The Existence of Oscillatory Solutions for a Coupled Complex-Valued Wilson-Cowan Neural Network Model with Delays

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Abstract: In this paper, a coupled complex-valued Wilson-Cowan neural network model with delays is investigated. By means of mathematical analysis method, some sufficient conditions to guarantee the existence of oscillatory solutions for the model are provided. Computer simulation is given to demonstrate the correctness of the criterion.

Keywords: complex-valued Wilson-Cowan network, delay, instability, oscillation.

1 Introduction

It is known that one can solve the XOR problem or the detection of symmetry problem by means of a complex-valued neural network which cannot be solved with a single real-valued neuron [1, 2]. Recently, various properties of complex-valued neural networks with delays have attracted great attention of many researchers [3-20]. For example, the delay-dependent sufficient conditions have been derived to guarantee the asymptotical stability of considered uncertain switched complex-valued neural networks based on suitable Lyapunov-Krasovskii functional [3]. A complex generalized Ito's formula to study the stability of complex-valued stochastic networks with Markovian switching on complex domain has been provided, which avoids separating the real and imaginary parts [4]. By constructing proper Lyapunov-Krasovskii functionals and using inequality techniques, some delay-dependent sufficient conditions in linear matrix inequality form were proposed to ascertain the global exponential convergence of the neural networks with two classes of complex-valued activation functions [5]. Criteria on uniqueness and global exponential stability of equilibrium point have been established for some impulsive complex-valued neural networks with time delay by using Lyapunov function method [6]. The existence, uniqueness, and globally asymptotical stability of the equilibrium point of complex-valued systems have been studied by separating complex-valued neural networks into real and imaginary parts, and constructing appropriate Lyapunov functional [7,8]. It is envisaged that the investigations of complex-valued neural networks not only included the properties of stability analysis but also other dynamical characteristics such as bifurcation, chaos and periodic solution. Noting that numerous results on the literature have mainly focused on stability when compared to periodic oscillatory behavior for any n dimensional systems. Recently, Ji *et al*. have considered a delayed complex-valued Wilson-Cowan neural network model as follows [13]:

$$\begin{cases} z_1'(t) = -z_1(t) + a_1 f(z_1(t)) + a_2 f(z_2(t-\tau)), \\ z_2'(t) = -z_2(t) + a_3 f(z_1(t-\tau)) + a_4 f(z_2(t)). \end{cases} \tag{1}$$

where $z_i(t), a_i$ are complex numbers, $f(z_i)$ are complex activation functions. By means of coordinate transformation $z = x + iy, a_i = a_i^R + ia_i^I$ and $f = f_R + if_I$, system (1) can be transformed into an equivalent system

$$\begin{cases} x_1' = -x_1 + a_1^R f_R(x_1, y_1) - a_1^I f_I(x_1, y_1) + a_2^R f_R(x_2(t\tau), y_2(t\tau)) - a_2^I f_I(x_2(t\tau), y_2(t\tau)), \\ y_1' = -y_1 + a_1^R f_I(x_1, y_1) + a_1^I f_R(x_1, y_1) + a_2^R f_I(x_2(t\tau), y_2(t\tau)) + a_2^I f_R(x_2(t\tau), y_2(t\tau)), \\ x_2' = -x_2 + a_3^R f_R(x_1(t\tau), y_1(t\tau)) - a_3^I f_I(x_1(t\tau), y_1(t\tau)) + a_4^R f_R(x_2, y_2) - a_4^I f_I(x_2, y_2), \\ y_2' = -y_2 + a_3^R f_I(x_1(t\tau), y_1(t\tau)) + a_3^I f_R(x_1(t\tau), y_1(t\tau)) + a_4^R f_I(x_2, y_2) + a_4^I f_R(x_2, y_2). \end{cases} \tag{2}$$

where $x_i(t\tau) = x_i(t-\tau), y_i(t\tau) = y_i(t-\tau), i = 1, 2$. For system (2), the sufficient conditions for Hopf bifurcation and its directions are provided through normal form theory and central manifold theorem. In this paper, we extend model (1) into a coupled complex-valued Wilson-Cowan neural network model with delays as the following [21]:

$$\begin{cases} z_1' = -z_1 - a_1 f(z_2(t-\tau_1)) + a_4 f(z_1(t-\tau_2)), \\ z_2' = -z_2 + a_2 f(z_1(t-\tau_1)) - a_3 f(z_4(t-\tau_2)) - a_5 f(z_2(t-\tau_2)), \\ z_3' = -z_3 - a_1 f(z_4(t-\tau_1)) + a_4 f(z_3(t-\tau_2)), \\ z_4' = -z_4 + a_2 f(z_3(t-\tau_1)) - a_3 f(z_2(t-\tau_1)) - a_5 f(z_4(t-\tau_2)). \end{cases} \tag{3}$$

Let $z_i(t) = x_i(t) + iy_i(t), a_i = a_i^R + ia_i^I, f(z_i) = f_R(x_i) + if_I(y_i)(i = 1, 2, 3, 4)$, by taking the real and imaginary parts from (3), we have an equivalent system as follows:

$$\begin{cases} x_1'(t) = -x_1(t) - a_1^R f_R(x_2(t-\tau_1)) + a_1^I f_I(y_2(t-\tau_1)) + a_4^R f_R(x_1(t-\tau_2)) \\ \qquad\qquad -a_4^I f_I(y_1(t-\tau_2)), \\ y_1'(t) = -y_1(t) - a_1^I f_R(x_2(t-\tau_1)) - a_1^R f_I(y_2(t-\tau_1)) + a_4^I f_R(x_1(t-\tau_2)) \\ \qquad\qquad +a_4^R f_I(y_1(t-\tau_2)), \\ x_2'(t) = -x_2(t) + a_2^R f_R(x_1(t-\tau_1)) - a_2^I f_I(y_1(t-\tau_1)) - a_3^R f_R(x_4(t-\tau_2)) \\ \qquad\qquad +a_3^I f_I(y_4(t-\tau_2)) - a_5^R f_R(x_2(t-\tau_2)) + a_5^I f_I(y_2(t-\tau_2)), \\ y_2'(t) = -y_2(t) + a_2^I f_R(x_1(t-\tau_1)) + a_2^R f_I(y_1(t-\tau_1)) - a_3^I f_R(x_4(t-\tau_2)) \\ \qquad\qquad -a_3^R f_I(y_4(t-\tau_2)) - a_5^I f_R(x_2(t-\tau_2)) - a_5^R f_I(y_2(t-\tau_2)), \\ x_3'(t) = -x_3(t) - a_1^R f_R(x_4(t-\tau_1)) + a_1^I f_I(y_4(t-\tau_1)) + a_4^R f_R(x_3(t-\tau_2)) \\ \qquad\qquad -a_4^I f_I(y_3(t-\tau_2)), \\ y_3'(t) = -y_3(t) - a_1^I f_R(x_4(t-\tau_1)) - a_1^R f_I(y_4(t-\tau_1)) + a_4^I f_R(x_3(t-\tau_2)) \\ \qquad\qquad +a_4^R f_I(y_3(t-\tau_2)), \\ x_4'(t) = -x_4(t) + a_2^R f_R(x_3(t-\tau_1)) - a_2^I f_I(y_3(t-\tau_1)) - a_3^R f_R(x_2(t-\tau_1)) \\ \qquad\qquad +a_3^I f_I(y_2(t-\tau_1)) - a_5^R f_R(x_4(t-\tau_2)) + a_5^I f_I(y_4(t-\tau_2)), \\ y_4'(t) = -y_4(t) + a_2^I f_R(x_3(t-\tau_1)) + a_2^R f_I(y_3(t-\tau_1)) - a_3^I f_R(x_2(t-\tau_1)) \\ \qquad\qquad -a_3^R f_I(y_2(t-\tau_1)) - a_5^I f_R(x_4(t-\tau_2)) - a_5^R f_I(y_4(t-\tau_2)). \end{cases} \tag{4}$$

Clearly, system (4) is a real differential equation with two delays. Assume that $f_R(0) = f_I(0) = 0, xf_R(x) > 0(x \neq 0), yf_I(y) > 0(y \neq 0)$. Both f_R and f_I are continuous differentiable bounded functions. It is clear that $x_i = 0, y_i = 0(i = 1, 2, 3, 4)$ is an equilibrium point of system (4). The linearized system of (4) at the origin leads to

$$\begin{cases} x_1'(t) = -x_1(t) - a_1^R f_R'(0)x_2(t-\tau_1) + a_1^I f_I'(0)y_2(t-\tau_1) + a_4^R f_R'(0)x_1(t-\tau_2) \\ \qquad\qquad -a_4^I f_I'(0)y_1(t-\tau_2), \\ y_1'(t) = -y_1(t) - a_1^I f_R'(0)x_2(t-\tau_1) - a_1^R f_I'(0)y_2(t-\tau_1) + a_4^I f_R'(0)x_1(t-\tau_2) \\ \qquad\qquad +a_4^R f_I'(0)y_1(t-\tau_2), \\ x_2'(t) = -x_2(t) + a_2^R f_R'(0)x_1(t-\tau_1) - a_2^I f_I'(0)y_1(t-\tau_1) - a_3^R f_R'(0)x_4(t-\tau_2) \\ \qquad\qquad +a_3^I f_I'(0)y_4(t-\tau_2) - a_5^R f_R'(0)x_2(t-\tau_2) + a_5^I f_I'(0)y_2(t-\tau_2), \\ y_2'(t) = -y_2(t) + a_2^I f_R'(0)x_1(t-\tau_1) + a_2^R f_I'(0)y_1(t-\tau_1) - a_3^I f_R'(0)x_4(t-\tau_2) \\ \qquad\qquad -a_3^R f_I'(0)y_4(t-\tau_2) - a_5^I f_R(0)x_2(t-\tau_2) - a_5^R f_I(0)y_2(t-\tau_2), \\ x_3'(t) = -x_3(t) - a_1^R f_R'(0)x_4(t-\tau_1) + a_1^I f_I'(0)y_4(t-\tau_1) + a_4^R f_R'(0)x_3(t-\tau_2) \\ \qquad\qquad -a_4^I f_I'(0)y_3(t-\tau_2), \\ y_3'(t) = -y_3(t) - a_1^I f_R'(0)x_4(t-\tau_1) - a_1^R f_I'(0)y_4(t-\tau_1) + a_4^I f_R'(0)x_3(t-\tau_2) \\ \qquad\qquad +a_4^R f_I'(0)y_3(t-\tau_2), \\ x_4'(t) = -x_4(t) + a_2^R f_R'(0)x_3(t-\tau_1) - a_2^I f_I'(0)y_3(t-\tau_1) - a_3^R f_R'(0)x_2(t-\tau_1) \\ \qquad\qquad +a_3^I f_I'(0)y_2(t-\tau_1) - a_5^R f_R'(0)x_4(t-\tau_2) + a_5^I f_I'(0)y_4(t-\tau_2), \\ y_4'(t) = -y_4(t) + a_2^I f_R'(0)x_3(t-\tau_1) + a_2^R f_I'(0)y_3(t-\tau_1) - a_3^I f_R'(0)x_2(t-\tau_1) \\ \qquad\qquad -a_3^R f_I'(0)y_2(t-\tau_1) - a_5^I f_R'(0)x_4(t-\tau_2) - a_5^R f_I'(0)y_4(t-\tau_2). \end{cases} \tag{5}$$

For convenience, set $b_i = a_i^R f_R'(0), c_i = a_i^I f_I'(0), i = 1, 2, \cdots, 4$. Then system (5) is equivalent to the following:

$$\begin{cases} x_1'(t) = -x_1(t) - b_1x_2(t-\tau_1) + c_1y_2(t-\tau_1) + b_4x_1(t-\tau_2) - c_4y_1(t-\tau_2), \\ y_1'(t) = -y_1(t) - c_1x_2(t-\tau_1) - b_1y_2(t-\tau_1) + c_4x_1(t-\tau_2) + b_4y_1(t-\tau_2), \\ x_2'(t) = -x_2(t) + b_2x_1(t-\tau_1) - c_2y_1(t-\tau_1) - b_3x_4(t-\tau_2) + c_3y_4(t-\tau_2) \\ \qquad\qquad -b_5x_2(t-\tau_2) + c_5y_2(t-\tau_2), \\ y_2'(t) = -y_2(t) + c_2x_1(t-\tau_1) + b_2y_1(t-\tau_1) - c_3x_4(t-\tau_2) - b_3y_4(t-\tau_2) \\ \qquad\qquad -c_5x_2(t-\tau_2) - b_5y_2(t-\tau_2), \\ x_3'(t) = -x_3(t) - b_1x_4(t-\tau_1) + c_1y_4(t-\tau_1) + b_4x_3(t-\tau_2) - c_4y_3(t-\tau_2), \\ y_3'(t) = -y_3(t) - c_1x_4(t-\tau_1) - b_1y_4(t-\tau_1) + c_4x_3(t-\tau_2) + b_4y_3(t-\tau_2), \\ x_4'(t) = -x_4(t) + b_2x_3(t-\tau_1) - c_2y_3(t-\tau_1) - b_3x_2(t-\tau_1) + c_3y_2(t-\tau_1) \\ \qquad\qquad -b_5x_4(t-\tau_2) + c_5y_4(t-\tau_2), \\ y_4'(t) = -y_4(t) + c_2x_3(t-\tau_1) + b_2y_3(t-\tau_1) - c_3x_2(t-\tau_1) - b_3y_2(t-\tau_1) \\ \qquad\qquad -c_5x_4(t-\tau_2) - b_5y_4(t-\tau_2). \end{cases} \tag{6}$$

The matrix form of system (6) is the follows:

$$U'(t) = PU(t) + QU(t-\tau_1) + RU(t-\tau_2) \tag{7}$$

where $U(t) = (x_1(t), y_1(t), \cdots, x_4(t), y_4(t))^T, U(t-\tau_i) = (x_1(t-\tau_i), y_1(t-\tau_i), \cdots, x_4(t-\tau_i), y_4(t-\tau_i))^T, i = 1, 2$. $P = (p_{ij})_{8\times 8} = diag(-1, -1, \cdots, -1)$, both $Q = (q_{ij})_{8\times 8}$ and $R = (r_{ij})_{8\times 8}$ are 8 by

8 matrices.

Based on the generalized Chafee's criterion [22, 23]: A time delay system which has a unique unstable equilibrium point, and all solutions of the system are bounded will generate a limit cycle, namely, a permanent oscillatory solution. In this paper we will discuss the existence of permanent oscillatory solutions for system (4) by means of the generalized Chafee's criterion.

2 Preliminaries

In this paper we adopt the following norms of vectors and matrices [24]: For a matrix $A = (a_{ij})_{n\times n}$, the norm $\|A\| = \max_{1\le j\le n}\sum_{i=1}^{n}|a_{ij}|$, the measure $\mu(A)$ is defined by $\mu(A) = \lim_{\theta\to 0^+}\frac{\|I+\theta A\|-1}{\theta}$, which for the chosen norms reduces to $\mu(A) = \max_{1\le j\le n}[a_{jj} + \sum_{i=1,i\ne j}^{n}|a_{ij}|]$.

Definition 1 The trivial solution of system (4) is called unstable if there exists at least one component which is unstable.

Lemma 1 Assume that the matrix $M(= P + Q + R)$ is a nonsingular matrix and $Q + R$ is not a positive definite matrix, then system (6)(or (7)) has a unique equilibrium, implying that system (4) has a unique equilibrium.

Proof Obviously, the zero point is an equilibrium of system (6)(or (7)). If system (6)(or (7)) has another nonzero equilibrium, say $u^* = [x_1^*, y_1^*, \cdots, x_4^*, y_4^*]^T$, then we have the following algebraic equation

$$Pu^* + Qu^* + Ru^* = (P + Q + R)u^* = Mu^* = 0. \tag{8}$$

According to the basic result of linear algebraic, if M is a singular matrix, equation (8) has only one solution, namely, the trivial solution, which contradicts u^* is a nonzero equilibrium. In other words, system (6)(or (7)) has a unique zero equilibrium. Since $xf_R(x) > 0(x \ne 0), yf_I(y) > 0(y \ne 0)$ and $Q + R$ is not a positive definite matrix, implying that system (4) has a unique equilibrium point.

Lemma 2 All solutions of system (4) are bounded.

Proof Noting that f_R and f_I are bounded continuous functions. Let $N_1 = |-a_1^R f_R(x_2(t-\tau_1)) + a_1^I f_I(y_2(t-\tau_1)) + a_4^R f_R(x_1(t-\tau_2)) - a_4^I f_I(y_1(t-\tau_2))|, N_2 = |-a_1^I f_R(x_2(t-\tau_1)) - a_1^R f_I(y_2(t-\tau_1)) + a_4^I f_R(x_1(t-\tau_2)) + a_4^R f_I(y_2(t-\tau_1))|, \cdots, N_8 = |a_2^I f_R(x_3(t-\tau_1)) + a_2^R f_I(y_3(t-\tau_1)) - a_3^I f_R(x_2(t-\tau_1)) - a_3^R f_I(y_2(t-\tau_1)) - a_5^I f_R(x_4(t-\tau_2)) - a_5^R f_I(y_4(t-\tau_2))|$. Then from (4) we have

$$\begin{cases} x_1'(t) = -x_1(t) + N_1, \\ y_1'(t) = -y_1(t) + N_2, \\ x_2'(t) = -x_2(t) + N_3, \\ y_2'(t) = -y_2(t) + N_4, \\ x_3'(t) = -x_3(t) + N_5, \\ y_3'(t) = -y_3(t) + N_6, \\ x_4'(t) = -x_4(t) + N_7, \\ y_4'(t) = -y_4(t) + N_8. \end{cases} \tag{9}$$

This leads that $|x_i(t)| \leq N, |y_i(t)| \leq N, (i = 1, 2, N = \max\{N_1, N_2, \cdots, N_8\}$. In other words, all solutions are bounded in system (4).

3 Existence of oscillatory solutions

Obviously, the instability of trivial solution of system (6) (or (7)) implies that the trivial solution of system (4) is unstable. Therefore, in the following we only consider the instability of trivial solution of system (6) (or (7)).

Theorem 1 Assume that system (6) (or (7)) has a unique equilibrium point for selected parameters. If the following condition holds

$$\parallel Q + R \parallel e\tau > e^{\tau}. \tag{10}$$

where $\tau = \min\{\tau_1, \tau_2\}$. Then the unique equilibrium point of system (6) (or (7)) is unstable, implying that system (4) generates a limit cycle, namely, a permanent oscillatory solution.

Proof We shall prove that the unique equilibrium point of system (6) (or (7)) which is exactly the zero point is unstable. Consider an auxiliary system of system (7) as follows:

$$U'(t) = PU(t) + (Q + R)U(t - \tau) \tag{11}$$

where $\tau = \min\{\tau_1, \tau_2\}$. Based on the theory of functional differential equation, if the trivial solution of system (11) is unstable, then the trivial solution of system (7) is unstable [25]. From (11) we get

$$\begin{cases} \frac{d|x_i(t)|}{dt} = -|x_i(t)| + \parallel Q + R \parallel |x_i(t - \tau)|, \\ \frac{d|y_i(t)|}{dt} = -|y_i(t)| + \parallel Q + R \parallel |y_i(t - \tau)| \ (i = 1, \cdots, 4). \end{cases} \tag{12}$$

If the unique equilibrium point of system (11) is stable, then the characteristic equation associated with (12) given by

$$\lambda = -1 + \parallel Q + R \parallel e^{-\lambda\tau} \tag{13}$$

will have a real negative root say λ_0, and we have from (13)

$$|\lambda_0| + 1 \geq \parallel Q + R \parallel e^{|\lambda_0|\tau}. \tag{14}$$

Using the formula $e^x/x \geq e$ for $x \geq 0$ one can get

$$1 \geq \frac{\parallel Q + R \parallel e^{|\lambda_0|\tau}}{1 + |\lambda_0|} = \frac{\parallel Q + R \parallel \tau e^{-\tau} e^{(1+|\lambda_0|)\tau}}{(1 + |\lambda_0|)\tau} \geq (\parallel Q + R \parallel e\tau)e^{-\tau} \tag{15}$$

The last inequality contradicts the equation (10). Hence, our claim regarding the instability of the equilibrium point of system (11) is valid, implying that the trivial solution of system (7) is unstable. This means that there exists at least one $x_i(t)$ or $y_i(t), i \in \{1, \cdots, 4\}$ is unstable. According to the definition 1, the instability of the component $x_i(t)$ or $y_i(t)$ implies that the trivial solution of (4) is unstable. Since all solutions of system (4) are bounded, the instability of the unique equilibrium point together with the boundedness of the solutions lead system (4) to generate a limit cycle, namely, a permanent oscillatory solution.

Theorem 2 Assume that system (4) has a unique equilibrium point for selected parameters. Let $\alpha_1, \alpha_2, \cdots, \alpha_8$ represent the eigenvalues of matrix $Q + R$. Assume that there is at least one eigenvalue, say α_j which has a positive real part $Re(\alpha_j) > 1$, satisfying that

$$Re(\alpha_j)\cos(\omega\tau) - Im(\alpha_j)\sin(\omega\tau) > 1. \tag{16}$$

where ω is a parameter. Then the unique equilibrium point of system (6) (or(7)) is unstable, which implies that system (4) generates a limit cycle.

Proof Consider system (11), the characteristic equation of system (11) is the follows:

$$det(\lambda I_{ij} + p_{ij} - (q_{ij} + r_{ij})e^{-\lambda\tau}) = 0 \tag{17}$$

where I_{ij} is an identity matrix. Since the eigenvalues of matrix $Q+R$ are $\alpha_1, \alpha_2, \cdots, \alpha_8$, so equation (17) changes to the following

$$\prod_{i=1}^{8}(\lambda + 1 - \alpha_i e^{-\lambda\tau}) = 0 \tag{18}$$

We are led to an investigation of the nature of the roots of the equation:

$$\lambda + 1 = \alpha_i e^{-\lambda\tau}, i = 1, 2, \cdots, 8. \tag{19}$$

Without loss of generality, assume that α_1 is a complex number which has a positive real part $Re(\alpha_1) > 1$, then we have

$$\lambda + 1 = \alpha_1 e^{-\lambda\tau}. \tag{20}$$

Assume that $\lambda = \sigma + i\omega$, $\alpha_1 = \alpha_{11} + i\alpha_{12}$, where $\sigma = Re(\lambda), \omega = Im(\lambda), \alpha_{11} = Re(\alpha_1), \alpha_{12} = Im(\alpha_1)$. From (19) we get

$$\sigma + i\omega + 1 = (\alpha_{11} + i\alpha_{12})e^{-(\sigma+i\omega)\tau}. \tag{21}$$

Separating the real and imaginary parts, we have

$$\sigma + 1 = \alpha_{11}e^{-\sigma\tau}\cos(\omega\tau) - \alpha_{12}e^{-\sigma\tau}\sin(\omega\tau). \tag{22}$$

We show that equation (22) has a positive real root. Let

$$f(\sigma) = \sigma + 1 - \alpha_{11}e^{-\sigma\tau}\cos(\omega\tau) + \alpha_{12}e^{-\sigma\tau}\sin(\omega\tau). \tag{23}$$

Thus $f(\sigma)$ is a continuous function of σ. Based on condition (16) we have $f(0) = 1 - \alpha_{11}\cos(\omega\tau) + \alpha_{12}\sin(\omega\tau) = 1 - (\alpha_{11}\cos(\omega\tau) - \alpha_{12}\sin(\omega\tau)) < 0$. Obviously, there exists a suitably large $\tilde{\sigma}(> 0)$ such that $f(\tilde{\sigma}) = \tilde{\sigma} + 1 - \alpha_{11}e^{-\tilde{\sigma}\tau}\cos(\omega\tau) + \alpha_{12}e^{-\tilde{\sigma}\tau}\sin(\omega\tau) > 0$ since $\lim_{\sigma\to+\infty} e^{-\sigma\tau} = 0$. By means of the Intermediate Value Theorem of continuous function, there exists a $\bar{\sigma} \in (0, \tilde{\sigma})$ such that $f(\bar{\sigma}) = \bar{\sigma} + 1 - \alpha_{11}e^{-\bar{\sigma}\tau}\cos(\omega\tau) + \alpha_{12}e^{-\bar{\sigma}\tau}\sin(\omega\tau) = 0$. This means that the characteristic value λ has a positive real part. Therefore, the trivial solution of system (11) is unstable, implying that the trivial solution of system (6) (or (7)) is unstable. Based on the generalized Chafee's criterion, there exists a limit cycle of system (4), namely, a permanent oscillatory solution.

4 Simulation Results

This simulation is based on system (4). We first select the activation function as $f(z_i) = \tanh(x_i) + i\tanh(y_i)(z_i = x_i + iy_i)$, time delays $\tau_1 = 0.8, \tau_2 = 0.5$, the other parameter values as $a_1^R = 1.5, a_1^I = 0.85, a_2^R = 1.25, a_2^I = 1.8,\ a_3^R = 0.5, a_3^I = 1.8,\ a_4^R = 0.2, a_4^I = 1.95, a_5^R = 1.2, a_5^I = 1.45$, the eigenvalues of matrix $Q + R$ are $0.8361 \pm 0.2507i$, $-0.5146 \pm 1.8516i$, $-0.9854 \pm 3.1516i$, $-1.3361 \pm 2.5507i$. Thus, $Q + R$ is not a positive definite matrix. We have $\parallel Q + R \parallel = 5.9$, and $\parallel Q + R \parallel e\tau = 5.9 * e * 0.5 = 8.0189 > 1.6487 = e^{0.5}$. Based on theorem 1, system (4) has an oscillatory solution (see Fig.1). In order to see the effect of time delay, we change delays as $\tau_1 = 1.5, \tau_2 = 1.2$, the other parameters are kept as the above, then $\parallel Q + R \parallel e\tau = 5.9 * e * 1.2 = 19.2454 > 3.3201 = e^{1.2}$. The restrictive conditions of the Theorem 1 are still satisfied. However, the oscillatory frequency and amplitude both are changed (see Fig.2). Then we select the activation function as $f(z_i) = \arctan(x_i) + i\arctan(y_i)(z_i = x_i + iy_i)$, time delays are $\tau_1 = 0.8, \tau_2 = 0.5$, the other parameter values are the same as in figure 1, we see that the dynamic behavior is almost the same as in figure 1. This means that the activation functions such as $\tanh(z)$ and $\arctan(z)$ affect the oscillatory frequency and amplitude slightly (see Fig.3). Then we keep activation function as $\arctan(z)$ and select parameters as $a_1^R = 1.85, a_1^I = 1.15, a_2^R = 0.85, a_2^I = 1.05,\ a_3^R = 1.2, a_3^I = 1.28, a_4^R = 1.24, a_4^I = 1.35, a_5^R = 0.85, a_5^I = 1.75$, time delays as $\tau_1 = 0.85, \tau_2 = 0.75$. In this case, $\tau = \min\{\tau_1, \tau_2\} = 0.75$, the eigenvalues of matrix $Q + R$ are: $1.6129 \pm 0.8500i$, $0.7092 \pm 1.0827i$, $-1.5193 \pm 2.7627i$, $-0.0229 \pm 1.7301i$. We see that the eigenvalue $1.6129 + 0.8500i$ satisfies the condition: $1.6129\cos(0.1 \times 0.75) - 0.8500\sin(0.1 \times 0.75)) = 1.5447 > 1$, where we select the parameter $\omega = 0.1$. Based on Theorem 2, system (4) has an oscillatory solution (see Fig.4).

5 Conclusion

In this paper, we have discussed the existence of permanent oscillatory solutions in a coupled complex-valued Wilson-Cowan neural network model with delays. Some simple criteria to guarantee the existence of oscillatory solutions have been proposed. By taking the real and imaginary parts from the coupled complex-valued Wilson-Cowan neural network model we obtained a set of real differential equations. This allowed to only focus on the instability of a unique equilibrium point for such a system. Some specific numerical simulations have been provided to demonstrate the result.

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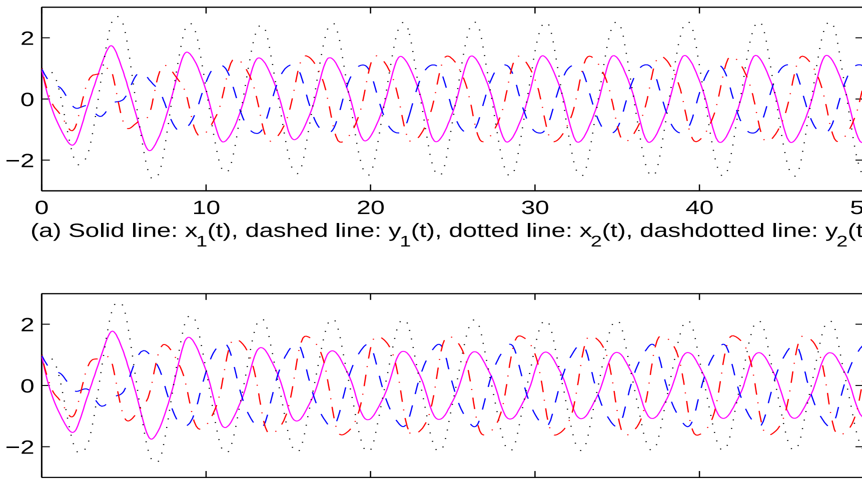
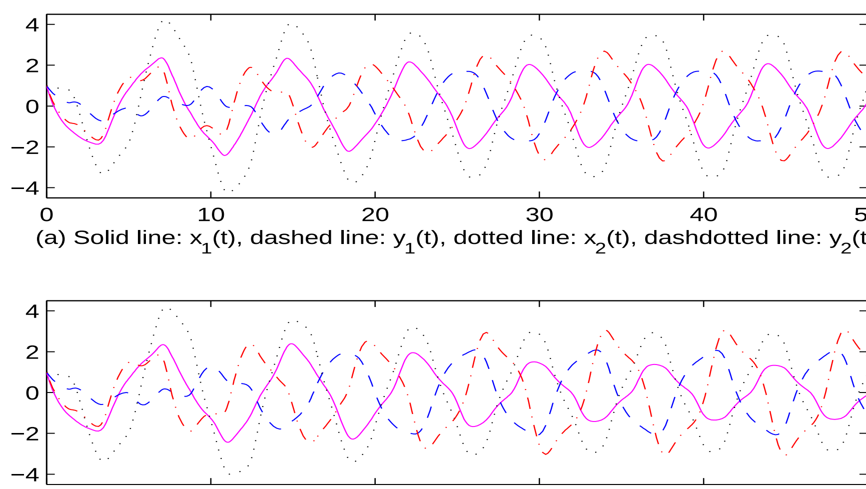
Fig.1 Oscillation of the solutions, delays: (0.8, 0.5), activation function: $\tanh(x_i)$+i $\tanh(y_i)$.



(a) Solid line: $x_1(t)$, dashed line: $y_1(t)$, dotted line: $x_2(t)$, dashdotted line: $y_2(t)$.

(b) Solid line: $x_3(t)$, dashed line: $y_3(t)$, dotted line: $x_4(t)$, dashdotted line: $y_4(t)$.

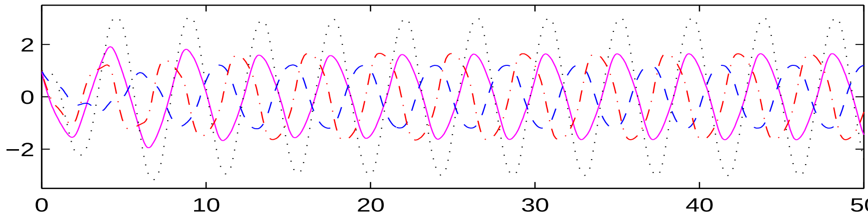
Fig.2 Oscillation of the solutions, delays: (1.5, 1.2), activation function: $\tanh(x_i)$+i $\tanh(y_i)$.



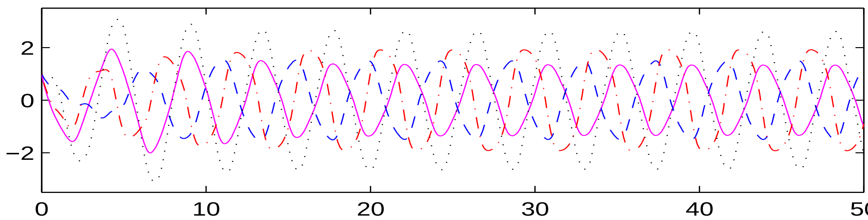
(a) Solid line: $x_1(t)$, dashed line: $y_1(t)$, dotted line: $x_2(t)$, dashdotted line: $y_2(t)$.

(b) Solid line: $x_3(t)$, dashed line: $y_3(t)$, dotted line: $x_4(t)$, dashdotted line: $y_4(t)$.

Fig.3 Oscillation of the solutions, delays: (0.8, 0.5),
activation function: $\arctan(x_i)+i\arctan(y_i)$.

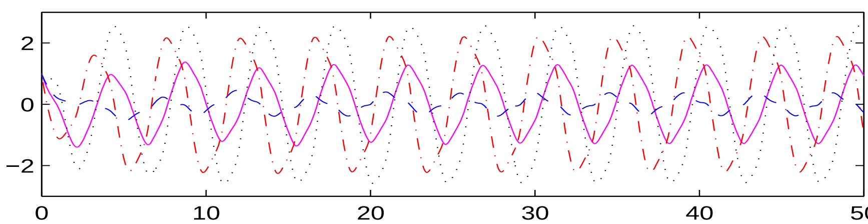


(a) Solid line: $x_1(t)$, dashed line: $y_1(t)$, dotted line: $x_2(t)$, dashdotted line: $y_2(t)$.

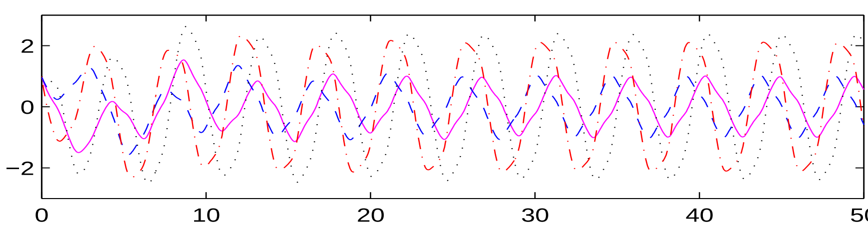


(b) Solid line: $x_3(t)$, dashed line: $y_3(t)$, dotted line: $x_4(t)$, dashdotted line: $y_4(t)$.

Fig.4 Oscillation of the solutions, delays: (0.85, 0.75),
activation function: $\arctan(x_i)+i\arctan(y_i)$.



(a) Solid line: $x_1(t)$, dashed line: $y_1(t)$, dotted line: $x_2(t)$, dashdotted line: $y_2(t)$.



(b) Solid line: $x_3(t)$, dashed line: $y_3(t)$, dotted line: $x_4(t)$, dashdotted line: $y_4(t)$.