping is one of the most powerful tool of modern mathematical analysis. Theorem concerning the existence and properties of fixed points are known as fixed point theorem. Fixed point theory with Pseudocontractive is a beautiful mixture of analysis, topology & geometry which has many applications in various fields such as mathematics engineering, physics, economics, game theory, biology, chemistry, optimization theory and approximation theory etc. Fixed point theory has its own importance and developed tremendously for the last one and half century. The purpose of the present paper is to study the history, development and application of Pseudocontractive mapping with fixed point theory.*

KEYWORDS: *Fixed Point, Banach Space, Contraction mapping, Pseudocontractive mapping.*

PRILIMINARIES: This section is devoted to some notation and basic definitions which are needed for the further study of this paper.
Throughout this paper we suppose that X is a real Banach space and that X^* is its topological dual. We use B_r to denote the closed ball centered at $x_0 \in X$ with radius $r > 0$.
Definition 1.1. Let X be a real Banach space and D be non empty subset of X then a mapping $T:D\rightarrow X$ is said to be pseudocontractive if for $r > 0$ and $x, y \in D$
$\|x-y\| \leq \|(1+r)(x-y) - r(T(x) - T(y))\|$
Definition.1. 2. A normed space X is called a Banach space if it is complete, i.e. if every Cauchy sequence in X is convergent.
Definition. 1.3. Let X be a Complete Metric Space.Then a map $T: X \rightarrow X$ is called a contraction mapping on X if there exists $k\in(0,1)$ such that $d(T(x),T(y))\leq k\, d(x\ y)$ for all x and y in X.
Definition. 1.4. Let X be a Banach space, D a subset of X.Then a mapping
$T:D\rightarrow X$ is said to be non expansive if for all $x,y\in D$
$\|T(x)-T(x)\| \leq\|x-y\|$
Definition. 1.5. A mapping $T: K\rightarrow E$ is said to be accretive if for all for all $x,y\in K$ and $t > 0$,
$\|x - y\|\leq\|x - y + (Tx - Ty)\|$ holds.

I. INTRODUCTION: Most important nonlinear problems of applied mathematics reduce to finding solutions of nonlinear functional equations (e.g. nonlinear integral equations, boundary value problems for nonlinear ordinary or partial differential equations, the existence of periodic solutions of nonlinear partial differential equations) It can be formulated in terms of finding the fixed points of a given nonlinear mapping on an infinite dimensional function space X into itself. Fixed point theorems give the conditions under which certain equation involving mappings (single or multivalued) have solutions. The theory itself is a beautiful mixture of analysis (pure and applied), topology and geometry. Over the last 50 years or so the theory of fixed points has been revealed as a very powerful and important tool in the study of nonlinear phenomena. In particular, fixed point techniques have been applied in such diverse fields as biology, chemistry, economics, engineering, game theory, physics, rocks waves in gases, movement of viscous fluids, chemical reactions, steady state tempreture distribution, Neutron transport theory, variational inequalities, economics theories, epidemics, complementary problems, optimal control, heat radiation, nonlinear oscillation in biology, etc. include such nonlinear problems which can be reduced in the form of nonlinear differential, integral and partial differential equations. Fixed Point Theory was motivated by the desire to study the existence and properties of boundary value problems for nonlinear partial differential equations. Its most classical tool in nonlinear functional analysis was developed in

the theory of compact nonlinear mappings in Banach spaces of the late 1920 and 1930s. The central role of compact mappings in this phase of development of nonlinear analysis has then given rise to early new theories for nonlinear noncompact operators because of their significant variety of applications. Such operators are namely (i) Monotone operators, (ii) Nonexpansive operators, (iii) Accretive operators and
(iv) Pseudocontractive operators.

II. HISTORY AND DEVELOPMENT : The definition of monotone operator intimately related to pseudocontractive mapping was first given by Kachurovski [10] and iterative methods for strongly monotone operators in Hilbert space satisfying a Lipschitz condition were first given by Zarantonello [11] . The first surjectivity theorem for monotone operators was given by Minty [12]._Later, Surjectivity results proved for different type of monotone operators were applied to get existence and uniqueness results for corresponding operator equations. This Theory is now widely developed and has found useful applications in the investigation of the solvability of nonlinear operator equations and in particular of partial differential equations and integral equations. The notion of monotone operators could be considered as a generalization of the concept of accretive operators in the Hilbert space H i.e. $F : H \to H$ is monotone if and only if $(I + \lambda F)$ is accretive for every). $\lambda > 0$. The accretive operators were introduced independently in 1967 by Browder [14] and Kato [8]. Interest in such mappings developed mainly due to their connection with equations of evolution. It is well known that many physically significant problems can be modeled by initial value problems of the form

$$x'(t) + Ax(t) = 0, \quad x(0) = Xo, \qquad (1.1.1)$$

where A is an accretive operator in an appropriate Banach space. Typical examples where such evolutions occur can be found in the heat, wave or schrodinger equations.The solutions of the equation $Ax = 0$ are precisely the equilibrium points of the system (1.1.1)

The class of nonexpansive mappings in Banach spaces has very interesting links with the theory of monotone and accretive mappings. The existence of fixed points of nonexpansive mapping was independently given by Browder [17], Gohde [7] and Kirk [13] in 1965. According to them nonexpansive, accretive and monotone mappings are related to each other as follows:

If T is a nonexpansive mapping, then U : I-T is monotone in Hilbert space from any subset D of H into H and accretive operator in Banach space into itself. But converse of this is not valid i.e. if U is monotone or accretive operator then T = I -U is not nonexpansive. This is the reason why pseudocontractive operator is introduced.

The class of pseudocontractive operator is introduced by Browder and Petryshyn [16] in 1967 in Hilbert space and proved that U is pseudocontractive operator if and only if T = I - U is monotone operator. They proved the existence results and then convergence results for this class of mappings in Hilbert space using Krasnoselskij [7] iteration. In the same year, Browder [14] alone gave the existence of fixed points of pseudocontractive mapping in real uniformly convex Banach space or real Banach space with uniform structure and proved tht the class of pseudocontractive operators includes the important class of nonexpansive operators and also showed that T is pseudocontractive if and only if A = I - T is accretive. Fixed point theorems for pseudocontractive mappings play an important role in the theory of nonlinear mappings because of their connection with the accretive operators (see Kirk and Schoneberg [9]). Browder [14] and Kato [88], independently of each other, characterized pseudocontractive mappings as those mappings T for which the mapping A = I - T is accretive. Consequently, considerable efforts have been devoted to the methods of approximating equilibrium points (when they exist) of the initial value problems (1.1.1). Since a map T is pseudocontractive if and only if A = I-T is accretive so that solution of $Ax = 0$ for accretive operator A corresponds to fixed point of T.

The existence of a fixed point is often useless in applications without an algorithm for calculating its value. Hence from the point of view of application, it is essential not only to show the existence of fixed points of such mappings under suitable hypothesis but also to develop systematic techniques for the construction or calculation of such points. This procedure or technique is called an iteration. Iterative procedures are used in nearly every branch of applied mathematics, Since the time of Archimedes, or perhaps even earlier, mathematicians have been studying approximation methods to estimate solutions of diverse types of equations. In fact, Archimedes introduced the method of exhaustion to estimate the length of a circumference. About 400 hundred years ago, Johannes Kepler encountered the need to approximate a fixed point for a function of the form a.sin(x) + b while he was investigating the orbits of the planets. Indeed, he found a rather delicate iteration method to solve this problem. Some 40 years later, Sir Isaac Newton proposed a powerful alternative method to approximate zeros using the derivative. This method was later systematized by Joseph Raphson. By 1890, the Picard method of successive approximation was introduced to prove the existence of fixed points for integral operators.

III. SOME FIXED POINT RESULTS

1. Let X be a Banach space with the (FPP). Suppose that $G:X\times X\to Z$ is a mapping satisfying conditions.

(g_1) $G(\lambda x,y) = \lambda G(x,y)$ for any $x,y\in X$ and $\lambda > 0$

(g_2) $\|x\|^2 \leq G(x,y)$ for any $x\in X$

Let C be a closed convex and unbounded subset of X with $x_0 \in C$. Let $T:C \to X$ be a continuous pseudocontractive mapping. Assume that the following conditions are satisfied. (a)T is weakly inward on C.(b)There exists $R > 0$ such that for every $x \in C$ with $\|x\| \geq R$ the inequality $G(T(x),x) \leq G(x,x)$ holds. Then T has a fixed point in C.

2. Let X be a Banach space with the Fixed point property. Suppose that $G:X \times X \to Z$ is a mapping satisfying conditions (g_1) and (g_2). Let C be a closed convex and unbounded subset of X with $x_0 \in C$ If $T:C \to X$ is a continuous pseudocontractive mapping weakly inward on C and

$$\lim_{x \in C \|x\| \to \infty} \frac{G(T(x)-x_0,x)}{G(x,x)} < 1$$

then T has a fixed point in C.

3. Let X be a Banach space with the Fixed point property. Suppose that $G:X \times X \to Z$ is a mapping satisfying conditions.

(g_1) $G(\lambda x,y) = \lambda G(x,y)$ for any $x,y \in X$ and $\lambda > 0$

(g_2) $\|x\|^2 \leq G(x,y)$ for any $x \in X$

(g_3) $G(x+y, z) \leq G(x,z) + G(y,z)$

(g_4) For each $y \in X$ there exist $t > 0$ (depends on y) such that $\|x\| \geq t$ then $|G(y,x)| < G(x,x)$

Let C be a closed convex and unbounded subset of X. If $T:C \to X$ is a continuous pseudocontractive mapping weakly inward on C and satisfies condition $G(T(x),x) \leq G(x,x)$. Then T has a fixed point in C.

4. Let X be a reflexive Banach space. Suppose that that $G:X \times X \to Z$ satisfies conditions.

(g_1) $G(\lambda x,y) = \lambda G(x,y)$ for any $x,y \in X$ and $\lambda > 0$

(g_2) $\|x\|^2 \leq G(x,y)$ for any $x \in X$

(g_3) $G(x+y, z) \leq G(x,z) + G(y,z)$

(g_4) For each $y \in X$ there exist $t > 0$ (depends on y) such that $\|x\| \geq t$ then $|G(y,x)| < G(x,x)$

Let $C \subseteq X$be a nonempty unbounded closed convex set. If $T:C \to X$ is a nonexpansive mapping such that $T(C) \subseteq C$,I - T is demiclosed and

$$\lim_{x \in C \|x\| \to \infty} \frac{G(T(x)-x_0,x)}{G(x,x)} < 1$$

for some $x_0 \in C$, then T has a fixed point in C.

5. Let X be a Banach space with the Fixed point property. Let C be a closed convex and unbounded subset of X. If $T:C \to X$ is a continuous pseudocontractive mapping weakly inward on C and for every $x \in C$ and $\|x\|$ large enough and $\|Tx - x_0\| \leq \| x - x_0 \|$ for some $x_0 \in X$, then T has a fixed point in C.

6. Let X be a Banach space with the Fixed Point Property. Let C be a closed convex and unbounded subset of X such that $x_0 \in C$. Let $T:C \to X$ be a continuous pseudocontractive mapping. Assume that the following conditions are satisfied.

(a) T is weakly inward on C.

(b) There exists R>0 such that for every $x \in C \setminus B_R$ and for every $\lambda > 1$,$T(x) \neq \lambda x$.

Then T has a fixed point in C**.**

IV. APPLICATIONS: There are so many applications of Pseudocontractive mapping with fixed point fixed point theory. Some of the applications are as follows:

1. Integral equations: These equations occur in applied mathematics, engineering and mathematical physics. They also arise as representation formulas in the solution of differential equations.

2. The Method of Successive Approximations: This method is very useful in determining solutions of integral, differential and algebraic equations.

3. Chemistry: We consider the mathematical model for an adiabatic tubular chemical reactor which processes an irreversible exothermic chemical reaction. For steady-state solutions, the model can be reduced to the ordinary differential equation

$u'' - \lambda u' + F(\lambda, u, \beta, u) = 0$ $u'' - \lambda u' + F(\lambda, u, \beta, u) = 0$

with boundary conditions

$u'(0) = \lambda u(0), u'(1) = 0$ $u'(0) = \lambda u(0), u'(1) = 0$

where

$F(\lambda, \mu, \beta, u) = \lambda \mu(\beta - u)\exp(u)$ $F(\lambda, \mu, \beta, u) = \lambda \mu(\beta - u)\exp(u)$

(The unknown u represents the steady-state temperature of the reaction, and the parameters λ, μ and β represent the Peclet number, the Damkohler number and the dimensionless adiabatic temperature rise respectively. This problem has been studied by various Authors who have demonstrated numerically the existence of solutions (sometimes multiple solutions) for particular parameter ranges.

4 Economics : In [2], Z.D. Mitrovic has used some results by S. Park [3] and derived a sufficient condition for existence of an equilibrium point in the economic model of supply and demand for finite dimensional topological vector space.

5 Game theory: We consider a game with $n \geq 2$ players, under the assumption that the players do not cooperate among themselves. Each players pursue a strategy, in dependence of the strategies of the other players. Denote the set of all possible strategies of the k^{th} player by k_k, and set $K = k_1 \times k_2 \ldots\ldots \times k_n$. An element $x \in K$ is called a strategy profile. For each k

let f_k: $K \to R$ be the loss function of the k^{th} player. If

$$\sum_{k=1}^{n} f_k(x) = 0, \qquad \forall\, x \in K$$

the game is said to be of zero-sum. The aim of each player is to minimize his loss, or, equivalently, to maximize his gain.

CONCLUSION : The contraction mapping is a distance diminishing mapping which is always continuous. Strikingly, pseudocontractive mapping is although distance diminishing mapping but it is not continuous that is, why fixed point does not exists for pseudocontractive mappings in arbitrary Banach space. Because, in Hilbert space, Lipschitzian or dernicontinuous condition is required and in uniformly convex Banach space or in Banach space with uniform structure, continuous condition is required. Thus, to get fixed point in arbitrary Banach space for pseudocontractive mappings more stronger conditions like Larey-Schauder or weakly inwardness conditions are used. Hence, it is still open question.

Finding fixed points of nonlinear mappings especially, nonexpansive mappings has received vast investigations due to its extensive applications in a variety of applied areas of inverse problem, partial differential equations, image recovery and signal processing. It is well known that pseudocontractive mappings have more powerful applications than nonexpansive mappings in solving different problems. In this paper, we devote to construct the methods for computing the fixed points of pseudocontractive mappings.

REFERENCES

[1] F. E. Browder, Nonexpansive nonlinear operators in a Banach space, Proc. Nat. Acad. Sci, U.S.A., 53 (1965), 1272-1276.
[2] Z.D. Mitrovic An application of a generalized KKM principle on the existence of an equilibrium point, Ser. Mat. 12 (2001), 64 - 67.
[3] S. Park, Generalizations of Ky Fan's matching theorems and their applications, J. Math. Anal. Appl. 41 (1989), 164 - 176.
[4] R. E. Bruck, Jr. A common fixed point theorm for a commuting family of nonexpansive mappings, Pacific J. Math., 53 (1974), 59-71.
[5] Gohde, D: Zum Prinzip der kontraktive Abbidung. Math.nachr 30,251-258(1965)
[6] Fan, Ky, Extensions of two fixed point theorems of F. E. Browder, Math. Z. 112 (1969), 234 - 240.
[7] M. A. Krasnoselskii, Two observations about the method of successive approximations,Usp.Mat.Nauk 10, 123-127(1955)
[8] T. Kato, Nonlinear semigroups and evolution equations, J. Math. Soc Japan, 19 (1967), 508-520.
[9] W. A. Kirk and R. Schonberg, Zeros of m-accretive operators in Banach spaces.Israel J. Mulh. 35 (1980). l-8.
[10] R. I.Kachurovsky. On monotone operators and convex functionals. Uspekhi Mat. 15 (1960), 213-215.
[11] E. H.Zarantonello. The closure of the numerical range contains the spectrum. Bull. Amer. Math. Sac. 70 (1964), 771-778.
[12] G. J.Minty. On nonlinear integral equations of the Hammerstein type. In "Nonlinear Integral Equations," pp. 99-154. Univ. of Wise. Press, Madison Wise., 1964.
[13] WA Kirk, Fixed point theorems for nonexpansive mappings satisfying certain boundary conditions. Proceedings of the American Mathematical Society 1975, 50: 143–149. 10.1090/S0002-9939-1975-0380527-7
[14] F.E Browder, Nonlinear mappings of nonexpansive and accretive type in Banach spaces, Bull. Amer. Math. Soc. 73 (1967), 875–882.
[15] T. Kato, Nonlinear semigroups and evolution equations, J. Math. Soc Japan, 19 (1967), 508-520.
[16] W. A. Kirk, A fixed point theorem for mappings which do not increase distances, Amer. Math. Monthly, 72 (1965), 1004-1006.
[17] F.E. Browder, W.V. Petryshyn, Construction of fixed points of nonlinear mappings in Hilbert space, J. Math. Anal. Appl. 20 (1967) 197–228