# Analysis of $M_1, M_2/G_1, G_2/1$ retrial queueing system with non-pre-emptive priority services, modified Bernoulli vacation, working breakdown, repair and reneging

J UDAYAGEETHA[#1], M ASAILA[*2]

[1]*Department of Mathematics, Perunthalaivar Kamarajar Arts College, Pududcherry, India.*

[2]*Department of Mathematics, Pondicherry Engineering College, Pududcherry, India.*

***Abstract*** - *This paper considers $M_1, M_2/G_1, G_2/1$ general retrial queueing system with non-pre-emptive priority services. The server serves two type of units subject to modified Bernoulli vacation. After completing the service, if there are no high priority units present in the system then the server may go for a vacation. The high priority unit who find the server busy are queued in the system and the low priority unit finds the server busy, they are routed to a retrial queue that attempts to get the service. The system may become defective at any point of time when it is in operation and does not sent for repair immediately. The remaining service will be given to the unit after that only the server go for repair station. After completing the repair and vacation if there is no high priority unit then the server becomes idle. The low priority unit renege the orbit during server's busy and unavailable periods. Using the supplementary variable technique, the steady-state distributions of the server state and stability condition are obtained.*

**Keywords -** *Priority Queueing Systems, Working Breakdown, modified Bernoulli Vacation, Retrial, Repair, Reneging.*

## I. INTRODUCTION

Retrial phenomenon naturally arises in various systems such as call centers, cellular networks and random access protocols in local area networks. Phung-Duc, T. [7] gives a comprehensive survey on theory and applications of retrial queues. Queues including impatient units have been dragged the notice of many authors and we see notable participation by numerous researchers in this area. The important works on impatient behaviour of the units was studied by Cheng-Dar Liou [2]. Dong-Yuh Yang and Ying-Yi Wu [3] analyzed customer's impatience behaviour with server vacations. In this paper, the low priority units renege the orbit during the server's busy and unavailable periods.

Studying queueing models with server breakdown, it is generally assumed that the server stops service when the server getting breakdown. However, in most of the models considered so far of queueing systems with server breakdowns, the underlying assumption has been that a server breakdown disrupts the service completely in the system. Kalidass and Kasturi [5] introduced the working breakdown policy, in which the server works at a lower service rate rather than stopping service during the breakdown period. In this policy the service can decrease complaints from the units who should wait for the server to be, repaired and reduces the cost of waiting units. Therefore, the working breakdown service is a more reasonable repair policy for unreliable queueing systems. Tao Li and Liyuan Zhang [8] and Zaiming liu and Yang Song [9] describes more about working breakdowns. Bo Keun Kim and Doo Ho Lee [1] studied the M/G/1 queueing system with disasters and working breakdowns.

In manufacturing systems based on the needs, the works are categorized and we give the preference as high and low priority one. Here, the correct assignment of work will reduce the time consumption and cost management in the service system. Kailash C. Madan [4] studied a non-preemptive priority queueing system with a single server serving two queues. Boutarfa L and Djellab L studied the performance of the $M_1, M_2/G_1, G_2/1$ retrial queue with pre-emptive resume policy [6].

This paper considers $M_1, M_2/G_1, G_2/1$ general retrial queueing system with priority services. Two different sort of units arrive at the system in two independent Poisson processes. Under the non-pre-emptive priority rule, the server providing general service to the arriving unit. Here, we propose a retrial queueing model with the additional characteristics of server's working breakdown, repair and reneging. Arriving high priority unit who find the server busy are queued and then are served in accordance with FCFS discipline. But the arriving low priority unit on finding the server busy cannot be queued and join the orbit as a retrial unit. After completing the service, if there are no high priority units present in the system then the server may go for a vacation. The system may become defective at any point of time when it is in operation due to normal breakdown. However, when the system is defective, it does not sent for repair immediately. The remaining service will be given to the interrupted unit. After that only the server go for repair station. After completing the repair and vacation if there is no high priority unit then the server becomes idle. We consider reneging to occur at the low priority unit while the server's busy and unavailable periods.

The remaining content of the paper is structured as follows: Section 2 provides a mathematical description of the model and governing equations of the model are formulated. The steady-state solutions of the system and conclusion are provided in sections 3 and 4 sequentially.

## II. MODEL DESCRIPTION

We analyze an unreliable server in a retrial queueing model with two sort of units. The basic operation of the model can be described as follows.

Two class of units arrive at the system in two independent Poisson processes with arrival rate $\lambda_1(>0)$ and $\lambda_2(>0)$ respectively. The high priority unit who find the server busy are queued in the system and the low priority unit find the server busy, they are routed to a retrial queue and follow constant retrial policy that tries to get the service. The retrial time is generally distributed with distribution function $I(s)$ and the density function $i(s)$.

After completing the service of all high priority units and every service completion of low priority unit the server may demand a vacation with probability $\theta$ or becomes idle. Vacation time is generally distributed with distribution function $V(s)$ and the density function $v(s)$ respectively .

During the busy period, the server may become inactive due to normal breakdown with breakdown rate $\alpha$, which is exponentially distributed. At the moment of breakdown, the unit who is in service will get the service continuously at a lower service rate which is known as working breakdown. After that the repair process starts so as to regain its functionality.The working breakdown time and repair time are generally distributed with distribution functions $W(s)$ and $R(s)$ and the density functions $w(s)$ and $r(s)$ respectively.

If the server is busy or unavailable in the system, the low priority unit may renege the orbit exponentially with rate $\xi$. After completing vacation and repair if there is no high priority unit present in the system then the server becomes idle. Several stochastic processes associated in the system are independent of each other.

Let $\mu_i(x)\,dx$, i =1,2,3, $\eta(x)\,dx$, $\beta(x)\,dx$ and $\gamma(x)\,dx$ be the conditional probability of completion rate of the service period of the high and low priority unit, working breakdown service, retrial period, vacation period, and repair period given that the elapsed time is x.

Also, $\mu_i(x)dx = \frac{dB_i(x)}{1-B_i(x)}$,i= 1, 2, 3, $\eta(x)dx = \frac{dI(x)}{1-I(x)}$, $\beta(x)dx = \frac{dV(x)}{1-V(x)}$, and $\gamma(x)dx = \frac{dR(x)}{1-R(x)}$ are the first order differential (hazard rate) functions of $B_i(.)$, I(.), V(.) and R(.) respectively.

## STATE OF THE SYSTEM

We get the probability generating function of the joint distribution of the state of the server including the number in the system by using $I^0(t)$, $B_1^0(t)$, $B_2^0(t)$ and $B_3^0(t)$ are the elapsed retrial time and service time of the high and low priority units at time t respectively. Also $V^0(t)$ and $R^0(t)$ are the elapsed vacation time and elapsed repair time of the server as supplementary variables at time t respectively. Assuming that the system is empty initially. $N_1(t)$ and $N_2(t)$ denotes the number of units in the queue (high priority units) and the orbit (low priority units) respectively and Y(t) be the state of the server at time t,

$$Y(t)=\begin{cases} 0, & if\ the\ server\ is\ in\ idle\ state; \\ 1, & if\ the\ server\ is\ in\ retrial\ state; \\ 2, & if\ the\ server\ is\ busy\ with\ high\ priority\ unit; \\ 3, & if\ the\ server\ is\ busy\ with\ low\ priority\ unit; \\ 4, & if\ the\ server\ is\ on\ vacation; \\ 5, & if\ the\ server\ is\ on\ working\ breakown\ state; \\ 6, & if\ the\ server\ is\ on\ repair\ process. \end{cases}$$

For the Markov process $\{Y(t); t \geq 0\}$ defined the probability as,

$$\overline{I}_{0,0}(s,t) = Pr\{Y(t) = 0, N_1(t) = 0, N_2(t) = 0\},$$

and the probability densities are,

$$\overline{I}_{0,k_2}(x,s,t)dx = Pr\{Y(t) = 1, N_1(t) = 0, N_2(t) = k_2\}, k_2 \geq 0$$
$$\overline{P}^{(1)}_{k_1,k_2}(x,s,t)dx = Pr\{Y(t) = 2, x < B_1^0(t) \leq x + dx, N_1(t) = k_1, N_2(t) = k_2\}, k_1, k_2 \geq 0,$$
$$\overline{P}^{(2)}_{k_1,k_2}(x,s,t)dx = Pr\{Y(t) = 3, x < B_2^0(t) \leq x + dx, N_1(t) = k_1, N_2(t) = k_2\}, k_1, k_2 \geq 0,$$
$$\overline{V}_{k_1,k_2}(x,s,t)dx = Pr\{Y(t) = 4, x < V^0(t) \leq x + dx, N_1(t) = k_1, N_2(t) = k_2\}, k_1, k_2 \geq 0,$$
$$\overline{W}_{k_1,k_2}(x,s,t)dx = Pr\{Y(t) = 5, x < B_3^0(t) \leq x + dx, N_1(t) = k_1, N_2(t) = k_2\}, k_1, k_2 \geq 0,$$
$$\overline{R}_{k_1,k_2}(x,s,t)dx = Pr\{Y(t) = 6, x < R^0(t) \leq x + dx, N_1(t) = k_1, N_2(t) = k_2\}, k_1, k_2 \geq 0,.$$

Further it is assumed that,

$$P^{(1)}_{k_1,k_2}(0) = P^{(2)}_{k_1,k_2}(0) = V_{k_1,k_2}(0) = W_{k_1,k_2}(0) = R_{k_1,k_2}(0) = I_{0,k_2}(0) = 0;\ k_1, k_2 \geq 0 \text{ and } I_{0,0}(0) = 1.$$

## EQUATIONS GOVERNING THE SYSTEM

The Kolmogorov differential-difference equations for the preceding model:

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda_1 + \lambda_2 + \alpha + \xi + \mu_1(x)\right)P^{(1)}_{k_1,k_2}(x,t) = (1-\delta_{k_10})\lambda_1 P^{(1)}_{k_1-1,k_2}(x,t) + (1-\delta_{0k_2})\lambda_2 P^{(1)}_{k_1,k_2-1}(x,t) + \xi P^{(1)}_{k_1,k_2+1}(x,t);\ k_1,k_2 \geq 0, \quad (1)$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda_1 + \lambda_2 + \alpha + \xi + \mu_2(x)\right)P^{(2)}_{k_1,k_2}(x,t) = (1-\delta_{k_10})\lambda_1 P^{(2)}_{k_1-1,k_2}(x,t) + (1-\delta_{0k_2})\lambda_2 P^{(2)}_{k_1,k_2-1}(x,t) + \xi P^{(2)}_{k_1,k_2+1}(x,t);\ k_1,k_2 \geq 0, \quad (2)$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda_1 + \lambda_2 + \xi + \beta(x)\right)V_{k_1,k_2}(x,t) = (1-\delta_{k_10})\lambda_1 V_{k_1-1,k_2}(x,t) + (1-\delta_{0k_2})\lambda_2 V_{k_1,k_2-1}(x,t) + \xi V_{k_1,k_2+1}(x,t);\ k_1,k_2 \geq 0, \quad (3)$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda_1 + \lambda_2 + \xi + \gamma(x)\right)R_{k_1,k_2}(x,t) = (1-\delta_{k_10})\lambda_1 R_{k_1-1,k_2}(x,t) + (1-\delta_{0k_2})\lambda_2 R_{k_1,k_2-1}(x,t) + \xi R_{k_1,k_2+1}(x,t);\ k_1,k_2 \geq 0, \quad (4)$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda_1 + \lambda_2 + \xi + \mu_3(x)\right)W_{k_1,k_2}(x,t) = (1-\delta_{k_10})\lambda_1 W_{k_1-1,k_2}(x,t) + (1-\delta_{0k_2})\lambda_2 W_{k_1,k_2-1}(x,t) + \xi W_{k_1,k_2+1}(x,t);\ k_1,k_2 \geq 0, \quad (5)$$

$$\left(\frac{d}{dt} + \lambda_1 + \lambda_2\right)I_{0,0}(t) = \int_0^\infty R_{0,0}(x,t)\,\gamma(x)dx + \int_0^\infty V_{0,0}(x,t)\,\beta(x)dx + (1-\theta)\left\{\int_0^\infty P^1_{0,0}(x,t)\,\mu_1(x)dx + \int_0^\infty P^2_{0,0}(x,t)\,\mu_2(x)dx\right\}, \quad (6)$$

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x} + \lambda_1 + \lambda_2 + \xi + \eta(x)\right)I_{0,k_2}(x,t) = 0;\ k_2 \geq 1. \quad (7)$$

with the boundary conditions

$$I_{0,k_2}(0,t) = \int_0^\infty R_{0,k_2}(x,t)\,\gamma(x)dx + \int_0^\infty V_{0,k_2}(x,t)\,\beta(x)dx + (1-\theta)\left\{\int_0^\infty P^1_{0,k_2}(x,t)\,\mu_1(x)dx + \int_0^\infty P^2_{0,k_2}(x,t)\,\mu_2(x)dx\right\}, \quad (8)$$

$$P^{(1)}_{k_1,k_2}(0,t) = \int_0^\infty P^{(1)}_{k_1+1,k_2}(x,t)\,\mu_1(x)dx + \int_0^\infty P^{(2)}_{k_1+1,k_2}(x,t)\,\mu_2(x)dx$$

$$+\int_0^\infty R_{k_1+1,k_2}(x,t)\,\gamma(x)dx + \int_0^\infty V_{k_1+1,k_2}(x,t)\,\beta(x)dx;\ k_1,k_2 \geq 0, \quad (9)$$
$$P^{(2)}_{0,k_2}(0,t) = \int_0^\infty I_{0,k_2+1}(x,t)\,\eta(x)dx + \lambda_2 I_{0,k_2}(x,t);\ k_2 \geq 0, \quad (10)$$
$$V_{0,k_2}(0,t) = \theta\{\int_0^\infty P^{(1)}_{0,k_2}(x,t)\mu_1(x)dx + \int_0^\infty P^{(2)}_{0,k_2}(x,t)\mu_2(x)dx\};\ k_2 \geq 0, \quad (11)$$
$$W_{k_1,k_2}(0,t) = \alpha\{\int_0^\infty P^{(1)}_{k_1,k_2}(x,t)(x)dx + \int_0^\infty P^{(2)}_{k_1,k_2}(x,t)dx\};\ k_1,k_2 \geq 0, \quad (12)$$
$$R_{k_1,k_2}(0,t) = \int_0^\infty W_{k_1,k_2}(x,t)\,\mu_3(x)dx;\ k_1,k_2 \geq 0. \quad (13)$$

The Probability Generating Function (PGF) of this model:

$$I_0(x,z_2,t) = \sum_{k_2=1}^\infty z_2^{k_2} I_{0,k_2}(x,t), \quad P^{(2)}(x,z_1,z_2,t) = \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty z_1^{k_1} z_2^{k_2} P^{(2)}_{k_1,k_2}(x,t),$$
$$R(x,z_1,z_2,t) = \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty z_1^{k_1} z_2^{k_2} R_{k_1,k_2}(x,t), \quad P^{(1)}(x,z_1,z_2,t) = \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty z_1^{k_1} z_2^{k_2} P^{(1)}_{k_1,k_2}(x,t),$$
$$W(x,z_1,z_2,t) = \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty z_1^{k_1} z_2^{k_2} W_{k_1,k_2}(x,t) \quad V(x,z_1,z_2,t) = \sum_{k_1=0}^\infty \sum_{k_2=0}^\infty z_1^{k_1} z_2^{k_2} V_{k_1,k_2}(x,t),$$
and $I_{0,0} = 1$.

Taking Laplace transform for the equations (1) - (13) and applying probability generating function for the equations we get,

$$\{\frac{\partial}{\partial x} + (s + \lambda_1[1-z_1] + \lambda_2[1-z_2] + \alpha + \xi(1-\frac{1}{z_2}) + \mu_1(x))\}\overline{P}^{(1)}(x,s,z_1,z_2) = 0, \quad (14)$$
$$\{\frac{\partial}{\partial x} + (s + \lambda_1[1-z_1] + \lambda_2[1-z_2] + \alpha + \xi(1-\frac{1}{z_2}) + \mu_2(x))\}\overline{P}^{(2)}(x,s,z_1,z_2) = 0, \quad (15)$$
$$\{\frac{\partial}{\partial x} + (s + \lambda_1[1-z_1] + \lambda_2[1-z_2] + \xi(1-\frac{1}{z_2}) + \beta(x))\}\overline{V}(x,s,z_1,z_2) = 0, \quad (16)$$
$$\{\frac{\partial}{\partial x} + (s + \lambda_1[1-z_1] + \lambda_2[1-z_2] + \xi(1-\frac{1}{z_2}) + \mu_3(x))\}\overline{W}(x,s,z_1,z_2) = 0, \quad (17)$$
$$\{\frac{\partial}{\partial x} + (s + \lambda_1[1-z_1] + \lambda_2[1-z_2] + \xi(1-\frac{1}{z_2}) + \gamma(x))\}\overline{V}(x,s,z_1,z_2) = 0, \quad (18)$$
$$\{\frac{\partial}{\partial x} + (s + \lambda_1 + \lambda_2 + \eta(x))\}\overline{I}_0(x,s,z_2) = 0. \quad (19)$$

Solving equations from (14) to (19),

$$\overline{P}^{(1)}(x,s,z_1,z_2) = \overline{P}^{(1)}(0,s,z_1,z_2)e^{\{-\varphi_1(s,z_1,z_2)x - \int_0^x \mu_1(t)dt\}}, \quad (20)$$
$$\overline{P}^{(2)}(x,s,z_1,z_2) = \overline{P}_0^{(2)}(0,s,z_2)e^{\{-\varphi_1(s,z_1,z_2)x - \int_0^x \mu_2(t)dt\}}, \quad (21)$$
$$\overline{V}(x,s,z_1,z_2) = \overline{V}(0,s,z_2)e^{\{-\varphi_3(s,z_1,z_2)x - \int_0^x \beta(t)dt\}}, \quad (22)$$
$$\overline{W}(x,s,z_1,z_2) = \overline{W}(0,s,z_1,z_2)e^{\{-\varphi_2(s,z_1,z_2)x - \int_0^x \mu_3(t)dt\}}, \quad (23)$$
$$\overline{R}(x,s,z_1,z_2) = \overline{R}(0,s,z_1,z_2)e^{\{-\varphi_2(s,z_1,z_2)x - \int_0^x \beta(t)dt\}}, \quad (24)$$
$$\overline{I}_0(x,s,z_2) = \overline{I}_0(0,s,z_2)e^{\{-\varphi(a,s)x - \int_0^x \eta(t)dt\}}. \quad (25)$$

Integrating equations (20) to (25), by parts with respect to x yields,

$$\overline{P}^{(1)}(s,z_1,z_2) = \overline{P}^{(1)}(0,s,z_1,z_2)[\frac{1-\overline{B}_1[\varphi_1(s,z_1,z_2)]}{\varphi_1(s,z_1,z_2}], \quad (26)$$
$$\overline{P}^{(2)}(s,z_1,z_2) = \overline{P}_0^{(2)}(0,s,z_2)[\frac{1-\overline{B}_2[\varphi_1(s,z_1,z_2)]}{\varphi_1(s,z_1,z_2}], \quad (27)$$
$$\overline{V}(s,z_1,z_2) = \overline{V}_0(0,s,z_2)\ [\frac{1-\overline{V}[\varphi_2(s,z_1,z_2)]}{\varphi_2(s,z_1,z_2)}], \quad (28)$$
$$\overline{W}(s,z_1,z_2) = \overline{W}(0,s,z_1,z_2)\ [\frac{1-\overline{B}_3[\varphi_2(s,z_1,z_2)]}{\varphi_2(s,z_1,z_2)}], \quad (29)$$
$$\overline{R}(s,z_1,z_2) = \overline{R}(0,s,z_1,z_2)\ [\frac{1-\overline{R}[\varphi_2(s,z_1,z_2)]}{\varphi_5(s,z_1,z_2)}], \quad (30)$$
$$\overline{I}_0(s,z_2) = \overline{I}_0(0,s,z_2)\ [\frac{1-\overline{I}[\varphi(a,s)]}{\varphi(a,s)}]. \quad (31)$$

Multiplying equations from (20) to (25) by, $\mu_1(x)$, $\mu_2(x)$, $\beta(x)$, $\mu_3(x)$, $\gamma(x)$ and $\eta(x)$ respectively and integrating with respect to $x$, we get,

$$\int_0^\infty \overline{P}^{(1)}(x,s,z_1,z_2)\,\mu_1(x)dx = \overline{P}^{(1)}(0,s,z_1,z_2)\overline{B}_1[\varphi_1(s,z_1,z_2)], \tag{32}$$
$$\int_0^\infty \overline{P}^{(2)}(x,s,z_1,,z_2)\,\mu_2(x)dx = \overline{P}_0^{(2)}(0,s,z_2)\overline{B}_2[\varphi_1(s,z_1,z_2)], \tag{33}$$
$$\int_0^\infty \overline{V}(x,s,z_1,z_2)\,\beta(x)dx = \overline{V}_0(0,s,z_2)\overline{V}[\varphi_2(s,z_1,z_2)], \tag{34}$$
$$\int_0^\infty \overline{W}(x,s,z_1,z_2)\,\mu_3(x)dx = \overline{W}(0,s,z_1,z_2)\overline{B}_3[\varphi_2(s,z_1,z_2)], \tag{35}$$
$$\int_0^\infty \overline{R}(x,s,z_1,z_2)\,\gamma(x)dx = \overline{R}(0,s,z_1,z_2)\overline{R}[\varphi_2(s,z_1,z_2)], \tag{36}$$
$$\int_0^\infty \overline{I}_0(x,s,z_2) = \overline{I}_0(0,s,z_2)\,\overline{I}[\varphi(a,s)]. \tag{37}$$

where,

$$\varphi(a,s) = s+\lambda_1+\lambda_2,\quad \varphi_1(s,z_1,z_2) = s+\lambda_1[1-z_1]+\lambda_2[1-z_2]+\alpha+\xi[1-\tfrac{1}{z_2}],$$
$$\varphi_2(s,z_1,z_2) = s+\lambda_1[1-z_1]+\lambda_2[1-z_2]+\xi[1-\tfrac{1}{z_2}],$$

Simplifying the boundary conditions we can get,

$$\begin{aligned} z_1\overline{P}^{(1)}(0,s,z_1,z_2) &= \overline{B}_1[\varphi_1(s,z_1,z_2)]\overline{P}^{(1)}(0,s,z_1,z_2)+\lambda_1 z_1\overline{I}_0(x,s,z_2)-\overline{P}_0^{(1)}(0,s,z_2)\\ &\times\overline{B}_1[\varphi_1(s,z_2)]+\overline{P}_0^{(2)}(0,s,z_2)\overline{B}_2[\varphi_1(s,z_1,z_2)]-\overline{P}_0^{(1)}(0,s,z_2)\overline{B}_2[\varphi_1(s,z_2)]\\ &+\overline{R}(0,s,z_1,z_2)\overline{R}[\varphi_2(s,z_1,z_2)]-\overline{R}_0(0,s,z_2)\overline{R}[\varphi_2(s,z_2)]+\\ &\times\overline{V}(0,s,z_1,z_2)\overline{V}[\varphi_2(s,z_1,z_2)]-\overline{V}_0(0,s,z_2)\overline{V}[\varphi_2(s,z_2)], \end{aligned} \tag{38}$$

$$\begin{aligned} \overline{I}_0(0,s,z_2) &= 1-(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s)+\int_0^\infty \overline{V}_0(x,s,z_2)\beta(x)dx+\int_0^\infty \overline{R}_0(x,s,z_2)\gamma(x)dx\\ &+(1-\theta)\{\int_0^\infty \overline{P}_0^1(x,s,z_2)\mu_1(x)dx+\int_0^\infty \overline{P}_0^2(x,s,z_2)\mu_2(x)dx\};\ k_2\ge 1, \end{aligned} \tag{39}$$

$$z_2\overline{P}_0^{(2)}(0,s,z_2) = \overline{I}_0(0,s,z_2)[\overline{I}[\varphi(a,s)]+\lambda_2 z_2[\tfrac{1-\overline{I}[\varphi(a,s)]}{\varphi(a,s)}]], \tag{40}$$
$$\overline{V}(0,s,z_1,z_2) = \theta\{\overline{P}^{(1)}(0,s,z_2)\overline{B}_1[\varphi_1(s,z_2)]+\overline{P}_0^{(2)}(0,s,z_2)\overline{B}_2[\varphi_2(s,z_2)]\}, \tag{41}$$
$$\overline{W}_0(0,s,z_1,z_2) = \alpha\{\overline{P}^{(1)}(0,s,z_1,z_2)\frac{[1-\overline{B}_1[\varphi_1(s,z_1,z_2)]]}{\varphi_1(s,z_1,z_2)}+\overline{P}^{(2)}(0,s,z_1,z_2)\overline{B}_2[\varphi_2(s,z_1,z_2)]\}, \tag{42}$$

$$\overline{R}(0,s,z_1,z_2) = \frac{\alpha\{\overline{P}^{(1)}(0,s,z_1,z_2)\frac{[1-\overline{B}_1[\varphi_1(s,z_1,z_2)]]}{\varphi_1(s,z_1,z_2)}+}{\overline{P}^{(2)}(0,s,z_1,z_2)\overline{B}_2[\varphi_2(s,z_1,z_2)]\}\overline{B}_3[\varphi_2(s,z_1,z_2)].} \tag{43}$$

By applying Rouche's theorem on (38), we get,

$$\overline{P}_0^{(1)}(0,s,z_2)\overline{B}_1[\varphi_1(s,z_2)] = \frac{\left\{\begin{aligned} &\lambda_1 g(z_2)\overline{I}_0(x,s,z_2)+\overline{P}_0^{(2)}(0,s,z_2)\{\overline{B}_2[\varphi_1(s,g(z_2))]-\overline{B}_2[\varphi_1(s,z_2)]\\ &+\alpha\tfrac{1-\overline{B}_2[\varphi_1(s,g(z_2))]}{\varphi_1(s,g(z_2))}\overline{B}_3[\varphi_1(s,g(z_2))]\overline{R}[\varphi_2(s,g(z_2))]\\ &-\alpha\tfrac{1-\overline{B}_2[\varphi_1(s,z_2)]}{\varphi_1(s,z_2)}\overline{B}_3[\varphi_1(s,z_2)]\overline{R}[\varphi_2(s,z_2)]+\theta\overline{B}_2[\varphi_1(s,z_2)]\\ &\times[\overline{V}[\varphi_2(s,g(z_2))]-\overline{V}[\varphi_2(s,z_2)]]\} \end{aligned}\right\}}{\left\{\begin{aligned} &\overline{B}_1[\varphi_1(s,z_2)]\{1-\theta\overline{V}[\varphi_2(s,g(z_2))]+\theta\overline{V}[\varphi_2(s,z_2)]\}+\alpha\tfrac{1-\overline{B}_2[\varphi_1(s,z_2)]}{\varphi_1(s,z_2)}\\ &\times\overline{B}_3[\varphi_1(s,z_2)]\overline{R}[\varphi_2(s,z_2)] \end{aligned}\right\}}. \tag{44}$$

By substituting (44) in required equations we get,

$$\overline{P}^{(1)}(0,s,z_1,z_2) = \frac{\left\{\begin{aligned} &\lambda_1[\tfrac{1-\overline{I}_0(0,s,z_2)}{\overline{\phi}(a,s)}]\{[\overline{\zeta}_4(s,z_1,z_2)-\overline{\zeta}_1(s,z_1,z_2)](1-(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s))\}\\ &+\overline{P}_0^{(2)}(0,s,z_2)\{\lambda_1[\tfrac{1-\overline{I}_0(0,s,z_2)}{\overline{\phi}(a,s)}][\overline{\zeta}_3(s,z_1,z_2)\overline{\zeta}_5(s,z_1,z_2)+\overline{\zeta}_4(s,z_1,z_2)\\ &\times\overline{\zeta}_6(s,z_1,z_2)-\overline{\zeta}_1(s,z_1,z_2)\overline{\zeta}_6(s,z_1,z_2)-\overline{\zeta}_2(s,z_1,z_2)\overline{\zeta}_5(s,z_1,z_2)]\\ &+\overline{\zeta}_1(s,z_1,z_2)\overline{\zeta}_3(s,z_1,z_2)+\overline{\zeta}_2(s,z_1,z_2)\overline{\zeta}_4(s,z_1,z_2)\} \end{aligned}\right\}}{\left\{\begin{aligned} &\{\overline{\zeta}_4(s,z_1,z_2)-\lambda_1[\tfrac{1-\overline{I}_0(0,s,z_2)}{\overline{\phi}(a,s)}]\}\{z_1-\overline{B}_1[\varphi_1(s,z_1,z_2)]\\ &-\alpha[\tfrac{1-\overline{B}_1[\varphi_1(s,z_1,z_2)]}{\varphi_1(s,z_1,z_2)}]\overline{B}_3[\varphi_1(s,z_1,z_2)]\overline{R}[\varphi_2(s,z_1,z_2)]\} \end{aligned}\right\}}, \tag{45}$$

$$\overline{P}_0^{(2)}(0,s,z_2)=\frac{\left\{\begin{array}{c}(1-(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s))\{\overline{\zeta}_4(s,,z_1,z_2)\\ \times\{\overline{I}[\varphi(a,s)]+\lambda_2 z_2[\frac{1-\overline{I}[\varphi(a,s)]}{[\varphi(a,s)]}]\}\}\end{array}\right\}}{\left\{\begin{array}{c}z_2\{\overline{\zeta}_4(s,z_1,z_2)-\lambda_1 g(z_2)[\frac{1-\overline{I}[\varphi(a,s)]}{\varphi(a,s)}]\overline{\zeta}_5(s,z_1,z_2)\}\\ -\{\overline{\zeta}_3(s,z_1,z_2))\overline{\zeta}_5(s,z_1,z_2)+\overline{\zeta}_4(s,z_1,z_2)\overline{\zeta}_6(s,z_1,z_2)\}\\ \times\{\overline{I}[\varphi(a,s)]+\lambda_2 z_2[\frac{1-\overline{I}[\varphi(a,s)]}{\varphi(a,s)}]\}\end{array}\right\}}, \tag{46}$$

$$\overline{I}_0(0,s,z_2)=\frac{\left\{\begin{array}{c}(1-(s+\lambda_1+\lambda_2)\overline{I}_{0,0}(s))\overline{\zeta}_4(s,z_1,z_2)+\overline{P}_0^{(2)}(0,s,z_2)\\ \times\{\overline{\zeta}_4(s,z_1,z_2)\overline{\zeta}_6(s,z_1,z_2)+\overline{\zeta}_3(s,z_1,z_2)\overline{\zeta}_5(s,z_1,z_2)\}\end{array}\right\}}{\left\{\overline{\zeta}_4(s,z_1,z_2)-\lambda_1(g(z_2))[\frac{1-\overline{I}[\varphi(a,s)]}{\varphi(a,s)}]\overline{\zeta}_5(s,z_1,z_2)\right\}}. \tag{47}$$

where,

$$\overline{\zeta}_1(s,z_1,z_2)=\overline{B}_1[\varphi_1(s,z_2)]\{1-\theta\overline{V}[\varphi_2(s,z_1,z_2)]+\theta\overline{V}[\varphi_2(s,z_2)]\}+\alpha[\frac{1-\overline{B}_1[\varphi_1(s,z_2)]}{\varphi_1(s,z_2)}]$$
$$\times\overline{B}_3[\varphi_1(s,z_2)]\overline{R}[\varphi_2(s,z_2)],$$
$$\overline{\zeta}_2(s,z_1,z_2)=\overline{B}_2[\varphi_1(s,z_1,z_2)]-\overline{B}_2[\varphi_1(s,z_2)]\theta\overline{B}_2[\varphi_1(s,z_2)]\{\overline{V}[\varphi_2(s,z_1,z_2)]$$
$$-\overline{V}[\varphi_2(s,z_2)]\}+\alpha[\frac{1-\overline{B}_2[\varphi_1(s,z_1,z_2)]}{\varphi_1(s,z_1,z_2)}]\overline{B}_3[\varphi_1(s,z_1,z_2)]\overline{R}[\varphi_2(s,z_1,z_2)]$$
$$-\alpha[\frac{1-\overline{B}_2[\varphi_1(s,z_2)]}{\varphi_1(s,z_2)}]\overline{B}_3[\varphi_1(s,z_2)]\overline{R}[\varphi_2(s,z_2)],$$
$$\overline{\zeta}_3(s,z_1,z_2)=\overline{B}_2[\varphi_1(s,g(z_2))]-\overline{B}_2[\varphi_1(s,z_2)]+\alpha\frac{1-\overline{B}_2[\varphi_1(s,g(z_2))]}{\varphi_1(s,g(z_2))}\overline{B}_3[\varphi_1(s,g(z_2))]$$
$$\times\overline{R}[\varphi_2(s,g(z_2))]-\alpha\frac{1-\overline{B}_2[\varphi_1(s,z_2)]}{\varphi_1(s,z_2)}\overline{B}_3[\varphi_1(s,z_2)]\overline{R}[\varphi_2(s,z_2)]$$
$$+\theta\overline{B}_2[\varphi_1(s,z_2)][\overline{V}[\varphi_2(s,g(z_2))]-\overline{V}[\varphi_2(s,z_2)]],$$
$$\overline{\zeta}_4(s,z_1,z_2)=\overline{B}_1[\varphi_1(s,z_2)]\{1-\theta\overline{V}[\varphi_2(s,g(z_2))]+\theta\overline{V}[\varphi_2(s,z_2)]\}+\alpha\frac{1-\overline{B}_2[\varphi_1(s,z_2)]}{\varphi_1(s,z_2)}$$
$$\times\overline{B}_3[\varphi_2(s,z_2)]\overline{R}[\varphi_2(s,z_2)],$$
$$\overline{\zeta}_5(s,z_2)=\overline{B}_1[\varphi_1(s,z_2)]\{1-\theta+\theta\overline{V}[\varphi_2(s,z_2)]\}+\alpha\frac{1-\overline{B}_1[\varphi_1(s,z_2)]}{\varphi_1(s,z_2)}\overline{B}_3[\varphi_1(s,z_2)]\overline{R}[\varphi_2(s,z_2)],$$
$$\overline{\zeta}_6(s,z_2)=\overline{B}_2[\varphi_1(s,z_2)]\{1-\theta+\theta\overline{V}[\varphi_2(s,z_2)]\}+\alpha\frac{1-\overline{B}_2[\varphi_1(s,z_2)]}{\varphi_1(s,z_2)}$$
$$\times\overline{B}_3[\varphi_1(s,z_2)]\overline{R}[\varphi_2(s,z_2)].$$

## III. STEADY-STATE ANALYSIS: LIMITING BEHAVIOUR

By using the property,

$$\lim_{s\to 0}s\overline{f}(s)=\lim_{t\to\infty}f(t).$$

to the equations from (26) to (31), we obtain the steady-state solutions of this model. The steady-state probability for the high priority and low priority units with vacation, working breakdown, repair and retrial time are given by

$$I_0(z_2)=I_0(0,z_2)[\frac{1-\overline{I}[\varphi(a)]}{\varphi(a)}], \tag{48}$$
$$P^{(1)}(z_1,z_2)=P^{(1)}(0,z_1,z_2)[\frac{1-\overline{B}_1[\varphi_1(z_1,z_2)]}{\varphi_1(z_1,z_2)}], \tag{49}$$
$$V(z_1,z_2)=\theta\{P^{(1)}(0,z_2)\overline{B}_1[\varphi_1(z_2)]+P_0^{(2)}(0,z_2)\overline{B}_2[\varphi_2(z_2)]\}[\frac{1-\overline{V}[\varphi_2(z_1,z_2)]}{\varphi_2(z_1,z_2)}], \tag{50}$$
$$P^{(2)}(z_1,z_2)=P_0^{(2)}(0,z_2)[\frac{1-\overline{B}_2[\varphi_1(z_1,z_2)]}{\varphi_1(z_1,z_2)}], \tag{51}$$
$$W(z_1,z_2)=\alpha\{P^{(1)}(z_1,z_2)[\frac{1-\overline{B}_1(\varphi_1(z_1,z_2))}{\varphi_1(z_1,z_2)}]+P^{(2)}(z_1,z_2)[\frac{1-\overline{B}_2[\varphi_1(z_1,z_2)]}{\varphi_1(z_1,z_2)}]\}$$
$$\times[\frac{1-\overline{B}_3[\varphi_2(z_1,z_2)]}{\varphi_2(z_1,z_2)}], \tag{52}$$
$$R(z_1,z_2)=\alpha\{P^{(1)}(z_1,z_2)[\frac{1-\overline{B}_1(\varphi_1(z_1,z_2))}{\varphi_1(z_1,z_2)}]+P^{(2)}(z_1,z_2)[\frac{1-\overline{B}_2[\varphi_1(z_1,z_2)]}{\varphi_1(z_1,z_2)}]\}$$

$$\times \overline{B}_3[\varphi_2(z_1,z_2)][\frac{1-\overline{R}[\varphi_2(z_1,z_2)]}{\varphi_2(z_1,z_2)}]. \tag{53}$$

Let $P_q(z_1,z_2)$ be the probability generating function of the queue size irrespective of the state of the system and combining equations from (48) to (53), we obtain,

$$P_q(z_1,z_2) = P^{(1)}(z_1,z_2) + V(z_1,z_2) + P^{(2)}(z_1,z_2) + W(z_1,z_2) + R(z_1,z_2) + I_0(z_2) \tag{54}$$

Using the normalization condition,

$$P_q(1,1) + I_{0,0} = 1.$$

we obtain $I_{0,0}$ as,

$$I_{0,0} = \frac{\{\quad \alpha(\lambda_1+\lambda_2)(\lambda_1+\lambda_2-\xi)\zeta_4(1,1)\}}{Dr}. \tag{55}$$

And the utilization factor is,

$$\rho = \frac{\left\{\begin{array}{l} P_0^{(2)}(0,1)\{(\overline{B}_2(\alpha))\{(\lambda_1+\lambda_2-\xi)+\alpha(E(W)+E(R))\}\zeta_4(1,1)(\lambda_1+\lambda_2) \\ +\alpha\theta E(V)(\lambda_1+\lambda_2)\{\overline{B}_2(\lambda_1+\alpha)\zeta_4(1,1)+\overline{B}_1(\lambda_1+\alpha)\zeta_3(1,1)\}\}-4\alpha\theta\xi \\ \times I_0(0,1)[1-\overline{I}[\varphi(\alpha)]]\overline{B}_1(\lambda_1+\alpha)E(V^2)+P^{(1)}(0,1,1)(1-\overline{B}_1(\alpha)) \\ \times\zeta_4(1,1)(\lambda_1+\lambda_2)\{(\lambda_1+\lambda_2-\xi)+\alpha(E(W)+E(R))\} \end{array}\right\}}{Dr}, \tag{56}$$

where

$$\begin{aligned} Dr &= \alpha(\lambda_1+\lambda_2)(\lambda_1+\lambda_2-\xi)\zeta_4(1,1) + P_0^{(2)}(0,1)\{(\overline{B}_2(\alpha))\{(\lambda_1+\lambda_2-\xi) \\ &+\alpha(E(W)+E(R))\}\zeta_4(1,1)(\lambda_1+\lambda_2) + \alpha\theta E(V)(\lambda_1+\lambda_2)\{\overline{B}_2(\lambda_1+\alpha) \\ &\times \zeta_4(1,1) + \overline{B}_1(\lambda_1+\alpha)\zeta_3(1,1)\}\} - 4\alpha\theta\xi I_0(0,1)[1-\overline{I}[\varphi(\alpha)]] \\ &\times \overline{B}_1(\lambda_1+\alpha)E(V^2) + P^{(1)}(0,1,1)(1-\overline{B}_1(\alpha))\zeta_4(1,1)(\lambda_1+\lambda_2) \\ &\times \{(\lambda_1+\lambda_2-\xi)+\alpha(E(W)+E(R))\}. \end{aligned}$$

and $\rho < 1$ is the stability condition under which steady state exists, for the model studied.

## IV. CONCLUSION

We have analysed an $M_1, M_2/G_1, G_2/1$ retrial queue with constant retrial policy and non-pre-emptive priority services under modified Bernoulli vacation subject to working breakdown and repair process are also investigated. In addition, the effect of impatient behaviour of the unit on a service system is studied. The joint steady-state probability generating functions of the server state and the stability condition are derived.

## REFERENCES


[1] Bo Keun Kim, Doo Ho Lee (2016), The M/G/1 queue with disasters and working breakdowns, *Applied Mathematical Modelling*, Vol. 4,437 – 459.

[2] Cheng-Dar Liou (2013), Markovian queue optimisation analysis with an unreliable server subject to working breakdowns and impatient customers, *International Journal of Systems Science* ,Vol. 46(12), 2165 – 2182.

[3] Dong-Yuh Yang, Ying-Yi Wu (2017), Analysis of a finite-capacity system with working breakdowns and retention of impatient customers, Journal of Manufacturing Systems, Vol. 44 207 – 216.

[4] Kailash C. Madan (2011), A Non-Preemptive Priority Queueing System with a Single Server Serving Two Queues M/G/1 and M/D/1 with Optional Server Vacations Based on Exhaustive Service of the Priority Units, *Applied Mathematics*, Vol. 2, 791-799.

[5] Kalidass K, Kasturi R (2012), A queue with working breakdowns, *Computers and Industrial Engineering*, Vol. 63, 779-783.

[6] Boutarfa L, Djellab N (2015), On the performance of the $M_1, M_2/G_1, G_2/1$ retrial queue with pre-emptive resume policy, *Yugoslav Journal of Operations Research*, Vol. 25(1), 153-164.

[7] Phung-Duc, T. (2019), Retrial Queueing Models: A Survey on Theory and Applications, *arXiv 2019, arXiv:1906.09560v1*.

[8] Tao Li, Liyuan Zhang (2017), An M/G/1 Retrial G-Queue with General Retrial Times and Working Breakdowns, *Math. Comput. Appl.* Vol. 22, 15.

[9] Zaiming liu, Yang Song (2014), The $M^X/M/1$ queue with working breakdown, *Rairo operations research*, Vol. 48, 339 – 413.