A Brief Survey on Finite Time and Fixed Time Synchronization of Complex Dynamical Networks and its applications

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Abstract: *Complex networks have, in recent years, brought many innovative impacts to large-scale systems. However, great challenges also come forth due to distinct complex situations and imperative requirements in human life now a days. This paper attempts to present an overview of recent progress of synchronization of complex dynamical networks and its applications. We focus on Finite-time and Fixed-time synchronization of complex dynamical networks with nonidentical discontinuous nodes, time delay, Class of Output-Coupling via continuous control and Markovian jump complex networks. Then, were view several applications of synchronization in complex networks, especially in neuroscience and power grids. The present limitations are summarized and future trends are explored and tentatively highlighted.*

Keywords: *Complex network, synchronization, finite-time and fixed-time, coupling, time delay, impulsive, adaptive, intermittent, Markovian jump.*

2010 Subject Classification: 34A34, 34A37, 92B20, 34K20, 93D05, 93D09, 93D20.

I. INTRODUCTION

Complex networks have been extensively studied in various fields in recent decades, including biology, chemistry, physics, and mathematics. A complex dynamic network is typically a broad collection of interconnected nodes, in which each node is a nonlinear dynamic system. For example, if there are several interconnected nodes in the neural network, It can also be thought of as complex networks. Many large-scale systems can be modelled by complex dynamic networks (CDNs) with the nodes in the system representing individuals and the edges representing the unique relation between them. Many of these networks display complexity in network nodes and coupled units 'overall topological and dynamic properties. Most natural and technological processes can be modelled as complex networks, which are very common in biological and physical structures such as genetic networks, metabolic pathways, social networks, delivery networks for electrical resources, and the World Wide Web (WWW), etc.

In complex networks one of the most important and fascinating phenomena is synchronization of all dynamic nodes. Synchronisation has been widely studied as the mutual action of complex networks. The synchronization phenomena are growing and significant in real-world networks, for example Internet synchronization, Transmission of digital or analog signals in communication networks and biological neural network synchronisation. Synchronization is unique in nature and can play an extremely important role in many science fields including genetics, climatic science, sociology and ecology. Consequently, study of synchronization of complex networks is essential in both theory and practice.

To analyze a complex network's synchronous behaviours, first, some useful modelling techniques need to be explored. Algebraic graph theory is commonly accepted as one of the most convenient approaches and was widely used to provide a universal analysis model for node-to-node interactions within a network. In particular, an undirected or a directed graph may define the communication topology of a complex network, where a vertex signifies an individual node within a network and an edge denotes a communication connection between nodes. Therefore, an individual system's intrinsic dynamics reflect its evolutionary law and strongly influence the dynamic processes of the entire network. It should also be taken into account in the modelling process. The developed model for a complex dynamic network will therefore concern both the nonlinear dynamic properties of nodes and the exchange of information between nodes.

In recent years, researchers have paid growing attention to the synchronization issue of complex dynamic networks in order to get a deep understanding of the synchronization process and to make good use of the synchronization behaviour. Due to its applications in safe communication and design of signal generators, the synchronization problems in coupled dynamic networks have been extensively investigated in the last few years. Over the past two decades, synchronization of complex networks with identical dynamical topology has been broadly studied in different fields of engineering and sciences with many useful applications in biological systems, secure communication, image processing and so on. For this purpose, several useful methods for the synchronization of complex networks without control [1–4] have been implemented. However, sometimes we cannot achieve the synchronization of network without adding any controller to the dynamics of individual node. Thus, to synchronize the complex networks by designing a suitable controller is seen to be a most significant topic in both theory and application. As a consequence, many useful methods were developed to achieve stability of chaos and synchronization of chaos, such as robust control, adaptive control, pinning control, impulsive control, event-triggered control, finite-time and fixed-time control, sliding mode control, Fuzzy Logic control and intermittent control.

Time delay is often the main cause of instability in the system and poor performance. Time delay is common in practical CDNs, such as the flow of steam and fluid in pipes, and the propagation of electrical signals along long lines. Nowadays the CDNs with time delays have gained more interest, influenced by the impact of time delays.

It is important to note that most of the current works on synchronization problems of complex dynamic networks were asymptotic synchronization [5-11], or exponential synchronization [12-13], which indicated that synchronization could only be realized in infinite time. The authors of [5], having found pinning power, studied the synchronization problem of general complex networks. The synchronization with impulses and time-varying delays of an array of linearly coupled memristor-based recurrent neural networks was investigated in [6]. Pinning control strategy was used in [9] to study the synchronization problems of linearly coupled complex networks in the clusters. In [12], the complex-valued dynamic networks 'exponential synchronization problems with time-varying delays and stochastic disruptions through time-delayed impulsive control and coupled stochastic memristor-based NNs with coupling, time-varying and impulsive delays in [13]. In practice, however, we still anticipate the synchronization to be accomplished as soon as possible, suggesting that synchronization can be realized by the system within a defined time span. This can be accomplished by the use of the finite-time synchronization strategy proposed in [14] that has been demonstrated with many remarkable advantages including improved robustness, higher precision and faster convergence rate, etc. The authors of [15], studied the results on cluster finite-time synchronization of coupled CNs via adaptive control were presented. The authors of [16] and [17] considered the finite-time synchronization of complex dynamical networks by periodically intermittent control. In [18], by employing pinning control the finite-time cluster synchronization problem for CNs were discussed. Nonetheless, in the above tests, the settlement time of finite-time synchronization depends heavily on the initial conditions of all subsystems, but in certain cases the initial state information or experience is uncertain or not available in advance. Therefore, if the initial conditions are very large, the settling time must be high enough. In fact, these restrictions can somewhat limit its broad application.

Here, we are trying to present a survey of recent major findings in finite-time and fixed-time synchronization of complex networks. Although it seems difficult to cover all the contributions, we are committed to explaining clear lines of research and helping to categorize problems and methodologies. The survey is structured according to the following. We mentioned the basic meanings and properties relating to SCDNs in section 2.Sections 3.1 – 3.5 provide an overview of the final and fixed time synchronization of complex networks with non-identical discontinuous nodes, synchronization of complex dynamic networks with time delay, synchronization of coupling networks through discontinuous controllers and finite-time synchronization of Markovi Section 4 focuses on the synchronization applications in complex networks, ranging from cancer therapy and power grids to neuroscience. The paper ends with a summary of the latest hot topics and possible directions of study in Section 5.

Notation: All the notements used in this paper are fairly ordinary. All along the book, R^n and $R^{n\times n}$ represent the *n*-dimensional Euclidean space and the space of $n\times n$ matrices, respectively. Superscript "T" is applied to express the transpose of a vector or a matrix. I_n is an $n\times n$ identity matrix, while $0_{n\times n}$ is an $n\times n$ zero matrix and 0_n is an *n*-dimensional column vector with all entries equal to 0. $|\cdot|$ refers to the absolute value of a real number, $\|\cdot\|$ represents the Euclidean norm in R^n , $\otimes$ stands for the Kronecker product, and *diag* $(x_1,..., x_n)$ serves as a diagonal matrix whose diagonal entries are $x_1,..., x_n$ successively. Denote *min* x_i and *max* x_i as the minimum and maximum value of variable x_i for all *i*, and let λ_{min} (A) and λ_{max} (A) be the minimal and maximal Eigenvalue of matrix A, respectively.

II. PRELIMINARIES

In this section we remember some concepts and synchronization properties of complex dynamic networks, which will be used throughout the paper.

Definition 1: The complex network $x'_i(t)$ Is said to be in alignment on $z'(t)$ within a finite time if, by adding a suitable designed controller $u_i(t)$ to system $x'_i(t)$, If there is a time of settlement $T > 0$ depending on the initial value $x(0)$, such that

$$\lim_{t\to t_1}\|c(t)\| = 0 \text{ and } \|c(t)\| \equiv 0,\ \forall t \geq T,\quad i \in \square$$

where $c(t) = x_i(t) - z(t)$ is the solution of error dynamical system.

Definition 2: The complex network $x'_i(t)$ is said to be synchronized onto $z'(t)$ within a fixed time if, by adding a suitable designed controller $u_i(t)$ to system $x'_i(t)$, if there exists a settling time $T > 0$ which is independent on the initial value, such that

$$\|c(t)\| \equiv 0,\ \forall t \geq T,$$

where $c(t) = x_i(t) - z(t)$ is the solution of error dynamical system.

Definition 3: The Filippov set-valued map of $f(x)$ at $x \in R^n$ is defined as follows:

$$f(x) = \bigcap_{\delta>0}\bigcap_{\mu(N)=0} \overline{co}\left[f\left(\beta(x,\delta)\setminus\Omega\right)\right],$$

where $\overline{co}[E]$ is the closure of the convex hull of the set E, $\beta(x,\delta) = \{y : \|y - x\| \leq \delta\}$, and $\mu(\Omega)$ is the Lebesgue measure of set Ω.

Definition 4: The set valued map $F : R^n \to R^n$ is said to satisfy the basic conditions in the domain $W \in R^n$, if for any $x \in W$, $F(x)$ is nonempty, bounded, closed, and convex, and F is upper semicontinuous in W.

Remark 1: The differential equation of dynamical system $\chi'(t) = f(\chi(t))$ has at least one Filippov solution on R^n, if the basic conditions in Definition 4 are satisfied.

Definition 5: A function $\chi : [0,T) \to R^n$, $T \in (0,\infty)$, is a solution (in the sense of Filippov) of the discontinuous system $\chi'(t) = f(\chi(t))$ on [0, T) if:

(i) $\chi(t)$ is absolutely continuous on [0, T);

(ii) There exists a measurable function $\gamma(t) : [0,T) \to R^n$, such that $\gamma(t) \in f(\chi(t))$ for almost all (a.a.) $t \in [0, T)$ and $\chi'(t) = \gamma(t)$ for a.a. $t \in [0, T)$.

Definition 6: On the probability space $(\Omega, F, \{F_t\}_{t\geq 0}, P)$, let $\{r_t, t \geq 0\}$ is a right-continuous Markov chain taking values in a finite state space $S = \{1, 2, ..., w\}$ with a generator $\Pi = (\pi_{ij})_{w\times w}$ given by:

$$P\{r_{t+\delta t} = j : r_t = i\} = \begin{cases} \pi_{ij}\delta t + o(\delta t), & \text{if } i \neq j \\ 1 + \pi_{ii}\delta t + o(\delta t), & \text{if } i = j \end{cases}$$

where $\delta t > 0$ and $\lim_{\delta t\to 0}\left(\frac{o(\delta t)}{\delta t}\right) = 0.$ Here, $\pi_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\pi_{ii} = -\sum_{j=1, j\neq i}^{w} \pi_{ij}$.

Definition 7: A weighted graph is a triple $G = \{V, E, A\}$, where $V = \{1, 2. . . N\}$ is a vertex set, $E = \{(i, j)\} \subseteq V \times V$ is an edge set, and $A = [a_{ij}]_{N\times N}$ is a weighted adjacency matrix for it.

Remark 2: In Definition 7, each entry a_{ij} of adjacency matrix A is equal to the weight on an edge between two vertices. For a weighted digraph, if there is an edge from vertex i to vertex j, then $a_{ji} > 0$ for $(i, j) \in E$; otherwise $a_{ji} = 0$ for $i \neq j$.

Definition 8: (Laplacian matrix). Given a graph G, its Laplacian matrix $L = [l_{ij}]_{N\times N}$ is defined as below:

$$\begin{cases} l_{ij} = -a_{ij}, & i \neq j \\ l_{ii} = \sum_{k=1, k\neq i}^{N} a_{ik}, & i \in \square \end{cases}$$

Remark 3: From Definition 8, it is clear that $l_{ij} \leq 0$ for $i \neq j$, and $\sum_{j=1}^{N} l_{ij} = 0$ for $i \in \square$.

Definition 9: (Connectivity).
(i) A digraph is strongly connected if there is a directed path between any two vertices.
(ii) A digraph is quasi-strongly connected if at least one root exists.

Definition 10: (M-matrix). A square matrix Z = [z_{ij}] is an M-matrix if $z_{ij} \leq 0$ for $i \neq j$ and $z_{ii} \geq \sum_{j\neq i} |z_{ij}|$; precisely, matrix Z concedes a decomposition $Z = sI - A$, where A is nonnegative and its spectral radius $\rho(A) \leq s$ whenever $s \geq max z_{ii}$.

Remark 4: From Definition 10, one can observe that the Laplacian matrix of a graph is an M-matrix. Another interesting finding is that the Laplacian matrix is irreducible if and only if the corresponding digraph is strongly connected. If a digraph is strongly connected, then its Laplacian matrix has a simple zero eigenvalue and a positive left eigenvector associated with the zero eigenvalue. It should be noted that the above statement is a sufficient condition, rather than a necessary condition. Moreover, the zero eigenvalue of the Laplacian matrix is simple if and only if the associated graph has a directed spanning tree (or quasi-strongly connected).

Assumption 1: Areal-valued function $f : R \to R$ is called Lipschitz continuous if there exists a positive real constant K such that, $|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$ for all real x_1 and $x_2 \in R$.

Assumption 2: Assume that there exists a positive definite diagonal matrix $P = diag(p_1, p_2, ..., p_n)$ and a diagonal matrix $Q = diag(\theta_1, \theta_2, ..., \theta_n)$ such that $f(*)$ satisfies the following inequality:

$$(y-x)^T P\big(f(y) - f(x) - Q(y-x)\big) \leq -\xi (y-x)^T (y-x),$$

for some $\xi > 0$, $\forall x, y \in R^n$ and $t > 0$.

Assumption 3: Two constant matrices do exist $\Theta = (\theta_{ij})_{n\times n}$ and $\phi = (\varphi_{ij})_{n\times n}$, in which $\theta_{ij} \geq 0$, $\varphi_{ij} \geq 0$ such that $\left| f_i(t, x(t), x(t-\tau)) - f_i(t, y(t), y(t-\tau)) \right| \leq \sum \left(\theta_{ij} |x_j(t) - y_j(t)| + \varphi_{ij} |x_j(t-\tau) - y_j(t-\tau)| \right)$

Assumption 4: Let $0 < \beta < 1$ and $\lambda > 0$ there exists a continuous function g:[0, ∞) → [0, ∞) with $g(0) > 0$, for any $0 \leq u \leq t$, such that $g(t) - g(u) \leq -\lambda \int_u^t (g(s))^\beta \, ds$ holds.

Lemma 1: If $A \in R^{N\times N}$ is a real symmetrical matrix, and then

$$\lambda_{\min}(A) x^T x \leq x^T A x \leq \lambda_{\max}(A) x^T x, \text{ for any } x \in R^n.$$

Lemma 2: Suppose a positive-definite function $U(t)$ which is continuous and satisfies the following inequality:

$$U'(t) \leq -\alpha U^p(t), \ \forall t \geq 0$$

where $\alpha > 0$, $0 < p < 1$. Then for any given t_0, $U(t) \equiv 0, \forall t \geq T(x_0)$, where $T(x_0) = \frac{U^{1-p}(0)}{\alpha(1-p)}$.

Lemma 3: Assume that a continuous, definite-positive function $U(t)$ satisfies the following inequality:

$$U'(t) \leq -\beta U^p(t) - \gamma U^q(t),$$

where $\beta > 0$, $\gamma > 0$, $0 < p < 1$, $q > 1$. Then, $U(t) \equiv 0, \forall t \geq T$ with $T = \frac{1}{\gamma(q-1)} + \frac{1}{\beta(1-p)}$.

Lemma 4: If $\xi_1, \xi_2, ..., \xi_n \geq 0$, $0 < t \leq 1$, $\kappa > 1$, then

$$\left(\sum_{i=1}^{n} \xi_i \right)^t \leq \sum_{i=1}^{n} \xi_i^t \quad \text{and} \quad n^{1-\kappa} \left(\sum_{i=1}^{n} \xi_i \right)^\kappa \leq \sum_{i=1}^{n} \xi_i^\kappa .$$

Lemma 5: Suppose that a continuous, positive-definite function $V(t)$ satisfies the following inequality:

$$V'(t) \le -\alpha V^{p}(t),\ \forall t \ge t_0,\ V(t_0) \ge 0,$$

where $\alpha > 0,\ 0 < p < 1$ are two constants. Then, for any given t_0, $V(t)$ satisfies the following differential inequality:

$$V^{1-p}(t) \le V^{1-p}(t_0) - \alpha(1-p)(t-t_0),\ t_0 \le t \le t_1 \text{ and } V(t) \equiv 0, \forall t \ge t_1, \text{with } t_1 = t_0 + \frac{V^{1-p}(t_0)}{\alpha(1-p)}.$$

Lemma 6: Let $x_1, x_2, ..., x_n \in \mathbb{R}^n$ are any vectors $0 < q < 2$ is a real number satisfying:

$$\left(\|x_1\|^2 + \|x_2\|^2 + ... + \|x_n\|^2\right)^{q/2} \le \|x_1\|^q + \|x_2\|^q + ... + \|x_n\|^q$$

Lemma 7: For $x_1, x_2, ..., x_n \in \mathbb{R}^n$ the following inequality holds:

$$\sqrt{\|x_1\|^2 + \|x_2\|^2 + ... + \|x_n\|^2} \le \|x_1\| + \|x_2\| + ... + \|x_n\|$$

Lemma 8: When $a \le b$ and $c < 1$ are all positive numbers, then the following inequality holds:

$$a^c \le b^c$$

III. MAIN SURVEY

This part is divided into five such parts including the finite-time and fixed time synchronization of Complex Networks with nonidentical discontinuous nodes, Synchronization for a Class of Output-Coupling Networks via Continuous control, Synchronization Control of complex dynamical networks with time delay, Synchronization of coupled networks via discontinuous controllers and Finite-time synchronization of Markovian jump complex networks.

A. Synchronization of Complex Networks with nonidentical discontinuous nodes

Finite-time synchronization implies optimum convergence time, and has greater robustness and rejection properties of disturbance. Xinsong Yang et al studied the problem of finite-time synchronization for linearly coupled complex networks with discontinuous non-identical nodes in the paper entitled "Finite-time synchronization of complex networks with non-identical discontinuous nodes." New simple conditions are suggested for the general chaotic discontinuous systems. A collection of new controllers are constructed in such a way that in finite-time the errors between the uncertain Filippov solutions caused by node states discontinuities approach to zero. The results obtained refer to both direct and undirected networks, and the coupled nodes and isolated nodes can be discontinuous and continuous, and even partial nodes are discontinuous.

Consider a complex N-non-identical node model with diffusively linear couplings in which each node is a dynamic *n*-dimensional structure, i.e.

$$x'_i(t) = C_i x_i(t) + f_i(x_i(t)) + \sum_{j=1}^{N} a_{ij} \Gamma x_j(t),\ i \in N \tag{1}$$

where $x_i(t) \in R^n$ represents the state vector of the i^{th} dynamical node, the dynamics of the uncoupled i^{th} node is $x'_i = C_i x_i + f_i(x_i)$, in which $C_i = \left(c_{jr}^i\right)_{n\times n} \in R^{n\times n}$, the nonlinear vector function $f_i(x_i) \in R^n$, the constant matrix $A = \left(a_{ij}\right)_{N\times N}$ describes the linear network coupling configuration that satisfies $a_{ij} \ge 0$, for $i \ne j$ and $a_{ii} = -\sum_{j=1,\, j\ne i}^{N} a_{ij}, i \in N$ and $\Gamma = \left(\gamma_{ij}\right)_{n\times n}$ is inner-coupling matrix between nodes. The author designed suitable controllers $R_i(t)$ So that complex network states (1) synchronize in a finite time with the state of the following process,

$$z'(t) = Dz(t) + g(z(t)) \tag{2}$$

where $D = \left(d_{jr}\right)_{n\times n} \in R^{n\times n}$, $g(z(t)) \in R^n$.

The controlled complex network (1) is

$$x'_i(t) = C_i x_i(t) + f_i(x_i(t)) + \sum_{j=1}^{N} a_{ij}\Gamma x_j(t) + R_i(t), \quad i \in N \tag{3}$$

where $R_i(t) = -\xi_i e_i(t) - (\eta_i + k)\operatorname{sgn}(e_i(t)), \quad i \in N$ (4)

Because although the sign function in (4) meets the basic requirements, the Filippov solutions of (2) and (3) are available and can be described as

$$z'(t) = D z(t) + \beta(t) \tag{5}$$

$$x'_i(t) = C_i x_i(t) + \gamma_i(t) + \sum_{j=1}^{N} a_{ij}\Gamma x_j(t) + R_i(t), \quad i \in N \tag{6}$$

where $\beta(t) \in g(z(t)), \quad \gamma_i(t) \in f_i(x_i(t))$

Let $e(t) = x_i(t) - z(t)$

Subtracting (5) from (6) produces the following error dynamical system:

$$e'_i(t) = C_i e_i(t) + F_i(t) + W_i(t) + \sum_{j=1}^{N} a_{ij}\Gamma e_j(t) + R_i(t), \quad i \in N \tag{7}$$

or $e'_i(t) = D e_i(t) + G_i(t) + E_i(t) + \sum_{j=1}^{N} a_{ij}\Gamma e_j(t) + R_i(t), \quad i \in N$ (8)

Under Description 1, finite-time convergence of the dynamic network (3) into (2) is analogous to the problem of finite-time stabilization of the dynamic error systems (7) or (8) at origin. The following theorems were presented to give the homogeneous trajectory (2) the finite-time synchronization for complex networks (3) by developing the Lyapunov function $V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t) e_i(t)$.

Theorem 1: Suppose that the control parameters η_i and ξ_i in the set of controllers (4) satisfy the following conditions:

$$\eta_i \geq M_i + M_i, \quad i \in N \tag{9}$$

$$\Xi \geq C + R + \theta A^s \tag{10}$$

where $M_i = \max\left\{\overline{M}_{ij} + M_j + \sum_{r=1}^{n}\left|c_{jr}^i - d_{jr}\right| H_r, j = 1,2,...,n\right\}$, $C = diag\left(\|C_1\|, \|C_2\|, ..., \|C_N\|\right)$,

$R = diag(\alpha_1, \alpha_2, ..., \alpha_N)$, $A = (a_{ij})_{N\times N}$, $a_{ij} = a_{ij}$, $i \neq j$, $a_{ii} = \frac{\rho_{\min}}{\theta} a_{ii}$, $\theta = \|\Gamma\|$, $\rho_{\min}$ is the minimum Eigenvalue of $\Gamma^s, \Xi = diag(\xi_1, \xi_2, ..., \xi_N)$. The complex network (3) must then be synchronized on (2) in a finite time $t_1 = \frac{\sqrt{2}}{k} V^{\frac{1}{2}}(0)$, where $V(0) = \sum_{i=1}^{N} e_i^T(0) e_i(0)$, $e_i(0)$ is the initial condition of $e_i(t) = x_i(t) - z(t)$.

Corollary 1: Suppose that the control parameters η_i and ξ_i in the set of controllers (4) satisfy the following conditions:

$$\eta_i \geq \delta, \tag{11}$$

$$\xi_i \geq \lambda_1 + \lambda_2 + \theta\lambda_3, \tag{12}$$

where $\delta = \max\{M_i + M_i, i \in N\}$, $\lambda_1 = \max\{\|C_i\|, i = 1,2,...,n\}$, $\lambda_2 = \max\{\alpha_i, i = 1,2,...,n\}$,

$\lambda_3 = \lambda_{\max}(A^s)$, the other parameters are defined in Theorem 1. Then the complex network (3) is synchronized onto (2) in a finite time t_1 defined in Theorem 1.

Theorem 2: Assume the control parameters η_i and ξ_i in the set of controllers (4) satisfy the following conditions:

$$\eta_i \geq K_i + \overline{M},\ i \in N \tag{13}$$

$$\Xi \geq \left(\|D\| + \overline{\alpha}\right) I_N + \theta A^s, \tag{14}$$

where $K_i = \max\left\{\overline{K}_{ij} + K_j + \sum_{r=1}^{n} \left|c_{jr}^i - d_{jr}\right| L_{ir},\ j = 1,2,...,n \in N\right\}$, the other criteria are laid down in Theorem 1. The complex network (3) must then be synchronized on (2) in a finite time t_1, for which the one above is the same.

Corollary 2: Suppose that the control parameters η_i and ξ_i in the set of controllers (4) satisfy the following conditions:

$$\eta_i \geq K + \overline{M}, \tag{15}$$

$$\xi_i \geq \|D\| + \overline{\alpha} + \theta\lambda_3, \tag{16}$$

where $K = \max\left\{K_i + M_i,\ i \in N\right\}$, λ_3 The other parameters are defined in Corollary 1, in Theorem 1. Then, in a finite time defined in Theorem 1, the complex network (3) is synchronized to (2)

If all of the system nodes (3) are similar to the isolate system (2),
i.e., $C_i = D,\ f_i(x_i) = g(x_i),\ i \in N$ the error mechanisms (7) and (8) then turn out to be as follows:

$$e'_i(t) = De_i(t) + G_i(t) + \sum_{j=1}^{N} a_{ij}\Gamma e_j(t) + R_i(t),\ \ i \in N \tag{17}$$

By contrasting the program (17) with (8), the following corollary can be easily obtained by taking $K_i = 0$ in Theorem 2.

Corollary 3: Assume $C_i = D,\ f_i(x_i) = g(x_i),\ i \in N$. Assume that the control parameters in the controllers set (4) meet the following conditions:

$$\eta_i \geq \overline{M},\ i \in N \tag{18}$$

$$\Xi \geq \left(\|D\| + \overline{\alpha}\right) I_N + \theta A^s, \tag{19}$$

Where in Theorem 1, the other parameters are defined, then, in a finite t_1 time, the complex network (3) with identical nodes is synchronized to (2) where t_1 is the same as above.

Theorems 1 and 2 are formulated in terms of LMIs, while the algebraic inequalities are used to describe Corollaries 1 and 2. While results with algebraic inequalities are more conservative than those with LMIs, computing using algebraic inequalities is simpler than using LMIs particularly for complex networks, as complex networks typically have a large number of nodes. The error mechanisms (7) and (8) derive specific synchronization parameters respectively. All the synchronization requirements in Theorems 1, 2, and Corollaries 1, 2 are valid for regulation of finite-time synchronization of complex networks with non-identical discontinuous nodes. Although operators should choose less restrictive parameters for synchronization according to the realistic situation in actual applications.

B. Synchronization for a Class of Output-Coupling Networks via Continuous control

Most studies have concentrated on state-coupling complex networks in recent years, while output-coupling complex networks attract comparatively less consideration, let alone research on fixed-time and finite-time synchronization issues.

As we all know, coupling power between the nodes of complex networks plays a very significant role in the issue of complex network synchronization. In general, reinforcing the coupling effect to understand the synchronization for CNs is a simple and prime concept. In this condition, however, the coupling force is also required to be as strong as possible.

It is therefore necessary and desirable to find a suitable coupling strength which is appropriate. Use of adaptive technique [21] is a simple and successful way of achieving this goal. The authors of [22] studied the problem of synchronization with delays of coupled connected NN. The authors investigated in [23] that a single controller used pinning control for complex networks, but the coupling strength was expected to be very high,

which was very rigorous. The problem of adaptive coupling power for fixed-time synchronization of complex networks with output nodes is less discussed in [24].

Zhiwei Li discussed the above problem in detail in his paper entitled "Fixed-Time and Finite-Time Synchronization for a Class of Output-Coupling Complex Networks by Continuous Control." He was investigating the problem of finite-time and fixed-time synchronization with output feedback nodes for a class of general output-coupling CNs by using the Lyapunov stability principle, LMI and adaptive methodology, many ample conditions ensuring fixed-time and finite-time synchronization are extracted.

Find output coupling complex networks with output nodes as observes:

$$\begin{cases} x'_i(t) = Ax_i(t) + f\left(y_i(t)\right) + c\sum_{j=1}^{N} g_{ij}\Gamma y_j(t) + u_i(t), \\ y_i(t) = Cx_i(t), \quad i \in N \end{cases} \tag{20}$$

where $x_i(t) \in R^n$ Is the function state variables of the i^{th} dynamical node, $y_i(t) \in R^m$ the output parameter is the i^{th} dynamical node, $f : R^n \to R^n$ is a continual function that regulates the dynamics of i^{th} isolated nodes, c > 0 is a coupling strength, $G = \left[g_{ij} \right]_{N\times N}$ is complex network weight configuration matrix, $\Gamma \in R^{n\times m}$ is the internal coupling matrix and $C \in R^{m\times n}$ is the output matrix, $u_i(t) \in R^n$ Is system-designed controller (20). For $i \neq j$, $g_{ij} > 0$ If there is a node relation and only if i to the node j, and the diagonals are known as $g_{ii} = -\sum_{j=1,\, j\neq i}^{N} g_{ij}$

The complex network initial value (1) is $x_i(0) = x_{i0}$, $i \in N$, the method (20) can therefore be used to define the complex networks, guided and undirected weighted.

If output matrix C is matrix of identity I_n, (1) then degrade to

$$x'_i(t) = Ax_i(t) + f\left(x_i(t)\right) + c\sum_{j=1}^{N} g_{ij}\Gamma x_i(t) + u_i(t) \tag{21}$$

This has been examined in detail by [25-27]. Hence, system (20) is wider than system (21).

The author developed appropriate controllers $u_i(t)$ in such a way that complex network states (20) synchronize in a finite time and fixed time into the state of the following target system;

$$\begin{cases} x'^*(t) = Ax^*(t) + f\left(y^*(t)\right), \\ y^*(t) = C\,x^*(t). \end{cases} \tag{22}$$

The following dynamical error systems are obtained by subtracting (22) from (20):

$$\begin{cases} e'_i(t) = Ae_i(t) + F\left(\psi_i(t)\right) + c\sum_{j=1}^{N} g_{ij}\Gamma \psi_j(t) + u_i(t), \\ \psi_i(t) = y_i(t) - y^*(t), \quad i \in N \end{cases} \tag{23}$$

where

$e_i(t) = x_i(t) - x^*(t)$, $F\left(\psi_i(t)\right) = f\left(y_i(t)\right) - f\left(y^*(t)\right)$,

$e_i(t)$ and $\psi_i(t)$ are the i^{th} node 'status error and output error, respectively. We definitely have

$$\psi_i(t) = C\,x_i(t) - C\,x^*(t) = C\,e_i(t). \tag{24}$$

a) Finite-Time Synchronization

The author developed suitable controllers $u_i(t)$ to synchronize complex networks (20) with finite time output-coupling into (22) such as:

$$u_i(t) = -\varepsilon_i e_i(t) - \beta e^{\theta/\phi}(t),\ i \in N \tag{25}$$

where $\varepsilon_i > 0\ (i \in N)$, θ and ϕ Are odd positive entries satisfying $\theta < \phi$ should be known, and $\beta > 0$ is a tunable constant.

The following theorem is given by constructing the Lyapunov function $V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t) e_i(t)$ to give the finite-time synchronization for complex networks (22) to the homogenous trajectory (21).

Theorem 3: Suppose that 1 holds that. If any positive constants exist $\varepsilon_i \in N$ such that $\left(\lambda_{\max}\left(A^s\right) + \delta\chi\right) I_N + G^* - \Pi \le 0$, where $\chi = \sqrt{\lambda_{\max}\left(C^T C\right)}$, $\Pi = diag\left[\varepsilon_i\right]_N$, $G^* = \left[g_{ij}^*\right]_{N\times N}$ with $g_{ii}^* = c\varpi g_{ii}$ and $g_{ij}^* = c\rho g_{ij}\ (i \ne j)$, where $\varpi = \sqrt{\lambda_{\max}(\Gamma C)^s}$, $\rho = \|\Gamma C\|$, then, the controlled network (20) Is synchronized in a finite time on (22) under the controller (25): $T_0 = \frac{2\phi V^{\frac{\phi-\theta}{2\phi}}(0)}{\beta 2^{\frac{\phi+\theta}{2\phi}}(\phi-\theta)}$, where $V(0) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(0) e_i(0)$.

Particularly if the output matrix C is a matrix of identity with correct dimensions, then we will have the results below.

Corollary 4: Assume the holding of Assumption 1 and the output matrix $C = I_N$. If positive constant exists $\varepsilon_i \in N$ such that $\left(\lambda_{\max}\left(A^s\right) + \delta\right) I_N + G^* - \Pi \le 0$, where $\Pi = diag\left[\varepsilon_i\right]_N$ and $G^* = \left(g_{ij}^*\right)$ with $g_{ii}^* = c\varpi g_{ii}$ and $g_{ij}^* = c\rho g_{ij}\ (i \ne j)$, where $\varpi = \sqrt{\lambda_{\max}(\Gamma)^s}$, $\rho = \|\Gamma\|$. Then complex network (20) must achieve finite-time synchronization under the controller (25).

Corollary 5: Suppose that Assumptions 1 and 2 holds, the inner coupling matrix $\Gamma = I_N$, output matrix $c = I_N$ and coupling strength c = 1. For the weight configuration matrix G of the complex networks, if there exist a constant $\varepsilon > 0$ such that $\left(\lambda_{\max}\left(A^s\right) + \delta - \varepsilon\right) I_N + G \le 0$, then complex network (20) can achieve finite-time synchronization under the controller (25).

b) Fixed-Time Synchronization

The author has developed suitable controllers $u_i(t)$ to synchronize complex output-coupling networks (1) into (3) within a specified time-limited settlement period as follows:

$$u_i(t) = -\xi_i e_i(t) - \beta e_i^{\theta/\phi}(t) - \gamma e_i^{k/l}(t) \tag{26}$$

where θ, ϕ, k, l all positive odd entries satisfy $\theta > \phi$ and $k < l$.

The following theorem is provided to give the synchronization of fixed time for complex networks (22) to the homogeneous trajectory (21) by building the Lyapunov function $V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t) e_i(t)$.

Theorem 4: Suppose Assumption 1 holds that. If any positive constants exist $\xi_i \in N$ such that $\left(\lambda_{\max}\left(A^s\right)+\delta\chi\right)I_N + G^* - \Xi \le 0$, where $\chi = \sqrt{\lambda_{\max}\left(C^T C\right)}$, $\Xi = diag\left[\xi_i\right]_N$, $G^* = \left[g_{ij}^*\right]_{N\times N}$ with $g_{ii}^* = c\varpi g_{ii}$ and $g_{ij}^* = c\rho g_{ij}$ $\left(i \ne j\right)$. Then, under the controller (26), the controlled network (20) is synchronized on (22) in a fixed time: $T = \dfrac{2l}{\gamma\left(l-k\right)2^{\frac{l+k}{2l}}\left(Nn\right)^{\frac{l+k}{2l}}} + \dfrac{2\phi}{\beta\left(\phi-\theta\right)2^{\frac{\phi+\theta}{2\phi}}}$.

From Theorem 4 results it can be found that the settling time is independent of the initial state value x_i (0) and x_0 (0). In addition, the settling time can be determined by the node dimension *n*, the design parameters and group order N.

c) Adaptive Adjustment of the Coupling Strength

Using adaptive technique, the adaptive coupling intensity problem for fixed-time synchronisation of complex networks with output nodes is investigated. The regulated complex network with output nodes is followed by the adaptive coupling rule to:

$$\begin{cases} x'_i\left(t\right) = Ax_i\left(t\right) + f\left(y_i\left(t\right)\right) + c\left(t\right)\sum_{j=1}^{N} g_{ij}\Gamma y_i\left(t\right) + u_i\left(t\right), \\ y_i\left(t\right) = Cx_i\left(t\right),\ i \in N, \\ c'\left(t\right) = \alpha\sum_{i=1}^{N} e_i^T \Gamma C e_i\left(t\right). \end{cases} \tag{27}$$

where α is a small positive constant.

The following theorem is presented to give the homogenous trajectory a fixed-time synchronization for adaptively controlled complex networks by creating the Lyapunov function

$$V\left(t\right) = \frac{1}{2}\sum_{i=1}^{N} e_i^T\left(t\right)e_i\left(t\right) + \frac{1}{2\alpha}\left(c\left(t\right)-\tilde{c}\right)^2.$$

Theorem 5: By the assumption of Assumption 1 holds, the adaptively controlled complex networks can achieve fixed-time synchronization with a desirable coupling strength and a positive definite matrix is ubiquitous.

C. Finite-time Synchronization Control of complex dynamical networks with time delay

"In recent years, several requirements have been provided for synchronization of complex dynamic networks with or without delays in time via impulsive control or intermittent control. Much of the research focussed on asymptotic or exponential network synchronization by impulsive control and sporadic control. In fact, however, we could always expect the networks to achieve synchronization as quickly as possible, particularly in engineering fields. Application of finite-time synchronization control techniques is an effective approach for achieving faster convergence rate in time-delay complex networks. Finite-time synchronization implies the Convergence-time optimality. And so far, there are few published papers that consider the finite-time synchronization of complex networks with time delays. It is therefore important to research the finite-time synchronization of time-delay complex networks, based on actual demands".

In the paper "Finite-time synchronization control of complex dynamic networks with time delay," Jun Mei et al addressed finite-time synchronization between two complex networks with non-delayed and delayed coupling through the use of impulsive control and periodically intermittent control through the use of finite-time stability theorem. Additionally, the finite-time synchronization requirements of systems are defined by applying Lyapunov theorem and inequality techniques in terms of linear matrix inequality, which are very simple to verify.

Consider a time delay complex dynamical network consisting of *N* nodes, in which each node is an *n*-dimensional dynamical system, i.e.,

$$x'_i(t) = f\left(x_i(t)\right) + c\sum_{j=1}^{N} a_{ij}\Gamma x_j(t) + c\sum_{j=1}^{N} b_{ij}\Gamma x_j(t-\tau), \quad i \in N \tag{28}$$

where $x_i(t) \in R^n$ represents the vector State of the i^{th} dynamical node, $f : R^n \to R^n$ stands for the activity function of i^{th} node, the constant c > 0 is a coupling strength, $\tau \geq 0$ is the coupling delay. $\Gamma = \left(\gamma_{ij}\right)_{n\times n}$ is inner-coupling matrix between nodes and $A = a_{ij}$, $B = b_{ij} \in R^{N\times N}$ Are the coupling matrices representing the coupling power and the underlying topology for the non-delayed configuration and one delayed at time t respectively. If the nodes bind to each other i to j ($j \neq i$), then $a_{ij} > 0, b_{ij} > 0$; otherwise, $a_{ij} = 0, b_{ij} = 0 (j \neq i)$ and the diagonal elements of matrices A, B are defined as

$a_{ii} = -\sum_{j=1, j\neq i}^{N} a_{ij}$, $b_{ii} = -\sum_{j=1, j\neq i}^{N} b_{ij}$, $i \in N$.

a) Synchronization of complex networks with time delay via impulsive control with finite-time

To evaluate finite-time impulsive synchronization with time delay process of general complex dynamic networks as follows:

The above complex dynamic network (28) can be rewritten in the following form of impulsive differential equation, without loss of generality:

$$y'_i(t) = f\left(y_i(t)\right) + c\sum_{j=1}^{N} a_{ij}\Gamma y_j(t) + c\sum_{j=1}^{N} b_{ij}\Gamma y_j(t-\tau) + u_i(t), \quad i \in N \tag{29}$$

where $y_i(t) \in \square^n$ denotes the response state vector of the i^{th} dynamical node and the controllers $u_i(t)$ are designed as follows:

$$\begin{cases} u_i(t) = -\eta_i e_i(t) - \bar{k}\dfrac{\sqrt{\lambda_{\max}(P)}}{\lambda_{\min}(P)} sign(e_i) - \bar{k}\left(q_i \int_{t-\tau}^{t} e_i^T(s) P e_i(s) ds\right)^{1/2} \left(\dfrac{e_i}{\|e_i\|^2}\right), \ i \in \square \\ u_i(t) = 0, \quad \|e_i\| = 0, \end{cases} \tag{30}$$

Where $\eta_i > 0$ are constants to be determined, $\bar{k}$ > 0 is a tunable constant and real number. Denote $\lambda_{\max}(P)$ $\lambda_{\min}(P)$ as the maximum (minimum) eigenvalue of the positive definite diagonal matrix P. And we choose the impulsive control gain B_{ik} which is a $n \times n$ constant matrix and the impulsive distances $\sigma_{k+1} = t_{k+1} - t_k < \infty (k \in l)$ such that the states of drive dynamical networks (28) synchronize with the state of response dynamical networks (29), that is, $\lim_{t\to T} \|e_i(t)\| = 0.$

Let $e(t) = y_i(t) - x_i(t), (1 \leq i \leq N)$ be synchronization errors and the following error dynamical system is obtained:

$$\begin{cases} e'_i(t) = f\left(y_i(t)\right) - f\left(x_i(t)\right) + c\sum_{j=1}^{N} a_{ij}\Gamma e_j(t) + c\sum_{j=1}^{N} b_{ij}\Gamma e_j(t-\tau) + u_i(t), \ t \neq t_k, \\ \Delta e_i(t) = B_{ik} e_i, \quad t = t_k, k \in l. \end{cases} \tag{31}$$

Theorem 6: Compare the system of errors (31) with impulsive controllers (30) and presume holds of Assumption 1. Suppose the positive constants are $\eta_1, \eta_2, ..., \eta_n$ satisfying:

$$\begin{bmatrix} \theta_j I_N - \Xi + \frac{1}{2}Q + c\gamma_j A & \frac{1}{2}c\gamma_j B \\ * & -\frac{1}{2}Q \end{bmatrix} \le 0,\ j \in N,$$

$$0 < \tau \le \inf_k \{t_{k+1} - t_k\},$$

$$\beta = \frac{\rho_k \|P\|}{\lambda_{\min}(P)} < 1,\ \rho_k = \max\left(\|I + B_{ik}\|^2\right),\ k \in l,$$

Where $\Theta = diag(\theta_1, \theta_2, ..., \theta_n)$, $\Xi = diag(\eta_1, \eta_2, ..., \eta_n) > 0$, $Q = diag(q_1, q_2, ..., q_n) > 0$,

$\Gamma = diag(\gamma_1, \gamma_2, ..., \gamma_n)$, $\xi = \min_{1 \le i \le N} \xi_i$ and I_N Is a suitable identity matrix. Then the error systems (31) are coordinated under the controllers (30) in a finite time $T_1 = \frac{2V^{1/2}(0)}{\sqrt{2k}}$, and $\sigma_{m+1} \le \frac{T_1}{m+1},\ m \in l,$

where $V(0) = \frac{1}{2}\sum_{i=1}^{N}\left[e_i^T(0)Pe_i(0) + q_i\int_{-\tau}^{0} e_i^T(s)Pe_i(s)ds\right]$, the initial condition is $e_i(0)$.

The theorem above was implemented by constructing the following LKF

$$V(t) = \frac{1}{2}\sum_{i=1}^{N}\left[e_i^T(t)Pe_i(t) + q_i\int_{t-\tau}^{t} e_i^T(s)Pe_i(s)ds\right].$$

Corollary 7: Consider the impulsive controller error systems (30), and conclude the Assumption 1 holds. Suppose that positive constants $\eta_1, \eta_2, ..., \eta_n$ satisfying:

$$\theta I_N - \Xi + c\gamma A \le 0,$$

$$\beta = \frac{\rho_k \|P\|}{\lambda_{\min}(P)} < 1,\ \rho_k = \max\left(\|I + B_{ik}\|^2\right),\ k \in l,$$

where $A = (a_{ij})_{N \times N}$, $\Xi = diag(\eta_1, \eta_2, ..., \eta_n) > 0$, $\Gamma = diag(\gamma_1, \gamma_2, ..., \gamma_n)$, $\Theta = diag(\theta_1, \theta_2, ..., \theta_n)$ and adequate identity matrix is I_N . Next, the error systems (31) are coordinated under the controllers (30) in a finite time $T_2 = \frac{2V^{1/2}(0)}{\sqrt{2k}}$, and $\sigma_{m+1} \le \frac{T_2}{m+1},\ m \in l,$

where $V(0) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(0)Pe_i(0)$, $e_i(0)$ is the initial condition of $e_i(t)$.

b) Synchronization of complex networks with time delay via intermittent control with finite-time

The intermittent controllers $u_i(t)$ of general complex dynamical networks with time delay to model is defined as

$$\begin{cases} u_i(t) = -\eta_i e_i(t) - \bar{k}\dfrac{\sqrt{\lambda_{\max}(P)}}{\lambda_{\min}(P)} sign(e_i) - \bar{k}\left(q_i\displaystyle\int_{t-\tau}^{t} e_i^T(s)Pe_i(s)ds\right)^{1/2}\left(\dfrac{e_i}{\|e_i\|^2}\right), & lT \le t < lT + \delta, \\ u_i(t) = 0, \quad \|e_i\| = 0, \end{cases} \tag{32}$$

Where $\eta_i > 0$ are constants called control gain, $T > 0$ is the control period, $\delta > 0$ is called the control width (control duration) and $k > 0$ is a tunable constant and real number.

Let $e(t) = y_i(t) - x_i(t), (1 \le i \le N)$ be synchronization errors and $\theta = \delta / T$ be the ratio of the control width δ to the control period T called control rate. According to the control law (32), they obtained the following error dynamical system:

$$\begin{cases} e'_i(t) = f(y_i(t)) - f(x_i(t)) + c\sum_{j=1}^{N} a_{ij}\Gamma e_j(t) + c\sum_{j=1}^{N} b_{ij}\Gamma e_j(t-\tau) + u_i(t), \\ lT \le t < lT + \theta T, \quad i \in \square \\ e'_i(t) = f(y_i(t)) - f(x_i(t)) + c\sum_{j=1}^{N} a_{ij}\Gamma e_j(t) + c\sum_{j=1}^{N} b_{ij}\Gamma e_j(t-\tau), \\ lT + \theta T \le t < (l+1)T, \quad i \in N \end{cases} \tag{33}$$

Theorem 7: Suppose Assumption 1 does hold. When a positive definite matrix exists $\Xi = diag(\eta_1, \eta_2, ..., \eta_n) > 0$ and a diagonal matrix $P > 0$ So the conditions are as follows: $\begin{bmatrix} \theta_j I_N - \Xi + \frac{1}{2}Q + c\gamma_j A & \frac{1}{2}c\gamma_j B \\ * & -\frac{1}{2}Q \end{bmatrix} \le 0,\ j \in \square,$

$$\begin{bmatrix} \theta_j I_N - \frac{\xi}{\lambda_{\max}(P)} I_N + \frac{1}{2}Q + c\gamma_j A & \frac{1}{2}c\gamma_j B \\ * & -\frac{1}{2}Q \end{bmatrix} \le 0,\ j \in \square,$$

Where $\Theta = diag(\theta_1, \theta_2, ..., \theta_n)$, $\Xi = diag(\eta_1, \eta_2, ..., \eta_n) > 0$, $Q = diag(q_1, q_2, ..., q_n) > 0$, $\Gamma = diag(\gamma_1, \gamma_2, ..., \gamma_n)$, $\xi = \min_{1 \le i \le N} \xi_i$ and and I_N is an appropriate identity matrix. The error systems (33) are then synchronized within the controllers in a finite time $T_3 = \frac{2V^{1/2}(0)}{\sqrt{2k\theta}}$,

where $V(0) = \frac{1}{2}\sum_{i=1}^{N}\left[e_i^T(0) P e_i(0) + q_i \int_{-\tau}^{0} e_i^T(s) P e_i(s) ds \right]$, $e_i(0)$ is the initial condition of $e_i(t)$.

The theorem above was implemented by constructing the following LKF

$$V(t) = \frac{1}{2}\sum_{i=1}^{N}\left[e_i^T(t) P e_i(t) + q_i \int_{t-\tau}^{t} e_i^T(s) P e_i(s) ds \right].$$

Corollary 8: Suppose B = 0 holds, and Assumption 1. When a positive definite matrix exists $\Xi = diag(\eta_1, \eta_2, ..., \eta_n) > 0$ and a diagonal matrix $P > 0$ so the conditions are as follows:

$$\theta_j I_N - \Xi + \frac{1}{2}Q + c\gamma_j A \le 0,\ j \in N,$$

$$\theta_j I_N - \frac{\xi}{\lambda_{\max}(P)} I_N + \frac{1}{2}Q + c\gamma_j A \le 0,\ j \in N,$$

Where $\Theta = diag(\theta_1, \theta_2, ..., \theta_n)$, $\Xi = diag(\eta_1, \eta_2, ..., \eta_n) > 0$, $Q = diag(q_1, q_2, ..., q_n) > 0$, $\Gamma = diag(\gamma_1, \gamma_2, ..., \gamma_n)$, ξ is a positive constant satisfies Assumption 1 and I_N is an appropriate identity matrix. Then, under the controllers the error systems (33), are synchronized under the periodically intermittent controllers in a finite time $T_4 = \frac{2V^{1/2}(0)}{\sqrt{2k\theta}}$, where $V(0) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(0) P e_i(0)$, $e_i(0)$ is the initial condition of $e_i(t)$.

The following theorem is formulated by considering the following periodic laws for updating

$$\begin{cases} r_i^{'} = -\lambda_i \left[\dfrac{\eta_i r_i e_i^T P e_i}{r_i^2} + e_i^T P e_i + \dfrac{\bar{k}}{\sqrt{\lambda_i}} sign(r_i) \right], \ lT \le t < lT + \delta, \\ r_i^{'} = 0, \quad \|r_i\| = 0 \quad \text{or} \quad lT + \delta \le t < (l+1)T, \end{cases} \tag{34}$$

where $\lambda_i > 0$ is arbitrary, and by considering the following LKF

$$V(t) = \frac{1}{2} \sum_{i=1}^{N} \left[e_i^T(t) P e_i(t) + \frac{1}{\lambda_i} r_i^2 + q_i \int_{t-\tau}^{t} e_i^T(s) P e_i(s) ds \right]$$

Theorem 8: Suppose Assumption 1 does hold. When a positive definite matrix exists

$\Xi = diag(\eta_1, \eta_2, ..., \eta_n) > 0$ and a positive diagonal matrix P > 0 which holds the following conditions:

$$\begin{bmatrix} \theta_j I_N - \Xi + \frac{1}{2} Q + c\gamma_j A & \frac{1}{2} c\gamma_j B \\ * & -\frac{1}{2} Q \end{bmatrix} \le 0, \ j \in N,$$

$$\begin{bmatrix} \theta_j I_N - \frac{\xi}{\lambda_{\max}(P)} I_N + \frac{1}{2} Q + c\gamma_j A & \frac{1}{2} c\gamma_j B \\ * & -\frac{1}{2} Q \end{bmatrix} \le 0, \ j \in N,$$

Where $\Theta = diag(\theta_1, \theta_2, ..., \theta_n), \Xi = diag(\eta_1, \eta_2, ..., \eta_n) > 0, Q = diag(q_1, q_2, ..., q_n) > 0,$ $\Gamma = diag(\gamma_1, \gamma_2, ..., \gamma_n), \xi$ is a positive constant satisfies Assumption 1 and I_N is an appropriate identity matrix. Then, under the controllers the error systems (4.3), are synchronized under the periodically intermittent controllers and intermittent updating laws in a finite time $T_5 = \dfrac{2V^{1/2}(0)}{\sqrt{2k\theta}}$,

where $V(0) = \frac{1}{2} \sum_{i=1}^{N} \left[e_i^T(0) P e_i(0) + q_i \int_{-\tau}^{0} e_i^T(s) P e_i(s) ds + \frac{1}{\lambda_i} (r_i(0))^2 \right]$, $e_i(0)$, $r_i(0)$ are the initial conditions of $e_i(t)$, $r_i(t)$.

D. Synchronization of coupled networks via discontinuous controllers with finite time

Some of the earlier research on the synchronization of coupled neural networks have implemented linear feedback controller and can only achieve exponential or asymptotic convergence. Nevertheless, in practical application the finite-time convergence is more significant. The finite- time synchronization advantage requires robustness and higher convergence rate against uncertainties. As is well known, certain discontinuous dynamic systems are more likely to achieve convergence of the finite-time. Discontinuous controllers are often deliberately configured to monitor and stabilize finite-time. For example, two discontinuous control algorithms have been developed in Cortes (2006), which achieve consensus on multi-agent systems in finite time. The authors discussed finite-time semi-stability for discontinuous dynamic systems in Hui et al. (2009, 2010), and extended this principle to finite-time consensus with topology switches.

Jun shen explores the finite- synchronization by discontinuous controllers of an array of coupled neural networks. Some appropriate parameters for finite-time synchronization are obtained based on the Lyapunov method. In addition, author suggested strategies for shifted control and adaptive tuning parameter to reduce the settling time. Additionally, the pinning control scheme is also configured for finite-time synchronization via a single controller. With the hypothesis that the topology of the coupling network includes a directed spanning tree and that each of the strongly connected components is detail-, it has been shown that finite- synchronization

can be accomplished by pinning. With the hypothesis that the topology of the coupling network includes a directed spanning tree and that each of the strongly connected components is detail-, it has been shown that finite- synchronization can be accomplished by pinning.
Find the following configuration for a neural network:

$$x'(t) = -Bx(t) + Tg(x(t)) + J(t) \tag{35}$$

where $x_i(t) \in R^n$ represents the state vector associated with the neurons, $B = diag\{b_1, b_2, ..., b_n\}$ is a positive diagonal matrix with b_i modelling self-inhibition of i^{th} neuron, $T = [t_{ij}] \in R^{n \times n}$, is the interconnection matrix, $J(t) = (J_1(t), J_2(t), ..., J_n(t)) \in R^n$ with $\|J(t)\| \le \bar{J}$ is a bounded external input, and $g(x) = (g_1(x_1), g_2(x_2), ..., g_n(x_n))^T$ with $g_i(.)$ modelling the input-output activation of the i^{th} neuron.

The following model of N linearly coupled neural networks can be considered for studying synchronization: $y'_i(t) = -By_i(t) + Tg(y_i(t)) + J(t) + c\sum_{j=1}^{N} a_{ij}(y_j - y_i) + u_i(t), \quad i \in N$ (36)

where $A = [a_{ij}]$ the adjacency matrix of the coupled neural networks, c > 0 denotes the coupling strength and $u_i(t) \in \square^n$ is the control input formulated as follows:

$$u_i(t) = -\dot{d}_i(y_i - x) - \beta sign(y_i - x), \; i \in N \tag{37}$$

Under the above control scheme (37), systems (35) and (36) can be represented as follows:

$$\begin{cases} x'(t) = -Bx(t) + Tg(x(t)) + J(t) \\ y'(t) = -(I_N \otimes B)y + (I_N \otimes T)G(y) + I_N \otimes J(t) - c(L \otimes I_N)y - (D \otimes I_N)(y - 1_N \otimes x) - \beta sign(y - 1_N \otimes x) \end{cases}$$

where I_N represents Dimensional identity matrix N and 1_N denote the column vector of n- with all entries equal to one, $D = diag\{d_1, d_2, ..., d_N\}$ and L denotes matrix Laplacian associated with matrix A of adjacence.

Theorem 9: Suppose the activation function $g(.)$ satisfies Assumption 1 and a positive constant ε exists such that the following inequality holds the following:

$$-(I_N \otimes B) - \left(c\frac{L + L^T}{2} + D\right) \otimes I_N + \varepsilon \|T\|^2 I_{Nn} + \frac{1}{4\varepsilon} I_N \otimes l^2 < 0 \tag{38}$$

where $l = diag\{l_1, l_2, ..., l_n\}$. Then the coupled neural networks (6) conduct finite-time synchronisation $t^* = \frac{\sqrt{2}V(0)^{1/2}}{\beta}$ where $V(t) = \frac{1}{2}e^T(t)e(t)$.

Adopt the following continuous feedback controller to reduce the settling time via switched control: $u_i(t) = -\dot{d}_i(y_i - x) - \beta sign^{\alpha}(y_i - x), \; i \in N$ (39)

It's clear to see that $V(t)$ decreases monotonically under condition (38). The initial error may be very large, so continuous controller (39) is easier to use. Then, after some time, the synchronization error gets smaller and the use of discontinuous controller (37) is easier. The following switched control technique, i.e. flipping the parameter α according to the synchronization error, is therefore fair to suggest:

$$\alpha(t) = \begin{cases} 1 - \dfrac{2}{\ln(2V(0))}, & \text{when } V(t) > \dfrac{1}{2}e^2 \\ 0, & \text{when } V(t) \le \dfrac{1}{2}e^2 \end{cases} \tag{40}$$

Theorem 10: Suppose the $g(.)$ activation function satisfies Assumption 1 and holds Condition (39). Then, under the switched control scheme (40), the coupled neural networks (35, 36) achieve finite-time synchronisation and the approximate settling time is

$$t^* = \frac{e\ln\left(2V(0)\right) - \ln\left(2V(0)\right)e^{\frac{2}{\ln(2V(0))}} + 2e}{2\beta} \text{ where } V(t) = \frac{1}{2}e^T(t)e(t).$$

The control parameter αis modified by the adaptive tuning parameter method according to the synchronisation error as follows:

$$\alpha(t) = \begin{cases} 1 - \dfrac{2k}{\ln\left(2V(t)\right)}(0 < k < 1), & \text{when } V(t) > \dfrac{1}{2}e^2, \\ 0, & \text{when } V(t) \le \dfrac{1}{2}e^2 \end{cases}. \quad (41)$$

Theorem 11: Suppose the activation function $h(.)$ fulfills Assumption 1 and holds condition (39). Then, under the switched control scheme (41) the coupled neural networks (35, 36) achieve convergence in finite time and the approximate settling time is

$$t^* = \frac{e^k\left(\ln\left(2V(0)\right) - 2\right) + 2e}{2\beta} \text{ where } V(t) = \frac{1}{2}e^T(t)e(t).$$

Nonlinear coupling and pinning control scheme: The finite-time synchronization of complex dynamic networks was investigated in Chen and Lu (2009) via nonlinear coupling and pinning control. It was suggested that suitable nonlinear coupling can often boost the efficiency of the network. Consider the problem of synchronization by pinning control of nonlinearly coupled neural networks. The nonlinear neural networks was constructed by the

$$y'_i(t) = -By_i(t) + Th\left(y_i(t)\right) + J(t) + c\sum_{j=1}^{N} a_{ij}\left(y_j - y_i + \sigma sign\left(y_j - y_i\right)\right) + u_i(t), \quad i \in N \quad (42)$$

The pinning function scheme considered as follows with a single controller:

$$u_i(t) = \begin{cases} -d_1(y_1 - x) - \beta sign(y_1 - x), & if\ i = 1; \\ 0, & \text{otherwise} \end{cases} \quad (43)$$

Within the aforementioned control scheme (43), the global nature of Filippov's solutions of the coupled system's initial value problem can be evaluated similarly as in Theorem 9. On the following we prove that if the coupling network topology G is closely connected and detail-balanced this pinning control scheme can work.

Theorem 12: Suppose the topology G of the coupling network is strongly connected and accurate. If the activation function $g(.)$ satisfies Assumption 1 and a positive constant ε exists, such that the following inequality is found:

$$\Xi \otimes \left(B + \varepsilon\|T\|^2 I_n + \frac{1}{4\varepsilon}l^2\right) - \left(\Xi\left(cL + \overline{D}\right)\right) \otimes I_N \le 0$$

where $l = diag\{l_1, l_2, ..., l_n\}$ and $\overline{D} = diag\{d_1, 0, 0, ..., 0\}$. The coupled neural networks (42) would then synchronize the isolated neural network (35) in a finite time, under control algorithm (43).

E. Finite-time synchronization of Markovian jump complex networks

Over the past decade [30–37], Markovian Jump Complex Networks (MJCNs) have provided a wide range of coverage. This is partly due to Markovian jump being a suitable mathematical pattern for representing a class of complex networks subject to spontaneous abrupt structural variations [31–35]. Additionally, MJCNs may be viewed as a special class of stochastic network systems. This class of network systems has finite modes

that move at various times from one to the next [38]. In addition, such a turn (or jump) can be controlled by a Markovian chain [37]. MJCNs occur in a variety of fields, such as NNs [35], genetic regulatory networks [36] and hopfield networks [33].The study of the dynamic and topological structure of MJCNs is therefore of fundamental importance for understanding the real-world functions.

Synchronization has been intensively studied in recent years for MJCNs, with or without delays, as one of the most important dynamic behaviours [31–36, 39–42]. For example, synchronization issues with mode-dependent mixed time delays in [39] were resolved for the discrete MJCNs. Exponential synchronization for MJCNs with mixed time delays has been studied in [32]. And synchronization of hybrid coupled MJCNs with mode-dependent mixed delays has studied in [41, 42]. In practice engineering, however, synchronization is often needed to be accomplished in a finite time. It thus becomes necessary to investigate the synchronization for MJCNs with time delays in the finite-time convergence range.

Wenxia Cui et al studied the synchronization of a class of Markovian jump complex networks (MJCNs) with largely unknown transition rates and time delays in the finite-time convergence process. By building the correct stochastic functional Lyapunov-Krasovskii, using the finite-time stability theorem, inequality techniques and pinning control techniques, several adequate parameters have been proposed to ensure finite-time synchronization with or without time delays for the MJCNs. Since finite-time synchronization implies optimal convergence time and has better robustness and disturbance rejection properties, this paper has important theoretical significance and practical value for application. Finally, he demonstrated by considering numerical simulations.

a) Finite-time Synchronization of MJCNs with partially unknown transition rates

Consider a complex Markovian jump network consisting of N identical nodes with diffusive couplings, in which each node is a dynamic system of n dimensions, i.e.

$$x'_i(t) = f_i(x_i(t)) + c\sum_{j=1}^{N}\alpha_{ij}(r(t))\Gamma x_j(t) + u_i(t), \quad i \in N \tag{44}$$

where $x_i(t) \in R^n$ represents the state vector of the i^{th} dynamical node, $f : R^n \to R^n$ stands for the activity function of i^{th} node, $\Gamma = (\gamma_{ij})_{n\times n}$ is inner-coupling matrix between nodes. Let $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions. Here, $A(r(t)) = [\alpha_{ij}(r(t))]_{N\times N}$ describes the linear network coupling configuration at time t on mode $r(t)$, and is defined as $a_{ij}(r(t)) \geq 0$, for $i \neq j$, $\alpha_{ii}(r(t)) = -\sum_{j=1, j\neq i}^{N}\alpha_{ij}(r(t))$, $i \in$. $u_i(t)$ denotes input control node i.

Let $\{r_t, t \geq 0\}$ be a continuous right-hand Markov chain on the probability space $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ putting principles into finite space $S = \{1, 2, ..., w\}$ with a generator $\Pi = (\pi_{ij})_{w\times w}$ indicated by:

$$P\{r_{t+\Delta t} = j : r_t = i\} = \begin{cases} \pi_{ij}\Delta t + o(\Delta t), & \text{if } i \neq j \\ 1 + \pi_{ii}\Delta t + o(\Delta t), & \text{if } i = j \end{cases} \tag{45}$$

Where $\Delta t > 0$ and $\lim_{\Delta t\to 0}\left(\frac{o(\Delta t)}{\Delta t}\right) = 0$. Here, $\pi_{ij} \geq 0$ Is the rate of change from i to j if $i \neq j$ while $\pi_{ii} = -\sum_{j=1, j\neq i}^{w}\pi_{ij}$.

Let us consider the transition rates $\pi_{ij} \geq 0$, the method of jumping to be reached partially and can be represented as a network system (44)

$$\begin{bmatrix} \pi_{11} & \pi_{11} & \dots & \pi_{1w} \\ \pi_{21} & \pi_{22} & \dots & \pi_{2w} \\ \dots & \dots & \dots & \dots \\ \pi_{w1} & \pi_{w1} & \dots & \pi_{ww} \end{bmatrix}.$$

The network's isolated node (or uncoupled node) (44) is supported by the

$$s'(t) = f(s(t), t), \tag{46}$$

where $s(t)$ 'is the System's basic solution (46).

The following dynamical error system is generated by subtracting (46) from

(44): $e'_i(t) = F(e_i(t)) + c\sum_{j=1}^{N} \alpha_{ij}(r(t))\Gamma e_j(t) + u_i(t), \quad i \in N$ (47)

where $F(e_i(t)) = f(x_i(t)) - f(s(t)), e_i(t) = x_i(t) - s(t), \quad i \in N.$

The following linear negative feedback-pinning controllers are used for the coupled device (47).

$$u_i(t) = -\varepsilon_i(r)\Gamma e_i(t) - k_i(r)\,sign(e_i(t))|e_i(t)|^{\beta}, \tag{48}$$

Where $\varepsilon_i(r) \geq 0, k_i(r) > 0, i \in J,$ otherwise $\varepsilon_i(r) = k_i(r) = 0, i \notin J, r \in S.$

The following theorem is provided to give the Markovian jump complex networks finite-time synchronization (44) with partly uncertain probabilities of transition and stochastic noise disruption to the homogenous trajectory (46).

Theorem 13: Let Hypothesis hold. When

$$\begin{cases} \left(\dfrac{2h+\pi_r}{2\lambda_{\min}(\Gamma)} I_N + cA(r) - \Xi(r)\right) \otimes \Gamma \leq 0, \\ q_i - a_j \leq 0 \ \text{if} \ i \neq j, \quad i \in S_2^T, \\ q_i - a_j \geq 0 \ \text{if} \ i = j, i \in S_2^T, \end{cases}$$

Where $\Xi(r) = diag\{\gamma_1(r), \gamma_2(r), ..., \gamma_N(r)\} \in R^{N\times N}$ is the diagonal matrix with entries $\gamma_i(r) > 0, i \in J,$ otherwise, $\gamma_i(r) = 0, i \notin J, \pi_r = \sum_{i \in S_1^T} \dfrac{\pi_{ji}(q_i - a_j)}{q_j} (a_j > 0, \ q_j > 1).$ Instead, under the controller set (48), the dynamic network (44) is finite-time synchronisation $t^* \leq \dfrac{V(0, r(0))^{1-(1+\beta)/2}}{2k\left(1 - \dfrac{1+\beta}{2}\right)}$, where 0 < β < 1, $k = \min(k_i(r)), i \in J, r \in S.$

$V(0, r(0)) = q_{r(0)} \sum_{i=1}^{N} e_i^T(0) e_i(0)$, $e_i(0)$ is the condition, i = 1, 2, ..., N.

b) Finite-time Synchronization of MJCNs with partially unknown transition rates and time delays

Time delays are well known to be inevitable when designing complex network models. Hence, considering the dynamics for the complex networks with time delays is very important.. In this section, the finite-time synchronization conditions for the presented MJCNs with time delays are obtained by the use of the finite-time stability theory.

Consider a complex Markovian jump network consisting of N equal nodes with time delays, in which each node is a dynamic system in *n* dimensions, i.e.

$$x'_i(t) = f(x_i(t), x_i(t-\tau)) + c\sum_{j=1}^{N} a_{ij}(r(t))\Gamma x_j(t) + u_i(t), \quad i \in N \tag{49}$$

where $\tau > 0$ is the time-delay of node *i*, $f(x_i(t), x_i(t-\tau))$ Is a continuous function indexed by vectors.

The Network Isolated Node (50) is given by $s'(t) = f(s(t), s(t-\tau)\, t),$ (50)

Where the basic device solution is (46).

The following dynamical error system is generated by subtracting (49) from

(48): $e'_i(t) = F(e_i(t), e_i(t-\tau)) + c\sum_{j=1}^{N} a_{ij}(r(t))\Gamma e_j(t) - \varepsilon_i(r)\Gamma e_i(t) - k_i(r)sign(e_i(t))|e_i(t)|\beta,\ i \in N$ (51)

Where $\varepsilon_i(r) > 0, k_i(r) > 0,\ i \in J,$ otherwise

$\varepsilon_i(r) = k_i(r) = 0,\ i \notin J,\ r \in S,$

$F(e_i(t), e_i(t-\tau)) = f(x_i(t), x_i(t-\tau)) - f(s(t), s(t-\tau)),\ e_i(t) = x_i(t) - s(t),\ i \in$.

The following theorem is presented to give Markovian jump complex networks finite-time synchronization (49) with some uncertain probabilities of transfer and time delays to the homogenous trajectory (50).

Theorem 14: Let Hypotheses 3 and 4 hold. When

$$\begin{cases} \left(\dfrac{(1-\sigma+\rho+\pi_r)I_N}{\lambda_{\min}(\Gamma)} + 2cA(r) - 2\Xi(r) \right) \otimes \Gamma \le 0, \\ (1-\sigma-\varphi)I_N \ge 0, \\ q_i - a_j \le 0 \ \ if\ \ i \ne j, \quad i \in S_2^T, \\ q_i - a_j \ge 0 \ \ if\ \ i = j, i \in S_2^T, \end{cases}$$

Where the diagonal matrix $\Xi(r) = diag\{\varepsilon_1(r), \varepsilon_2(r), ..., \varepsilon_N(r)\} \in$ $^{N \times N}$ with entries $\varepsilon_i(r) > 0,\ i \in J,$

otherwise, $\varepsilon_i(r) = 0,\ i \notin J,\ \pi_r = \sum_{i \in S_1^T} \dfrac{\pi_{ji}(q_i - b_j)}{q_j} (b_j > 0,\ q_j > 1), 0 < \sigma < 1.$

Therefore, under the controls set (49), the complex network (50) is finite time synchronization

$t^* \le \tau + \dfrac{V(0, r(0))^{1-(1+\beta)/2}}{\gamma\left(1 - \dfrac{1+\beta}{2}\right)}$, where 0 < β < 1, $\gamma = \min(2k, \lambda\sigma),\ k = \min(k_i(r)),\ i \in J,\ r \in S.$ 、

$V(0, r(0)) = q_{r(0)} \sum_{i=1}^{N} e_i^T(0) e_i(0),\ e_i(0)$ is the condition, i = 1, 2, ..., N.

IV. APPLICATIONS OF SYNCHRONIZATION OF COMPLEX NETWORKS

Complex network synchronization or controllability has been widely applied in many fields such as power grids and neuroscience and other fields which are briefly evaluated as follows:

1. The synchronization or controllability of complex networks has been commonly observed in natural systems such as transport systems, chemical reactions and communications (Arenas et al., 2008; Pikovsky et al., 2001; Vicsek & Zafeiris, 2012) as well as in other fields such as serving as a novel model for drug discovery in molecular networks (Csermely et al., 2013), understanding cancer progression .
2. Due to increasingly complex interconnections at continent scale and environmental opportunities, the modern power grid faces diverse challenges (Giannakis et al., 2014). The optimal power grid is designed to provide unparalleled visibility and controllability of its facilities and assets in order to provide rapid and reliable diagnosis / prognosis, operational flexibility to contingencies and deliberate attacks (Pasqualetti, Döfler & Bullo, 2013) and continuous incorporation of distributed renewable energy resources (Giannakis et al., 2014).
3. Power grid synchronization has been a classic engineering subject for more than three decades (Ribbens-Pavella & Evans, 1985) and is nowadays a longstanding and continuing research endeavour (Chiang et al., 1995; Machowski et al., 2008), which is an imperative element in the functioning of a power grid.
4. A study for transient stability analysis of large-scale electrical power systems with two distinct methodologies is presented in 1985 (Ribbens-Pavella & Evans, 1985). The first approach explores the application of the Lyapunov direct method to the study of traditional transient stability. The second approach focuses on deriving stability indexes, which are targeted at online surveillance, risk assessment and security regulation. Following this study, Chiang et al (1995) offers a corresponding

systematic review for the transient stability analysis. Synchronization of power grids networks has also been studied in Menck, Heitzig, Kurths, and Schellnhuber (2014), based on the basin stability suggested in Menck et al. (2013), to preserve robustness. Basin stability actually falls within the framework of transient stability which aims to characterize the attraction area. The method used in Menck et al. (2013), Menck et al. (2014) for basin stability is in fact the time-domain approach used in Chiang et al. (1995) or the numerical integration approach used in Ribbens-Pavella and Evans (1985). Based on the graph theory, it is found that dead ends and dead trees significantly reduce power grid stability. The principle of basin stability refers to the Northern European power structure, which supports this finding and verifies that the opposite holds as well.

5. In Motter, Myers, Anghel, and Nishikawa (2013), easy-to-verify conditions are obtained by a linearization approach for spontaneous synchrony in power grid networks, which is in reality low signal stability of power grids (Machowski et al., 2008). Nonetheless, new conditions for synchronization are introduced due to the implementation of the graph theory and the master stability function, and the optimization of synchronization efficiency is also investigated (Motter et al., 2013).
6. Latest developments in structural and functional magnetic resonance imaging, diffusion tensor imaging, magneto encephalography and electro encephalography, as well as modern methods of dynamic network theory, facilitate research into the structural and functional structures of the brain. Brain network has been shown to have a spatial topology and representative properties of complex networks, such as the presence of strongly connected hubs, small-world topology, and modularity-both at the entire brain (a macroscopic level) and cellular (a microscopic level) scale (Bullmore & Sporns, 2009; Bullmore & Sporns, 2012; Gu et al., 2014; Tang et al., 2013a; Zamora-López et al., 2010).
7. Synchronization of distributed brain activity has been confirmed to play a key role in the processing and synchronization of neural information (Engel, Fries, & Singer, 2001; Palva, Monto, Kulashekhar, & Palva, 2010; Uhlhaas & Singer, (2006); Abnormal neural synchronization is found from studies to be closely linked to schizophrenia, depression, autism, Alzheimer's disease, and Parkinson's disease as demonstrated in Palva et al. (2010), Tang et al. (2012c, 2012b, 2013a), Dahlem et al. (2013), studying the synchronization or controllability of a neuronal network remains of great importance, not only having a thorough understanding of the intrinsic features of weighted and guided synchronization or control networks, but also having some suggestions to avoid irregular synchronization;

A 2-D SCDN adaptive synchronization to image encryption based on the proposed spatiotemporal Crypto system in [19, 20].

V. CONCLUSIONS

A brief review is presented in this paper on recent developments in finite-time and fixed-time synchronization of complex dynamic networks with non-identical discontinuous nodes, time delay, output-coupling class by continuous control and complex Markovian jump networks. Then, several synchronization applications in complex networks, especially in neuroscience and power grids, were viewed. This paper's main aim is to make some new innovations and serve as good advice for people working in the area. If any of the latest published studies on the subject are missed, we give the writers and readers our apologies. There are some issues that should be addressed in future research given diverse findings. Some of them we high-light as follows:

(i) One can investigate the effect of communication network topologies and network-induced constraints on pinning controllability efficiency, pinning observability and pinning synchronization for a complex dynamic network with general topology. Since the shared communication network is vulnerable to malicious attacks and exploits, the issue of cyber security has received increasing interest in research and needs to be investigated in depth, especially for application-level multi-agent systems. In addition, problems related to stochastic complex network and randomly occurring pinning control strategies are interesting topics for study. It is also interesting and important to consider a complex dynamic network with non-identical nodes and the case that the pinning cost and the quantity of pinned nodes are finite for a large-scale, distributed, directed network.

(ii) It is still important to further examine the design of controllers with delays to achieve finite-time synchronization of coupled neural networks.

(iii) Another interesting but challenging problem is the analysis of finite-time synchronization of coupled neural networks with discontinuous activation functions through discontinuous controllers.

(iv) Power grids are becoming more distributed, smarter and more versatile. Nowadays, as small distributed power generators and decentralized energy storage systems need to be connected to the power network, smart grids are being introduced to supply electricity from producers to customers in order to conserve resources, thus lowering costs and increasing efficiency and transparency.

(v) Due to the advent of micro grids in power grids nowadays, it is becoming more important to use a droop controller to prevent the propagation of currents between converters without any essential communication). It is therefore imperative that complex network theory and control theory be used to enhance the controllability of power grids by the use of hierarchical control.

(vi) In the case of complex network synchronization studies, it is important to incorporate distinct network-induced constraints into the synchronization structures, such as time delays, time-varying sampling intervals, packet dropouts, saturations, communication noises and quantization errors, where network-induced constraints can be modelled either deterministically or stochastically.

(vii) Methods from control theory to the synchronization of complex networks, such as linear system theory (Rough, 1996), nonlinear system theory (Khalil, 2002) and stochastic system theory (Mao, 2007) are promising. In addition, the synchronization of complex networks can be considered under different output indices. Furthermore, statistical knowledge in various scales (microscopic, mesoscopic and macroscopic scales) can also be incorporated into the conventional control theory to capture the main points of complex systems, thereby promoting a detailed understanding of large-scale networked structures. Using these tools will provide solutions that not only deal with different types of problems in complex network functions, but also provide a more accurate way to understand the coordination of complex networks in the real world.

(viii) Including model-based methods, data-driven control uses the knowledge gathered from the available measurements to explain specific complex behaviours (Yin, Ding et al, 2014), thereby providing an effective control strategy for complex engineering applications. As for heterogeneous complex networks (Zhang et al, 2014), in complicated circumstances the self-dynamics of such nodes may be uncertain and time-varying. Hence, developing controllers is important in realizing synchronization based on historical data analysis.

(ix) Intelligent methods for modelling dynamics of complex networks, such as neural networks and fuzzy structures can be adopted. In addition, single objective or multi-objective evolutionary algorithms and constraint evolutionary algorithms are promising to serve as a candidate for managing complicated problems of optimization in complex networks such as problems of controllability.

(x) It should be noted that comprehensive studies incorporating some of the above topics are not yet appropriate, especially for the controllability of interdependent, complex networks and the robustness of complex network control (Bakule, 2014). Considering controllability, robustness, multiple layers, particularly applying the results in robotic systems, neuroscience and power grids would be challenging and promising at the same time.

Experimental findings, including a decrease in histogram variance, low PSNR, entropy closeness to 8 and a small association between plain images and ciphered images, indicate successful implementation of the theoretical results obtained in (Tengda Wei et al, 2017). Future work will concentrate on implementing permutation operation in encryption scheme, as NPCR (number of pixel change rate) and UACI (unified average change intensity) cannot achieve desired performance and hopefully the permutation operation will solve the problems.

ACKNOWLEDGMENT

Authors would like to thank the anonymous referees for their valuable comments and careful reading to the improvement of the manuscript.

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