The Partial Orderings of m-Symmetric Fuzzy Matrices

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Abstract

In this paper, I introduce the concept of partial orderings on fuzzy matrices using m-symmetric matrix. By using various generalized inverses, we show that the left-star and the right-star partial orderings are identical for certain class of fuzzy matrices. Also I derive the equivalent conditions for the existence of the generalized inverses in fuzzy matrices.

Keywords and Phrases: Partial order set, m-symmetric matrix, generalized inverse.

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1 Introduction

Let $F^{m\times n}$ will denote the set of $m \times n$ fuzzy matrices and in short $F_{n\times n}$ is denoted as F_n . For $A \in F_n$, the symbols A^*, $A^{\sim}$, A^m, $A^{\dagger}$, $R(A)$ and

$N(A)$ denote the conjucate transpose, Minkowski adjoint, Minkowski inverse, Moore-Penrose inverse, Range space and Null space of A respectively. A is said to be regular if $AXA = A$ has a solution. Thus $A\{1\}$ denotes the set of all g-inverses of an invertible fuzzy matrix A. Further $A\{1\}$ and $A\{2\}$ will denote the set of all $\{1\}$ and $\{2\}$ inverses of A defined as

$$A\{1\} = \{X \in F_{n,m} / AXA = A\} \qquad (1)$$

and

$$A\{2\} = \{X \in F_{n,m} / XAX = X\}, \qquad (2)$$

while $A\{3\}$ and $A\{4\}$ will denote the set of all right and left symmetrizers of A, defined as

$$A\{3\} = \{X \in F_{n,m} / (AX)^* = AX\} \qquad (3)$$

and

$$A\{4\} = \{X \in F_{n,m} / (XA)^* = XA\} \qquad (4)$$

constitute the classes of generalized inverses of A. In particular, the unique member of $A\{1,2,3,4\}$ is the Moore-Penrose inverse of A. However unlike the Moore-Penrose inverse of a matrix, the Minkowski inverse does not exists always. In [8], Meenakshi showed that the Minkowski inverse of a matrix $A \in C^{m\times n}$ exists if and only if $rk(AA^{\sim}) = rk(A^{\sim}A) = rk(A)$, where $A^{\sim} = GA^*G$ is called the Minkowski adjoint of the matrix A and G is the Minkowski metric matrix $G = \begin{bmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{bmatrix}$ satisfies $G = G^*$ *and* $G^2 = I_n$. Partial ordering plays an important role in fuzzy matrices. The concept of partial orderings on fuzzy matrices which is the analogue of the star ordering on complex matrices was initiated by Jian Miao Chen [3]. Meenakshi.AR and Inbam.C [7] studied the minus ordering is a partial ordering in the set of all regular fuzzy matrices. Furthermore, she defines space ordering [6] on fuzzy matrices, which is a partial ordering on the set of all idempotent matrices in F_n. In [9], the author developed the concept of Matrix Partial Orderings using Minkowski adjoint. In this paper I study the Partial orderings of m-symmetric fuzzy matrices using Minkowski metric tensor G.

2 Preliminaries

Definition 2.1. ([1]) *For every matrix* $A \in C^{m\times n}$, $A^{\dagger}$ *is the Moore-Penrose inverse of* A *if* $AA^{\dagger}A = A$, $A^{\dagger}AA^{\dagger} = A^{\dagger}$, $AA^{\dagger}$ *and* $A^{\dagger}A$ *are hermitian.*

Definition 2.2. *For* $A \in F^{n\times n}$ *is said to be m-symmetric if* $A = A^{\sim}$ *where* $A^{\sim} = GA^{*}G$.

Definition 2.3. A^{m} *is the Minkowski inverse of* A *if* $AA^{m}A = A$, $A^{m}AA^{m} = A^{m}$, AA^{m} *and* $A^{m}A$ *are m-symmetric.*

Theorem 2.4. $(Lemma 2.7\ [6])$ *Let* $A \in C^{n\times n}$, *if* A^{m} *exists in* F_{n} *then* $R(A^{m}) = R(A^{\sim})$.

Definition 2.5. *A matrix* $A \in F_{mn}$ *is said to be invertible if there exists* $A^{m} \in F_{nm}$ *such that* $AA^{m}A = A$. *In this case* A^{m} *is called the Minkowski inverse of* A.

Since the Minkowski inverse A^{m} is also a g-inverse of A.

3 Tilde-ordering on m-symmetric fuzzy matrices

Definition 3.1. *For* $A, B \in F_{m,n}$, *the Tilde ordering* $A \leq^{\sim} B$ *is defined as* $A \leq^{\sim} B$ *iff* $A^{\sim}A = A^{\sim}B$ *and* $AA^{\sim} = BA^{\sim}$.

Theorem 3.2. *Let* $A, B \in F_{n\times n}$ *and let* P *be the permutation matrix then* $A \leq^{\sim} B$ *iff* $PA \leq^{\sim} PB$ *iff* $AP \leq^{\sim} BP$.

Proof : *Let* $A \leq^{\sim} B$ *and* P *be any permutation matrix, then*
$A \leq^{\sim} B$ *iff* $A^{\sim}A = A^{\sim}B$ *and* $AA^{\sim} = BA^{\sim}$
iff $A^{\sim}PPA = A^{\sim}PPB$ *and* $PAA^{\sim}P = PBA^{\sim}P$
iff $(PA)^{\sim}PA = (PA)^{\sim}PB$ *and* $PA(PA)^{\sim} = PB(PB)^{\sim}$
iff $PA \leq^{\sim} PB$.

Similarly $A \leq^{\sim} B$ *iff* $A^{\sim}A = A^{\sim}B$ *and* $A^{\sim}A = AA^{\sim} = BA^{\sim}$
iff $PA^{\sim}AP = PA^{\sim}BP$ *and* $APPA^{\sim} = BPPA^{\sim}$
iff $(PA)^{\sim}AP = (PA)^{\sim}BP$ *and* $AP(AP)^{\sim} = BP(AP)^{\sim}$
iff $AP \leq^{\sim} BP$.

Theorem 3.3. *If* $A \in F_{m,n}$, *the tilde ordering is a partial ordering.*

Proof : *Clearly* $A \leq^{\sim} A$. *If* $A \leq^{\sim} B$, $B \leq^{\sim} A$ *then* $A = BA^m A$, $B = BB^m A$.
Hence by a theorem, $B = BB^m A = BA^m A = A$. *(By a theorem).*
If $A \leq^{\sim} B, B \leq^{\sim} A$, *then* $A = BA^m A$ *and* $B = CB^m B$. *By a theorem we have,*
$A = BA^m A = CB^m BA^m A = CB^m A = CA^m A$. *Similarly we have* $A = AA^m C$.
Thus $A \leq^{\sim} C$. *(By a theorem).*

Lemma 3.4. *For* $A \in F_{m,n}$ *and* $B \in F_{m,n}$ *the following are equivalent.*
$(i) A \leq^{\sim} B$
$(ii) A^m A = A^m B; AA^m = BA^m$
$(iii) A = AA^m B = BA^m A = BA^m B$

Proof : $(i) \implies (ii)$
$A \leq^{\sim} B \implies AA^m = BB^m$ *and* $A^m A = A^m B$ *for some* $A^m \in A\{1\}$.
consider $A = A(A^m A) = AA^m B$
$A = (AA^m)A = BA^m A$
$A = B(A^m A) = BA^m B$
$(ii) \implies (iii)$
$A^m A = A^m B$. *This gives* $A = AA^m A = AA^m B$
Also, from $AA^m = BA^m$, $A = AA^m A = BA^m A$.
Thus $A = AA^m B = BA^m A$.
$(iii) \implies (i)$
Let $X = A^m AA^m$

$$\begin{aligned} AXA &= A(A^m AA^m)A \\ &= (AA^m A)A^m A \\ &= AA^m A = A \text{ implies that } X \in A\{1\} \end{aligned}$$

Now consider

$$\begin{aligned} XA &= (A^m AA^m)AA^m B \\ &= A^m(AA^m A)A^m B \end{aligned}$$

$= (A^m A A^m)B = XB$

Similarly $AX = BX$. *Thus* $A \leq^{\sim} B$ *with respect to* $X \in A\{1\}$.

Theorem 3.5. *If* $A, B \in F_{m,n}$, *then the following are equivalent.*

$(i) A \leq^{\sim} B$

$(ii) A = AB^m B = BB^m A = AB^m A$ *for all* $B^m \in B\{1\}$

$(iii) R(A) \subset R(B)$ *and* $AB^m A = A$

Proof : $(i) \implies (ii)$

$A = BA^m B$ *(By lemma)*

$= BA^m (BB^m B)$

$= (BA^m B)B^m B)$

$= AB^m B$ *(By lemma)*

$Thus\ A = AB^m B\ for\ each B^m \in B\{1\}$

Similarly, we have $A = BB^m A$ *for each* $B^m \in B\{1\}$.

4 Left-Tilde and Right-Tilde Partial Orderings

In this section I introduce the concept of Left-Tilde and Right-Tilde orderings for fuzzy matrices as an analogue of left-star and right-star partial orderings for complex matrices. We show that these ordering preserves its Moore-Penrose inverse property. By using minkowski inverses, the Tilde-orderings are discussed. Thus these orderings are identical for certain class of fuzzy matrices.

Definition 4.1. *Let* $A, B \in F_{m,n}$. *We say that* A *is below* B *with respect to the left tilde ordering if* $A^{\sim} A = A^{\sim} B$ *and it is denoted as* $A \leq \sim B$. *We say that* A *is below* B *with respect to the right tilde ordering if* $AA^{\sim} = BA^{\sim}$ *and it is denoted as* $A \sim \leq B$.

Theorem 4.2. *Let* $A \in F_{m,n}$, $B \in F_{m,n}$, $A \sim \leq B$ *and* $A \leq \sim B$ *if and only if* $A \leq^{\sim} B$.

Proof : $A{\sim}{\leq}B$ and $A{\leq}{\sim}B$ implies that $A^{\sim}A = A^{\sim}B$ and $AA^{\sim} = BA^{\sim}$ implies that $A{\sim}{\leq}B$. Conversely, $A{\leq}^{\sim}B$ implies that $A^{\sim}A = A^{\sim}B$ and $AA^{\sim} = BA^{\sim}$ implies that $A^{m}A = A^{m}B$ and $AA^{m} = BA^{m}$.

$A{\leq}^{\sim}B$ implies that $A^{m}A = A^{m}B$ implies that $AA^{m}A = AA^{m}B$ implies that $A = XB$ where $X = AA^{m}$.

This implies that $R(A) \subset R(B)$

Similarly, $AA^{m} = BA^{m}$ implies that $A = BA^{m}A$ implies that $A = BY$

This implies that $C(A) \subset C(B)$.

Thus $A{\leq}^{\sim}B$ implies that $A{\sim}{\leq}B$ and $A{\leq}{\sim}B$.

Hence, for $A \in F_{mn}$, $B \in F_{mn}$, $A{\sim}{\leq}B$ and $A{\leq}{\sim}B$ if and only if $A{\leq}^{\sim}B$.

Theorem 4.3. *For $A, B \in F_{mn}$, if A^{m} and B^{m} exists then*

(i) $A{\sim}{\leq}B$ iff $A^{m}{\sim}{\leq}B^{m}$

(ii) $A{\leq}{\sim}B$ iff $A^{m}{\leq}{\sim}B^{m}$.

Proof : $A{\sim}{\leq}B$ implies that $A^{m}A = A^{m}B$

Since $A = BB^{m}A$ implies that $A = BB^{\dagger}A$

implies that $A = BB^{T}A$

implies that $A^{T} = A^{T}BB^{T}$

$= (B^{T}A)B^{T}$

$= B^{T}(AB^{T})$

implies that $A^{T}A = A^{T}B$

implies that $A(A^{T}A)B^{T} = A(A^{T}B)B^{T}$

implies that $AB^{T} = A(BB^{T}A)^{T}$

implies that $AB^{T} = AA^{T}(by A = BB^{T}A)$

implies that $AB^{\dagger} = AA^{\dagger}$.

implies that $AB^{m} = AA^{m}$.

Thus $(A^{m})^{*}B^{m} = (A^{m})^{*}A^{m}$ implies that $A^{m}{\leq}{\sim}B^{m}$.

Converse follows from the above part by using $(A^{m})^{m} = A$.

5 Conclusion

It is well known that the comparability relation on Fuzzy matrices is a partial *ordering. We study various orderings on fuzzy matrices using various generalized inverses such as g-inverse, Moore-Penrose inverses, Minkowski inverses and discuss their relationship between these orderings with Tilde-ordering.*

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