On Lorentzian para-Sasakian manifolds

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Abstract

In this paper we study some basic results on Lorentzian para Sasakian manifold and Conformally flat LP-sasakian manifold.

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I. Introduction

Sasakian manifold was firstly studied by the famous geometer Sasaki in 1960. By the analogy with Sasakian manifold in 1989 K.Matsumoto, [5] introduced the notion of LP-Sasakian manifold. Then the same notion has been introduced by Mihai [6,7,8] and obtained interesting results. These manifolds are also studied by U.C.De [4] SS Chern [10] and others.

The Present paper organized as follows: Section 1. Introduction. Section 2. Define LP Sasakian Manifold and proved some basic results. In section 3. Conformally flat LP Sasakian manifold.

II. Lorentzian para sasakian manifolds

An n-dimentional differentiable manifold is called Lorentzian para-sasakian manifold if the following conditions hold:

(2.1) $\phi^2 = I + \eta(X)\xi, \quad \eta(\xi) = -1,$
$g(\xi,\xi) = \epsilon, \quad \eta(X) = g(X,\xi),$

(2.3) $g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y)$
where ξ is space-like or time-like vector field. Also in Lorentzian para-Sasakian manifold, we have

(2.4) $(\nabla_X \phi) Y = g(X,Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y),$

Where ∇ denotes the operator of covariant differentiation with respect to the Lorentizian metric g.

Definition 2.1. An LP –Sasakian manifold will be called a manifold of quasi-constant curvature if the curvature tensor R of type (0,4) satisfies the condition
(2.5) $R(X,Y,Z,W) = a[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)] + b[g(X,W)T(Y)T(Z)$
$-g(X,Z)T(Y)T(W) + g(Y,Z)T(X)T(W) - g(Y,W)T(X)T(Z)],$
Where, $R(X,Y,Z,W) = g(R(X,Y)Z,W)$, R is the curvature tensor of type (1,3); a,b are scalar functions and ρ is a unit vector field defined by

(2.6) $g(X,\rho) = T(X).$
The notion of quasi-constant curvature for Riemannian manifolds were given by Chen and Yano [2].

Definition 2.2: An LP-Sasakian manifold will be called an η-Einstein manifold if the Ricci tensor S of type (0, 2) satisfies

$$S(X,Y) = a\,g(X,Y) + b\,\eta(X)\eta(Y),$$

where a and b are scalar functions.

Deftnition 2.3: An LP-Sasakian manifold will be called Weyl-semisymmetric if it satisfies
$(R.(X,Y).C)(Y,Z)W = 0$, where $R(X,Y)$ denotes the curvature operator and $C(Y,Z)W$ is the Weyl-conformal curvature tensor.

Lemma 2.1: An contact metric manifold is an LP- Sasakian manifold iff

(2.8) $\nabla_X \xi = \Phi x$

Proof: Let the manifold be an LP saskian manifold. Then from the equation (2.4) it follows that

$$\nabla_X \phi Y - \phi \nabla_X Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi.$$

Putting Y = ξ, we get

$$-\phi \nabla_X \xi = -(X + \eta(X)\xi),$$

Or

$$\phi \nabla_X \xi = \phi^2(X),$$

which implies, $\nabla_X \xi = \Phi x$.

Lemma 2.2. : In an LP-Sasakian manifold $(\nabla_X \eta)(Y) = g(\phi X, Y)$.

Proof: $(\nabla_X \eta)(Y) = \nabla_X \eta(Y) - \eta(\nabla_X Y)$

$$= \nabla_X g(Y, \xi) - g(\nabla_X Y, \xi) - g(Y, \nabla_X \xi) + g(Y, \nabla_X \xi).$$

Using the value of $\nabla_X \xi$, we have

$$(\nabla_X \eta)(Y) = g(\phi X, Y).$$

Lemma 2.3.: In a LP sasakian manifold

(2.11) $R(X,Y)\xi = \eta(Y)X - \eta(X)Y.$

Proof: $R(X, Y)\xi = \nabla_X \nabla_Y \xi - \nabla_Y \nabla_X \xi - \nabla_{[X,Y]}\xi$

$$= \nabla_X(\phi Y) - \nabla_Y(\phi X) - \phi([X, Y]).$$

The above relation after simplification gives

$$R(X,Y)\xi = \eta(Y)X - \eta(X)Y.$$

NOTE:

From the equation (2.11) it follows that in an LP- sasakian manifold,

(2.12) $R(\xi, X)Y = g(X,Y)\xi - \eta(Y)X.$

Also in an LP sasakian manifold

(2.13) $\eta(R(X,Y)Z) = g(Y,Z)\eta(X) - g(X,Z)\eta(Y).$

Lemma 2.4: In an LP Sasakian manifold

(2.14) $S(X,\xi) = (n-1)\eta(X).$

Proof: From the equation (2.13) we have

$$g(R(X, Y)Z, \xi) = g(Y, Z)\, g(X, \xi) - g(X, Z)\, g(Y, \xi).$$

Putting $Y=Z = e_i$, where $\{ e_i \}$ is an orthogonal basis of the tangent space at each point of the manifold, and taking summation over i where i=1,2,3,……n we get

$$S(X,\xi) = (n-1)\eta(X).$$

III. Conformally flat LP Sasakian Manifold

The Weyl conformal curvature tensor C of type (1,3) of an n-dimensional Riemannian manifold is given by

(3.1) $C(X,Y)Z = R(X,Y)Z - \frac{1}{n-2}[\, S(Y,Z)X - S(X,Z)Y + g(Y,Z)QX - g(X,Z)Y] +$

$\frac{r}{(n-1)(n-2)}$ [g(Y,Z)X – g(X,Z)Y],

Where Q is the Ricci operator defined by g(QX,Y) = S(X,Y) and r is the scalar curvature. Let us suppose that the manifold is conformally flat. Then from the above equation, we have

(3.2) g(R(X,Y)Z,W) = $\frac{1}{(n-2)}$[S(Y,Z)g(X,W) – S(X,Z)g(Y,W)+g(Y,Z)S(X,W)- $\frac{r}{(n-1)(n-2)}$[g(Y,Z)g(X,W)- g(X,Z)g(Y,W)].

Putting W= ξ and using the equation (2.14),the above equation gives

(3.3) η(R(X,Y)Z) = $\frac{1}{(n-2)}$ [S(Y,Z) η(X)- S(X,Z)η(Y)+(n-1)g(Y,Z) η(X)-(n-1)g(X,Z) η(Y)]

$\frac{r}{(n-1)(n-2)}$ [g(Y,Z) η(X) - g(X,Z) η(Y)].

In view of the equation (2.13) and the above equation yields

(3.4) S(Y,Z) η(X) = S(X,Z) η(Y) + $\frac{r}{(n-1)}$[g(Y,Z) η(X) - g(X,Z) η(Y)].

(3.5) S(Y,Z) = [$\frac{r}{(n-1)}$]g(Y,Z) – [$\frac{r+n(1-n)}{n-1}$] η(Y) η(Z).

Hence we can state the following:

Theorem 3.1. An (2n+1)- dimensional(n>1) conformally flat Lorentzian Para Sasakian manifold is an η-Einstein manifold.

Proof: Using the equation (3.5) in (3.2), we get

g(R(X,Y)Z,W) = $\frac{1}{n-2}[(\frac{2r}{n-1}-2)g(Y,Z)g(X,W)-[(\frac{2r}{n-1}-2)g(X,Z)g(Y,W)]-(\frac{r+n(1-n)}{(n-1)(n-2)})$ [η(Y)η(Z)g(X,W)- η(X)η(Z)g(Y,W) + η(X)η(W)g(Y,Z) - η(Y)η(W)g(X,Z)] - $\frac{r}{(n-1)(n-2)}$ [g(Y,Z)g(X,W)- g(X,Z)g(Y,W)].

The above relation can be written as g(R(X,Y)Z,W) = $\frac{r-2n+2}{(n-1)(n-2)}[g(Y,Z)g(X,W) - g(X,Z)g(Y,W)]$ – $(\frac{r+n(1-n)}{(n-1)(n-2)})$ [η(X)η(Z)g(Y,W)+ η(Y)η(W)g(X,Z) - η(X)η(W)g(Y,Z) - η(Y)η(Z)g(X,W)].

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