On Hypercyclic Operators

Varughese Mathew[#1]
[#]Associate Professor & Department of Mathematics & Mar Thoma College Tiruvalla
Kerala, India – 689 103.

Abstract — *The set $\{T^n x: n \geq 0\}$ is called the orbit of a vector x in a linear topological space X under a linear map T and is denoted by $Orb(T, x)$. If dense orbits exist, T is called a hypercyclic operator. In this paper, a hypercyclic vector is constructed based on the existing sufficient condition for hypercyclicity of an operator T. From a separable Banach space Y and a sequence of real numbers $\{\beta(n)\}$, a sequence space $(Y)_\beta$ is defined and proved that the backward shift operator acting on $(Y)_\beta$ is hypercyclic.*

Keywords — *Orbit of a vector, Hypercyclic operator, Invariant subspace problem.*

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I. INTRODUCTION

In this paper by an operator we shall mean a linear map from a linear space into itself. Let T be an operator on a linear topological space X. The orbit of the vector x in X under the operator T is denoted and defined as $\mathrm{Orb}(T, x) = \{T^n x: n \geq 0\}$. A vector x in X is called 'hypercyclic' under the operator T if the orbit of x under T is dense in X. The operator T is said to be 'hypercyclic on X' if there is at least one vector x in X whose orbit under T is dense in X.

If x is a hypercyclic vector for an operator T, then since $\mathrm{Orb}(T, x)$ is dense in X, every vector in X is a limit point of the orbit of x. The first hypercyclicity theorem was obtained by Birkhoff [1]. He proved that translation operators other than the identity operator acting on $H(\square)$, the space of entire functions of one complex variable are hypercyclic. Later MacLane[8], showed that differentiation operator acting on $H(\mathbb{C})$ is hypercyclic. The study of hypercyclicity for Banach and Hilbert space operators are originated with Rolewicz[9], who showed that for a scalar $\alpha > 1$, the scalar multiple of the backward shift operator on $l^p, 1 \leq p < \infty$, the space of all p-summable sequence spaces and on c_0, the separable Banach space of all real sequences (u_n) such that $\lim_{n\to\infty} u_n = 0$ with norm $\| u_n \|_\infty = \sup_{n\geq 1}\{|u_n|\}$ has a hypercyclic vector. The following sufficient condition for hypercyclicity of an operator was proved independently by Kitai[7] and Gethner and Shapiro[3] and usually call it as KGS sufficient condition for hypercyclicity.

Theorem 1.1

Suppose T is a bounded linear operator on a separable Banach space X. Suppose there exists a dense subset D of X and a right inverse S for T (TS=identity of X) such that $\|T^n x\| \longrightarrow 0$ and $\|S^n x\| \longrightarrow 0$ for every $x \in D$. Then T is hypercyclic on X.

In the next section, we construct the form of the hypercyclic vector guaranteed in Theorem 1.1. We construct a common hypercyclic vector for T and T^{-1}, under certain conditions. We define a sequence space $(Y)_\beta$, from a separable Banach space and proved that the Backward shift operator acting on $(Y)_\beta$ is hypercyclic.

II. MAIN RESULTS

In the proof of Theorem 1.1, the authors guaranteed the existence of a hypercyclic vector using Baire Category Theorem. But they didn't tell explicitly, the form of that vector. In the following theorem we construct a hypercyclic vector, under almost the same hypotheses of Theorem 1.1.

Theorem 2.1

Suppose T is a bounded linear operator on a separable Banach space X. Suppose there exists a countable dense subset D=$\{y_n\}$ of X and a right inverse S of T ($TS = I$) such that the sequence of powers T^n and S^n tends pointwise to zero on D. Then T is hypercyclic on X.

Proof.

We have $T^m y_1 \longrightarrow 0$ and $S^m y_1 \longrightarrow 0$ as $m \longrightarrow \infty$. Therefore for $\frac{1}{2^{n+1}}$ there is a positive integer $r(1)$ such that $\|T^n y_1\| < \frac{1}{2^{n+1}}$ and$\|S^m y_1\| < \frac{1}{2^{n+1}}$ for all $m \geq r(1)$.

Also $T^m y_2 \longrightarrow 0$ and $S^m y_2 \longrightarrow 0$.Therefore, for $\frac{1}{2^{n+2}}$ there is a positive integer N_1 such that $\|T^m y_2\| < \frac{1}{2^{n+2}}$ and $\|S^m y_2\| < \frac{1}{2^{n+2}}$ for all $m \geq N_1$.

Let $r(2) = \max(N_1, r(1))$. Then $\|T^m y_2\| < \frac{1}{2^{n+2}}$, $\|S^m y_2\| < \frac{1}{2^{n+2}}$ for all $m \geq r(2)$.
Since $T^m y_n \longrightarrow 0$ and $S^m y_n \longrightarrow 0$ for all $n \geq 1$ as $m \longrightarrow \infty$, proceeding the above process indefinitely, we get a sequence $\{r(i)\}$ of positive integers such that $r(1) < r(2) < r(3) < \cdots < r(n) < \cdots$ and $\|T^m y_i\| < \frac{1}{2^{n+i}}$ and $\|S^m y_i\| < \frac{1}{2^{n+i}}$ for all $m \geq r(i)$, $i = 1,2,3, \ldots$
Let $p(n) = \sum_{i=1}^{n} r(i)$.
Consider the series $\sum_{n=1}^{\infty} S^{p(n)} y_n$.
Now $\| \sum_{n=1}^{\infty} S^{p(n)} y_n \| \leq \sum_{n=1}^{\infty} \| S^{p(n)} y_n \| < \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \cdots = \frac{1}{2^{n+1}} \times 2 = \frac{1}{2^n} < \infty$.
Let $x_0 = \sum_{n=1}^{\infty} S^{p(n)} y_n$.
Claim:- x_0 is a hypercyclic vector for T.
Now $T^{p(n)} x_0 = T^{p(n)} \sum_{m=1}^{n-1} S^{p(m)} y_m + y_n + T^{p(n)} \sum_{m=n+1}^{\infty} S^{p(m)} y_m$.
For, $1 \leq m \leq n-1$ we have $p(n) > p(m)$ and so $T^{p(n)} S^{p(m)} = T^{p(n)-p(m)} = T^{r(m+1)+r(m+2)+\cdots r(n)}$.
Also, for, $m > n$ we have $p(m) > p(n)$ and so $T^{p(n)} S^{p(m)} = S^{p(m)-p(n))} = S^{r(n+1)+r(n+2)+\cdots r(m)}$.
Hence, $T^{p(n)} x_0 - y_n = \sum_{m=1}^{n-1} T^{r(n)+r(n-1)+r(n-2)+\cdots r(m+1)} y_m + \sum_{m=n+1}^{\infty} S^{r(n+1)+r(n+2)+\cdots+r(m)} y_m$.
Then, $\|T^{p(n)} x_0 - y_n\| \leq \sum_{m=1}^{n-1} \| T^{r(n)+r(n-1)+\cdots+r(m+1)} y_m \| + \sum_{m=n+1}^{\infty} \| S^{r(n+1)+r(n+2)+\cdots+r(m)} y_m \|$

$$< \sum_{m=1}^{n-1} \frac{1}{2^{n+m}} + \sum_{m=n+1}^{\infty} \frac{1}{2^{n+m}}$$

$$< \sum_{m=1}^{\infty} \frac{1}{2^{n+m}} = \frac{1}{2^n}.$$

Let x be an arbitrary element of X.
Let $\varepsilon > 0$ be given. Choose a positive integer n_1, such that $\frac{1}{2^{n_1}} < \frac{\varepsilon}{2}$.
Since the sequence $\{y_n : n \geq 1\}$ is dense in X, we can choose a y_m with $m > n$, such that $\|x - y_m\| < \frac{\varepsilon}{2}$.
Now, $\|T^{p(m)}{}_{x_0} - x\| \leq \|T^{p(m)} x_0 - y_m\| + \|y_m - x\| < \frac{1}{2^m} + \frac{\varepsilon}{2} < \varepsilon$.
Hence $\{T^n{}_{x_0} : n \geq 1\}$ is dense in X so that, x_0 is a hpercyclic vector for T. Hence the theorem.

Remarks 2.2

1) In Theorem 2.1, the condition $TS = I$ can be replaced by $T^k S^k = I$ for some $k > 0$. For, in the proof of the Theorem 2.1, take $p(n) = k \sum_{i=1}^{n} r(i)$.
2) In Theorem 1.1, the condition TS = I cannot be replaced by TS+K, where K is a compact operator. To support this claim, see the following example.

Example 2.3

Consider the sequence space $H = l^2$.
Let $\{e_n\}$ be an orthonormal basis of H. Let B and S be the backward and forward shift operators acting l^2 respectively. That is, B$e_n = e_{n-1}$, B$e_1 = 0$ and S$e_n = e_{n+1}$. Then D=Span$\{e_n\}$ is dense in H.
Let $T = \lambda S$ and $F = \frac{B}{\lambda}$, $0 < \lambda < 1$.
Cleary, $T^n x \longrightarrow 0$ and $F^n x \longrightarrow 0$ for all $x \in D$.
Now $TF = SB = I + K$ where K is the compact operator on l^2 defined by $K(x_0, x_1, x_2, ..) = (-x_0, 0, 0, \ldots)$.
But T is not hypercyclic on l^2, since it is a contraction.

Kitai[7] proved the existence of a common hypercyclic vector for T and T^{-1}, where T is an invertible, bounded, hypercyclic operator on a separable Hilbert space H. Can we construct a common hypercyclic vector for T and T^{-1}?. The following theorem, answer this question partially.

Theorem 2.4

Let T be an invertible, bounded linear operator on a separable Hilbert space H such that the sequence of powers T^n and T^{-n} tends pointwise to zero on a countable dense subset $D = \{y_n\}$ of H. Then T and T^{-1} have a common hypercyclic vector.

Proof.

By the given hypothesis, there is a sequence $\{r(i)\}$ of positive integers such that $r(1) < r(2) < \cdots < r(i-1) < r(i) < \cdots < \cdots$ and for all $m \geq r(i)$, $1 \leq i \leq \infty$,
we have $\|T^m y_i\| < \frac{1}{2^{n+1+i}}$, $\|T^{-m} y_i\| < \frac{1}{2^{n+1+i}}$.
Let $p(n) = \sum_{i=1}^{n} r(i)$.
Take $x_0 = \sum_{n=1}^{\infty} T^{-p(n)} y_n$ and $y_0 = \sum_{n=1}^{\infty} T^{p(n)} y_n$.
Let $z_0 = x_0 + y_0$. Then z_0 is a hypercyclic vector for T and T^{-1}.
For, $T^{p(n)} z_0 = T^{p(n)} \sum_{n=1}^{\infty} T^{-p(n)} y_n + T^{p(n)} \sum_{n=1}^{\infty} T^{p(n)} y_n$.
Then $\|T^{p(n)} z_0 - y_n\| < \frac{1}{2^{n+1}} + \frac{1}{2^{n+1}} = \frac{1}{2^n}$.

Similarly, $\left\|T^{-p(n)}z_0 - y_n\right\| < \frac{1}{2^n}$.
Thus z_0 is a hypercyclic vector for T and T^{-1}.

Definition 2.5 [5]
Let H_1 and H_2 be two Hilbert spaces and $T: H_1 \longrightarrow H_2$ be a bounded linear operator, which has a closed range, then the generalized inverse (Moore-Penrose inverse) $T^\dagger$ is the unique operator from H_1 to H_2 which is also bounded, linear and satisfying.

1. $TT^\dagger = (TT^\dagger)^*$ 2. $T^\dagger T = (T^\dagger T)^*$
3. $TT^\dagger T = T$ 4. $T^\dagger T\, T^\dagger = T^\dagger$

Remark 2.6
Kitai[7] proved that an invertible, bounded linear operator T on a separable Hilbert space H is hypercyclic $\Leftrightarrow$ T^{-1} is hypercyclic. We ask the same result for a bounded linear operator T on a separable Hilbert space H, whose generalized inverse $T^\dagger$ exist. We show in the following example that this is not true in the case of generalized inverse.

Example 2.7
Let X=l^2. Let $T = \lambda \mathrm{B}, \lambda > 1$ where B is the backward shift operator on X. Then $T^\dagger = \frac{1}{\lambda}S$, where S is the forward shift operator on X. For,

1). $TT^\dagger T = \lambda \mathrm{B}\frac{1}{\lambda}S\,\lambda \mathrm{B} = T$.
2). $T^\dagger T T^\dagger = \frac{1}{\lambda}S\,\lambda \mathrm{B}\frac{1}{\lambda}S = T^\dagger$.
3). $(TT^\dagger)^* = \left(\lambda \mathrm{B}\frac{1}{\lambda}S\right)^* = (\frac{1}{\lambda}S)^*(\lambda \mathrm{B})^* = \frac{1}{\lambda}\mathrm{B}\,\lambda S = \lambda \mathrm{B}\frac{1}{\lambda}S = TT^\dagger$ and
4). $(T^\dagger T)^* = \left(\frac{1}{\lambda}S\lambda \mathrm{B}\right)^* = (\lambda S\frac{1}{\lambda}\,\mathrm{B}) = \frac{1}{\lambda}S\lambda \mathrm{B} = T^\dagger T$.

But $\| T^\dagger \| < 1$, and so $T^\dagger$ is a contraction mapping on X. Therefore $T^\dagger$ is not hypercyclic.

Problem 2.1. If T and its generalised inverse $T^\dagger$ are hypercyclic on a separable Hilbert space H, then is T invertible?

Remark 2.8
Rolewicz[9], proved that the scalar multiple of the Backward shift operator by a scalar greater than one is hypercyclic on the sequence spaces $(Y)_{l^p}$ and $(Y)_{c_0}$ where Y being an arbitrary separable Banach space. Also, he remarked that the above result is true for the sequence space $(Y)_s$, Y being an arbitrary F-space under the F-norm, for the scalar multiple of the Backward shift by a scalar greater than zero. Now we define a sequence space $(Y)_\beta$ from an arbitrary separable Banach space and a sequence $\{\beta(n)\}$ of positive numbers. We will show that the backward shift operator B acting on $(Y)_\beta$ is hypercyclic if $\beta(n) \longrightarrow 0$.

Definition 2.9
Let Y be an arbitrary separable Banach space. Let $\{\beta(n)\}$ be a sequence of positive numbers with $\beta(0) = 1$. Define the sequence space $(Y)_\beta$ as $(Y)_\beta = \{y = (y_n): y_n \in Y$ and $\| y \|_\beta{}^2 = \sum_{n=0}^{\infty} \| y_n \|_Y{}^2\, \beta(n) < \infty\}$, where $\|.\|_Y$ denotes the norm on Y. It can be shown that, $(Y)_\beta$ is a Banach space.

Proposition 2.10
Let Y be an arbitrary separable Banach space. Let $\{\beta(n)\}$ be a bounded sequence of positive numbers with $\beta(0) = 1$ and $M = \max_n\{\beta(n)\}$. Let E be a countable dense subset of Y. Then the set N of all sequences of the form $\tilde{x} = (x_0, x_1, x_2, \dots, x_n, 0, 0, \dots.)$ with all but finitely many co-ordinates are zero is dense in $(Y)_\beta$, where $(x_n) \in E$.
Proof.
Let $\tilde{y} = (y_0, y_1, y_2, \dots) \in (Y)_\beta$. So, $\| \tilde{y} \|_\beta{}^2 = \sum_{k=0}^{\infty} \| y_k \|_Y{}^2\, \beta(k) < \infty$.

Therefore, for every $\varepsilon > 0$ there is an n (depending on ε) such that $\sum_{k=n+1}^{\infty} \| y_k \|_Y{}^2\, \beta(k) < \frac{\varepsilon^2}{2}$, since the left side of this inequality is the remainder of a convergent series.

Since E is dense in Y, for each $y_i \in Y, 1 \le i \le n$ there exists elements $x_i \in E$ such that $\| y_k - x_k \|_Y{}^2 < \frac{\varepsilon^2}{2nM}$.

Therefore, $\| y_k - x_k \|_Y{}^2\, \beta(k) < \frac{\varepsilon^2 M_0}{2nM}$, where $M_0 = \max_{1 \le k \le n}\{\beta(k)\}$.

Thus $\sum_{k=1}^{n} \| y_k - x_k \|_Y{}^2\, \beta(k) < \frac{\varepsilon^2}{2}$.

Let $\tilde{x} = (x_0, x_1, x_2, \dots, x_n, 0, 0, \dots)$.
Now, $\| \tilde{y} - \tilde{x} \|_\beta^{\,2} = \sum_{k=1}^{n} \| y_k - x_k \|_Y^{\,2}\, \beta(k) + \sum_{k=n+1}^{\infty} \| y_k \|_Y^{\,2}\, \beta(k) < \frac{\varepsilon^2}{2} + \frac{\varepsilon^2}{2} = \varepsilon^2$.
That is, $\| \tilde{y} - \tilde{x} \|_\beta < \varepsilon$. Hence N is a countable dense subset in $(Y)_\beta$.

Definition 2.11
Let X be a normed linear space. A mapping $T: X \longrightarrow X$ is said to be Lipschitzian of order 1 if there exist $M > 0$ such that $\| Tx - Ty \| \le M \| x - y \|$ for all $x, y \in X$. The Lipschitzian mapping is non-expansive if $M \le 1$ and contraction if $M < 1$.

There are hypercyclic operators which are not non-expansive. For example, the operator $T = aB$, $a > 1$,where B is the backward shift operator on $l^p, p \ge 1$ is hypercyclic but not non-expansive.

Lemma 2.12
The backward shift operator B acting on $(Y)_\beta$ is non-expansive.
Proof.
Consider $(Y)_\beta$. Let $y, x \in (Y)_\beta$.
Now $\| By - Bx \|_\beta^{\,2} = \sum_{n=0}^{\infty} \| y_{n+1} - x_{n+1} \|_Y^{\,2}\, \beta(n) = \sum_{n=0}^{\infty} \| y_{n+1} - x_{n+1} \|_Y^{\,2}\, \beta(n) \frac{\beta(n+1)}{\beta(n+1)}$

$$\le \sup_{n \ge 0} \frac{\beta(n)}{\beta(n+1)} \sum_{n=0}^{\infty} \| y_{n+1} - x_{n+1} \|_Y^{\,2}\, \beta(n+1)$$

$$\le \sum_{n=1}^{\infty} \| y_n - x_n \|_Y^{\,2}\, \beta(n) = \| y - x \|_\beta^{\,2}.$$

That is, $\| By - Bx \|_\beta \le \| y - x \|_\beta$. So B is a non-expansive mapping.

In the next theorem, we prove that even though the backward shift operator B is non-expansive on the sequence space $(Y)_\beta$, it is hypercyclic on $(Y)_\beta$.

Theorem 2.13
Let X be the sequence space $(Y)_\beta$, where Y is a separable Banach space and $\{\beta(n)\}$ be a sequence of bounded positive numbers with $\beta(0) = 1$. In addition if $\beta(n) \longrightarrow 0$ as $n \longrightarrow \infty$ then the backward shift operator B acting on X is hypercyclic.
Proof.
Let E be a countable dense subset of Y.
By the Proposition 2.10, the set of all sequences N of the form $\tilde{x} = (x_0, x_1, x_2, \dots, x_m, 0, 0, \dots)$ with all but finitely many co-ordinates are zero is dense in $(Y)_\beta$, where $(x_m) \in E$.
Let $y = (y_0, y_1, y_2, \dots) \in (Y)_\beta$.
Define the backward shift operator B on $(Y)_\beta$ by $B(y_0, y_1, y_2, \dots) = (y_1, y_2, y_3, \dots)$.
$\| By \|_\beta^{\,2} = \sum_{k=1}^{\infty} \| y_k \|_Y^{\,2}\, \beta(k) \le \| y \|_\beta^{\,2}$. Thus B is a bounded linear operator on $(Y)_\beta$.
Let $\tilde{x} = (x_0, x_1, x_2, \dots, x_m, 0, 0, \dots) \in N$. Then $B\tilde{x} = (x_1, x_2, \dots, x_m, 0, 0, \dots)$.
Thus $B^n \tilde{x} \longrightarrow 0$ for all sufficiently large n. That is, $B^n \longrightarrow 0$ pointwise on the dense subset N of $(Y)_\beta$.
Define the forward shift operator S on $(Y)_\beta$ by $S(y_0, y_1, y_2, \dots) = (0, y_0, y_1, y_2, \dots)$.
Then $BS = I$, the identity operator on N.
Also for $\tilde{x} = (x_0, x_1, x_2, \dots, x_m, 0, 0, \dots) \in N$, we have $S^n \tilde{x} = (\overbrace{0, 0, \dots, 0}^{n\ zeroes}, x_0, x_1, x_2, \dots, x_m, 0, 0, \dots)$.
Thus $\| S^n \tilde{x} \|_\beta^{\,2} = \sum_{k=0}^{m} \| x_k \|_Y^{\,2}\, \beta(k+n) \| y \|_\beta^{\,2}$, where $\|.\|_\beta$ and $\|.\|_Y$ denotes the norms on $(Y)_\beta$ and Y respectively. Since $\beta(n) \longrightarrow 0$ as $n \longrightarrow \infty$, we get $\| S^n \tilde{x} \|_\beta \longrightarrow 0$ for every $\tilde{x} \in N$.
Thus $S^n \longrightarrow 0$ pointwise on the dense subset N.
Thus by Theorem 2.1, the operator B is hypercyclic on $(Y)_\beta$. Hence the theorem.

Proposition 2.14 [2]
Suppose T is a bounded linear operator on a Hilbert space. If the adjoint of T has an eigenvalue, then T is not hypercyclic.

Remarks 2.15
1. Let $X = l^2$, the space of all 2-summable sequences. Define the operator A on X by $A(e_n) = \frac{e_n}{n}$ where $e_n, n = 1,2,3, \dots$ are the Schauder basis of X. It can be shown that the adjoint of A has an eigenvalue. So by

the Proposition 2.14, the operator A is not hypercyclic. Further A is non-expansive. That is, there are non-expansive mappings which are not hypercyclic.

2. Let $X = l^p, 1 \leq p \leq \infty$. For $0 < \alpha < 1$, define the operator $T = \alpha B$ on X, where B is the backward shift operator. Clearly, T is a contraction mapping on X. Thus $T^n x \longrightarrow 0$ for all x in X and so T is not hypercyclic on X. In general contraction mappings are not hypercyclic.

III. CONCLUSIONS

As already mentioned in the introduction, the study of hypercyclic operators seems to be of interest in the view of approximation theory. These well known concepts originated in connection with the famous Invariant Subspace Problem (Godefroy and Shapiro[4]). Does every continuous operator $\boldsymbol{T}$ on Hilbert space H have a nontrivial invariant subspace. That is, does there exist a closed subspace $\mathbf{0} \neq \boldsymbol{W} \subset \boldsymbol{H}$ such that $\boldsymbol{T}(\boldsymbol{W}) \subset \boldsymbol{W}$?. In the case of Banach spaces there do exist operators with only trivial invariant subspaces. For, Hilbert space, the Invariant Subspace Problem (ISP) remains open. For the following applications of hypercyclic operators we refer to Kuikui Liu[6]. Applications include dynamical systems and, in particular, chaos Dynamics is the study of the evolution of the states of a system. The states are elements of some metric space $\boldsymbol{X}$ and the evolution is described by a continuous operator $\boldsymbol{T}: \boldsymbol{X} \longrightarrow \boldsymbol{X}$. In the discrete case, the progression of the states of the system are indexed by a countable set. If $\boldsymbol{x_n} \in \boldsymbol{X}$ is the $\boldsymbol{n^{th}}$ stage of a system, then $\boldsymbol{x_{n+1}} = \boldsymbol{T}\boldsymbol{x_n}$ gives its $(\boldsymbol{n}+\mathbf{1})^{\boldsymbol{th}}$ stage. So, if there is no confusion regarding $\boldsymbol{X}$, the dynamical system $(\boldsymbol{X}, \boldsymbol{T})$ is simply written as $\boldsymbol{T}: \boldsymbol{X} \longrightarrow \boldsymbol{X}$. Chaotic system is another subtopic in linear dynamics. Small changes in the initial condition of the system may lead to behave differently. Chaotic systems are of great interest as there are many naturally occurring phenomena that exhibit chaotic behaviour and have applications to other fields. Dynamical systems have several applications as they provide tools to model various physical phenomena. They can describe the motion of various objects in Euclidean space, the flow of fluids, various engineering systems etc.

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