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On the Global Asymptotic Stability of a Fourth-Order Rational Difference Equation

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Abstract

In this work, we investigate the global stability of the following fourth- order rational difference equation

$$x_{n+1} = \frac{x_{n-1}^b x_{n-2}^b x_{n-3} + x_{n-1}^b + x_{n-3} + x_{n-2}^b + a}{x_{n-1}^b x_{n-2}^b + x_{n-1}^b x_{n-3} + x_{n-2}^b x_{n-3} + 1 + a}, \qquad (1)$$

where $a, b \in [0, \infty)$ and the initial values $x_{-3}, x_{-2}, x_{-1}, x_0 \in (0, \infty)$.

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1 Introduction

Recently there been a great interest in studying the qualitative properties of rational difference equations for the systematical studies of rational and nonrational difference equations, one can refer to the monographs $[1, 2]$ and the papers $[3 - 10]$ and references therein.

Rational difference equations can be look very simple, but in fact their global behaviors are mostly very complicated. Li and Zhu [1] are obtained a sufficient condition to quarente the global asymptotic stability of the following recursive sequence.

$$x_{n+1} = \frac{x_n x_{n-1}^b + x_{n-2}^b + a}{x_{n-1}^b + x_n x_{n-2}^b + a}, n = 0, 1, 2, 3... \qquad (2)$$

Li[2] use a new method to investigate the qualitative properties of the following rational differ

$$x_{n+1} = \frac{x_n x_{n-1} x_{n-3} + x_n + x_{n-1} + x_{n-3} + a}{x_n x_{n-1} + x_{n-1} x_{n-3} + x_n x_{n-3} + 1 + a} \qquad (3)$$

By the method of Li.V.V.Khuong[21,22,23] investigated the global behavior of the following fourth-order rational difference equation.

$$x_{n+1}=\frac{x_nx_{n-1}^bx_{n-3}+x_n+x_{n-1}^b+x_{n-3}+a}{x_nx_{n-1}^b+x_{n-1}^bx_{n-3}+x_nx_{n-3}+1+a},n=0,1,2,... \tag{4}$$

To be motivated by the above studies, in this paper, we consider the following nonlinear differece equation

$$x_{n+1}=\frac{x_{n-1}^bx_{n-2}^bx_{n-3}+x_{n-1}^b+x_{n-2}^b+x_{n-3}+a}{x_{n-1}^bx_{n-2}^b+x_{n-2}^bx_{n-3}+x_{n-1}^bx_{n-3}+1+a},n=0,1,2,3... \tag{5}$$

Where $a,b\in[0,\infty)$ and the inital values $x_{-3},x_{-2},x_{-1},x_0\in(0,\infty)$
We review some results which will be useful in our investigation.

Definition 1.1. *Let $I\subset R$ and $f:I^{k+1}\to I$ be a continously differentiable, then for every set of intital conditions $x_{-k},x_{-k+1},...,x_{-3},x_{-2},x_{-1},x_0\in I$ then the difference equation.*

$$x_{n+1}=f(x_n,x_{n-1},...,x_{n-k}),n=0,1,2,3,... \tag{6}$$

Has an unique solution $\{x_n\}_{n=-k}^{\infty}$ A point $\overline{x}\in I$ is called an equilibrium point of equation (6) if
$\overline{x}=f(\overline{x},\overline{x},...,\overline{x})$

Definition 1.2. *let $\overline{x}$ be the equilibrium point of the Eq. (6)*
(i) The equilibrium point $\overline{x}$ of Eq.(6) is locally stable if for every $\varepsilon>0$ there exists $\delta>0$ such that for all
$x_{-k},x_{-k+1},...,x_0\in I$, with

$$|x_{-k}-\overline{x}|+|x_{-k+1}-\overline{x}|+...+|x_0-\overline{x}|<\delta$$

We have

$$|x_n-\overline{x}|<\varepsilon \tag{7}$$

for all $n\geqslant-k$ (ii) The equilibrium point $\overline{x}$ of Eq.(6) is called a global attractor if for every $x_{-k},x_{-k+1},...,x_0\in I$, we have

$$\lim_{x\to\infty}x_n=\overline{x} \tag{8}$$

(iii) The equilibrium point $\overline{x}\in Eq.(6)$ is called a global asymptotically stable of it is locally stable and a global attractor.

Definition 1.3. *A positive semicycle of a solution* $\{x_n\}_{n=-k}^{\infty}$ *consists of a "string" of terms* $\{x_l, x_{l+1}, ..., x_m\}$*, all greater than or equal to the equilibrium* $\overline{x}$*, with* $l \geqslant -3$ *and* $m \leqslant \infty$ *and such that.*

Either $l = -3$*, or* $l > -3$ *and* $x_{l-1} < \overline{x}$*.*
and either $m = \infty$*, or* $m < \infty$ *and* $x_{m+1} < \overline{x}$*.*

A negative semycycle of a solution $\{x_n\}$ *consists of a "string" of terms* $\{x_l, x_{l+1}, x_{l+2}, ..., x_m\}$ *all less than to* $\overline{x}$*, with* $l \geqslant -3$ *and* $m \leqslant \infty$ *and such that. Either* $l = -3$ *or* $l > -3$ *and* $x_{l-1} \geqslant \overline{x}$

and either $m = \infty$ *or* $m < \infty$ *and* $x_{m+1} \geqslant \overline{x}$ *The lengh of a semicycle is the number of the total terms contained in it.*

2 Several Lemmas

It is easy to see that the positive equilibrium point $\overline{x}$ of Eq(1) satisfies

$$\overline{x} = \frac{\overline{x}^{1+2b} + \overline{x} + 2\overline{x}^b + a}{2\overline{x}^{1+b} + \overline{x}^{2b} + 1 + a} \tag{9}$$

From which one can see that Eq(9) has an unique positve equilibrium $\overline{x} = 1$

Lemma 2.1. *A positive solution* $\{x_n\}_{n=-3}^{\infty}$ *of Eq.(1) is eventually equal to 1 if and only if*

$$(x_{-2} - 1)(x_{-3} - 1)(x_{-1} - 1) = 0 \tag{10}$$

Proof. Assume the (10) holds. Then according to Eq.(1), it is easy to see that $x_n = 1$ *for* $n \geqslant 2$*.*
Conversely, assume that

$$(x_{-2} - 1)(x_{-3} - 1)(x_{-1} - 1)(x_1 - 1) \neq 0 \tag{11}$$

Then one can show that $x_n \neq 1$ *for any* $n \geqslant 2$
Assume the contrary that for some $N \geqslant 2$
$x_n = 1$ *and that* $x_n \neq 1$ *for* $-3 \leqslant n \leqslant N - 1$
It is easy to see that

$$1 = x_N = \frac{x_{N-2}^b x_{N-3}^b x_{N-4} + x_{N-2}^b + x_{N-3}^b + x_{N-4} + a}{x_{N-2}^b x_{N-3}^b + x_{N-3}^b x_{N-4} + x_{N-2}^b x_{N-4} + 1 + a}$$

Which implies $(x_{N-4} - 1)(x_{N-3}^b - 1)(x_{N-2}^b - 1) = 0$*. Obviously, this contradicts (10).*

Remark 2.1 If the initial anditons do not satisfy (10), then, the for any solution $\{x_n\}_{n=-3}^{\infty}$ *of Eq.(1),* $x_n \neq 1$ *for* $n \geqslant -3$.
Here, the solution is a nontrivial one.
Definition 1.4 A solution $\{x_n\}_{n=-3}^{\infty}$ *of Eq.(6) is said to be eventually trivial if* x_n *eventually equal to* $\overline{x} = 1$*; otherwise the solution is said to ve nontrivial.*

Lemma 2.2. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be an nontrivial positive solution of Eq.(1). Then the following conclusions are true for* $n \geqslant 0$.
a) $(x_{n+1}-1)(x_{n-1}^b-1)(x_{n-2}^b-1)(x_{n-3}-1) > 0$
b) $(x_{n+1}-x_{n-1}^b)(x_{n-1}^b-1) < 0$
c) $(x_{n+1}-x_{x-2}^b)(x_{n-2}^b-1) < 0$
d) $(x_{n+1}-x_{n-3})(x_{n-3}-1) < 0$
Proof, It follows in light of Eq.(1) that
$x_{n+1}-1 = \frac{(x_{n-1}^b-1)(x_{n-2}^b-1)(x_{n-3}-1)}{x_{n-1}^b x_{n-2}^b + x_{n-2}^b x_{n-3} + x_{n-1}^b x_{n-3} + 1 + a}$, n=0,1,2,3,...
and
$x_{n+1}-x_{n-2}^b = \frac{(1-x_{n-2}^b)[x_{n-1}^b(1+x_{n-2}^b)+x_{n-3}(1+x_{n-2}^b)]+a}{x_{n-1}^b x_{n-2}^b + x_{n-2}^b x_{n-3} + x_{n-1}^b x_{n-3} + 1 + a}$, n=1,2,3,...
and
$x_{n+1}-x_{n-1}^b = \frac{(1-x_{n-1}^b)[x_{n-2}^b(1+x_{n-1}^b)+x_{n-3}(1+x_{n-1}^b)+a]}{x_{n-1}^b x_{n-2}^b + x_{n-2}^b x_{n-3} + x_{n-1}^b x_{x-3} + 1 + a}$, n= 0,1,2,3,...
and
$x_{n+1}-x_{n-3} = \frac{(1+x_{n-3})[x_{n-1}^b(1+x_{n-3})+x_{n-2}^b(1+x_{n-3})+a]}{x_{n-1}^b x_{n-2}^b + x_{n-2}^b x_{n-3} + x_{n-1}^b x_{n-3} + 1 + a}$, n=0,1,2,3...

3 Main results

First we analyse the structure of the semicycles of nontrivial solution of Eq.(1). Here. we confine us to consider the situation of the strietly oscillatory of Eq.(1). From [6,7], we have the following theorem.

Theorem 3.1. *Let* $\{x_n\}_{n=-3}^{\infty}$ *be a strietly oscillatory of Eq.(1). Then the "rule for the trajectory structure" of nontrivial solution of Eq.(1) is*
or $..., 3^-, 1^+, 1^-, 2^+, 3^-, 1^+, 1^-, 2^+, ...$
or $..., 3^+, 1^-, 1^+, 2^-, 3^+, 1^-, 1^+, 2^-, ...$

Theorem 3.2. *Assume* $a, b \in [0, \infty)$. *Then the positive equilibrium of Eq.(1). is globally asymptotically stable and globally attractor.*
Proof: The linearized equation of Eq.(1). about the positive equilibrium point $\overline{x} = 1$ *is.*
$y_{n+1} = 0y_n + 0y_{n+1} + 0y_{n-3}$, n=0,1,2,3,...

by virtue of ([2], remark 1.2.7) $\overline{x}$ is locally asymptotically stable.
It remains to verify the every positive solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1) converges to $\overline{x}$ as $n \to \infty$. Namely, we want to prove

$$\lim_{n\to\infty} x_n = \overline{x} = 1 \tag{12}$$

If the intial values of the solution satisfy (10), then Lemma 2.1 says the solution is eventually equal to 1 and, of course (11) holds. Therefore, we assume in the following that the initial values of the solution do not satisfy (10). Then, Remark 2.1 we know, for any solution $\{x_n\}_{n=-3}^{\infty}$ of Eq.(1), $x_n \neq 1$ for $n \geqslant -3$

If the solution is nonoscillatory about the positive equilibrium point $\overline{x} = 1$ of Eq.(1)., then we know from Lemma 2.2 (a) that the solution is actually an eventually positive one. According to Lemma 22(b), we see that $\{x_{2n}\}$, $\{x_{2n-1}\}$, are eventually decreasing and bounded from below by 1. So, the lemits $\lim\limits_{n\to\infty} x_{2n} = L$, $\lim\limits_{n\to\infty} x_{2n-1} = M$ exist and are finite. Note

$x_{2n+1} = \frac{x_{2n-1}^b x_{2n-2}^b x_{2n-3} + x_{2n-1}^b + x_{2n-2}^b + x_{2n-3} + a}{x_{2n-1}^b x_{2n-2}^b + x_{2n-3} x_{2n-2}^b + x_{2n-1}^b x_{2n-3} + 1 + a}$, n=1,2,3,...

$x_{2n+2} = \frac{x_{2n}^b x_{2n-1}^b x_{2n-2} + x_{2n}^b + x_{2n-1}^b + x_{2n-2} + a}{x_{2n}^b x_{2n-1}^b + x_{2n-2} x_{2n-1}^b + x_{2n}^b x_{2n-2} + 1 + a}$, n=0,1,2,3,...

Take the limits on both sides of the above equations, we obtain.

$M = \frac{M^{1+b}L^b + M + L^b + M^b + a}{ML^b + L^b M^b + M^{1+b} + 1 + a}$

$L = \frac{M^b L^{b+1} + L + M^b + L^b + a}{LM^b + L^b M^b + L^{1+b} + 1 + a}$

We have M=L=1, which shows (12) is true. Thus, it suffices to prove that (12) hold for the solution to be the trictly oscillatory.
Assume now $\{x_n\}_{n=-3}^{\infty}$ to be strictly oscillatory solution about the positive equilibrium point $\overline{x} = 1$ of Eq.(1).
By virtue of Theorem (3.1), one understands that the lengths of positive and nagative semicycles which occur successively is
or $..., 3^-, 1^+, 1^-, 2^+, 3^-, 1^+, 1^-, 2^+, ...$
or $..., 3^+, 1^-, 1^+, 2^-, 3^+, 1^-, 1^+, 2^-, ...$

First, we investigate the case where the rule for the lengths of positive and negative semicycles which occur successively is $..., 3^-, 1^+, 1^-, 2^+, 3^-, 1^+, 1^-, 2^+, ...$
The rule for the nagative and positive semicycles to occur succesively can be periodically expressed as follows
$\{x_{p+7n}, x_{p+7n+1}, x_{p+7n+2}\}^-$, $\{x_{p+7n+3}\}^+$, $\{x_{p+7n+4}\}^-$, $\{x_{p+7n+5}, x_{p+7n+6}\}^+$, n=0,1,2,3,...

We have easily the followings inequalities

$$\frac{1}{x_{p+7(n-1)+3}} < x_{p+7n} < x_{p+7n+2} < x_{p+7n+4} < x_{p+7n+8} < \frac{1}{x_{p+7(n+1)+3}} < 1$$

We can see that $\frac{1}{x_{p+7(n+1)+3}}$ *is increasing with upper bound 1. So, the limits*

$$\lim_{n\to\infty} x_{p+7n} = \lim_{n\to\infty} x_{p+7n+2} = \lim_{n\to\infty} x_{p+7n+4} = \lim_{n\to\infty} x_{p+7n+8} = \lim_{n\to\infty} \frac{1}{x_{p+7(n+1)+3}} = L$$

exist and finite.
Nothing that

$$x_{p+7n+4} = \frac{x^b_{p+7n+2}x^b_{p+7n+1}x_{p+7n} + x^b_{p+7n+2} + x^b_{p+7n+1}x_{p+7n} + a}{x^b_{p+7n+2}x^b_{p+7n+1} + x^b_{p+7n+1}x_{p+7n} + x^b_{p+7n+2}x_{p+7n} + 1 + a}$$

Taking the limits on both sides of this equality, we obtain

$L = \frac{L^{1+2b}+2L^b+L+a}{2L^{1+b}+L^{2b}+1+a} \Rightarrow L = 1$

Next, we also have

$$1 < x_{p+7n+5} < x_{p+7n+3} < x_{p+7(n-1)+6} < \frac{1}{x_{p+7(n-1)+2}}$$

Taking the limits on both sides of this equality, we have

$$\lim_{n\to\infty} x_{p+7n+5} = \lim_{n\to\infty} x_{p+7n+3} = \lim_{n\to\infty} x_{p+7n+6} = 1$$

Up to now, un the first case we have shown .
$\lim_{n\to\infty} x_{p+7n+k} = 1$, $k = \overline{0,6}$
So, we have (9). In the second case, one can prove by the semilar way, it is omited. The proof is complete reference.

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