Super Fair Domination in the Corona and Lexicographic Product of Graphs

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Abstract: *A fair dominating set $S \subseteq V(G)$ is a super fair dominating set (or SFD-set) if for every $u \in V(G) \setminus S$, there exists $v \in S$ such that $N_G(v) \cap (V(G) \setminus S) = \{u\}$. The minimum cardinality of an SFD-set, denoted by $\gamma_{sfd}(G)$, is called the super fair domination number of G. In this paper, we characterize the super fair dominating set of the corona and lexicographic product of two graphs.*

Keywords: *dominating set, fair dominating set, super dominating set, corona, lexicographic*

I. Introduction

Domination in a graph has been a huge area of research in graph theory. Let G be a simple connected graph. A subset S of a vertex set $V(G)$ is a dominating set of G if, for every vertex $v \in V(G) \setminus S$, there exists a vertex $x \in S$ such that xv is an edge of G. The domination number $\gamma(G)$ of G is the smallest cardinality of a dominating set S of G. Domination in graph was introduced by Claude Berge in 1958 and Oystein Ore in 1962 [1]. Some related graph domination studies are found in [2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19].

One variant of domination in a graph is the fair domination in graphs [20]. A dominating subset S of $V(G)$ is a fair dominating set of G if all the vertices not in S are dominated by the same number of vertices from S, that is, $|N(u) \cap S| = |N(v) \cap S|$ for every two distinct vertices u and v from $V(G) \setminus S$ and a subset S of $V(G)$ is a k-fair dominating set in G if for every vertex $v \in V(G) \setminus S$, $|N(v) \cap S| = k$. The minimum cardinality of a fair dominating set of G, denoted by $\gamma_{fd}(G)$, is called the fair domination number of G. A fair dominating set of cardinality $\gamma_{fd}(G)$ is called γ_{fd}-set. A related paper on fair domination in graphs is found in [21,22]. Other variant of domination in a graph is the super dominating sets in graphs initiated by Lemanska et.al. [23]. A set $D \subset V(G)$ is called a super dominating set if for every vertex $u \in V(G) \setminus D$, there exists $v \in D$ such that $N_G(v) \cap (V(G) \setminus D) = \{u\}$. The super domination number of G is the minimum cardinality among all super dominating set in G denoted by $\gamma_{sp}(G)$. Variation of super domination in graphs can be read in [24,25,26,27].

A fair dominating set $S \subseteq V(G)$ is a super fair dominating set (or SFD-set) if for every $u \in V(G) \setminus S$, there exists $v \in S$ such that $N_G(v) \cap (V(G) \setminus S) = \{u\}$. The minimum cardinality of an SFD-set, denoted by $\gamma_{sfd}(G)$, is called the super fair domination number of G. The super fair dominating set was initiated by Enriquez [28]. In this paper, we extend the idea of super fair dominating set by characterizing the super fair dominating sets of the corona and lexicographic product of two graphs. For general concepts we refer the reader to [29].

II. Results

Remark 2.1 *A super fair dominating set is a super dominating and a fair dominating set of a nontrivial graph G.*

Since the minimum super dominating set S of a nontrivial complete graph K_n is $n-1$, it follows that $\gamma_{sfd}(K_n) = n-1$. With this observation, the following remark holds.

Remark 2.2 Let *G be a nontrivial connected graph G of order n. Then* $1 \le \gamma_{fd}(G) \le \gamma_{sfd}(G) \le n-1$.

Let G and H be graphs of order m and n, respectively. The corona of two graphs G and H is the graph $G \circ H$ obtained by taking one copy of G and m copies of H, and then joining the ith vertex of G to every vertex of the ith copy of H. The join of vertex v of G and a copy H^v of H in the corona of G and H is denoted by $v + H^v$.

The next result is the characterization of the super fair dominating set in the corona of two graphs.

Theorem 2.3 *Let G and H be nontrivial connected graphs. Then a proper subset S of $V(G \circ H)$ is a super fair dominating set of $G \circ H$ if and only if one of the following statements holds:*

(i) $S = \bigcup_{v \in V(G)} V(H^\wedge v)$.

(ii) $S = S_G \cup \bigcup_{v \in V(G)} V(H^v)$ where $S_G \subset V(G)$ is a fair dominating set of G.

(iii) \$$S = V(G) \cup \left(\bigcup_{v \in V(G)} S_v\right)$ where S_v is a super fair dominating set of H^v for all $v \in V(G)$

(iv) $S = V(G) \cup (\bigcup_{v \in X} S_v) \cup (\bigcup_{v \in V(G)\setminus X} V(H^v))$ where S_v is a super fair dominating set of H^v for each $v \in X$ and $X \subseteq V(G)$.

Proof: Suppose that a proper subset S of $V(G \circ H)$ is a super fair dominating set of $G \circ H$. Then S is a fair dominating set of $G \circ H$ such that for every $u \in V(G \circ H) \setminus S$, there exists $v \in S$ such that $N_{G \circ H}(v) \cap V(G \circ H) \setminus S = \{u\}$. Consider the following cases:

Case 1. Suppose that $V(G) \cap S = \emptyset$. Then $S \subseteq V(G \circ H) \setminus V(G) = \bigcup_{v \in V(G)} V(H^v)$, that is, $S \subseteq \bigcup_{v \in V(G)} V(H^v)$. Suppose that there exists $x \in \bigcup_{v \in V(G)} V(H^v)$ such that $x \notin S$. Then $x \in V(G \circ H) \setminus S$. Since H is nontrivial connected graphs, let $y \in V(H^v) \setminus \{x\}$ for some $v \in V(G)$ such that $xy \in E(G \circ H)$ and $y \in S$. Then $N_{G \circ H}(y) \cap V(G \circ H) \setminus S = \{x, v\}$ contradict to our assumption that S is a super dominating set of $G \circ H$. Thus, there does not exist $x \in \bigcup_{v \in V(G)} V(H^v)$ such that $x \notin S$. This implies that for all $x \in \bigcup_{v \in V(G)} V(H^v)$, $x \in S$, that is, $\bigcup_{v \in V(G)} V(H^v) \subseteq S$. By principle of set equality, $S = \bigcup_{v \in V(G)} V(H^v)$ proving statement (i).

Case 2. Suppose that $V(G) \cap S \neq \emptyset$. Let $S_G \subseteq V(G) \cap S$. Then $S_G \subseteq V(G)$ and $S_G \subseteq S$. Consider the following subcases:

Subcase 1. Suppose that $S_G \subset V(G)$. Then there exists $x \in V(G) \setminus S_G$ such that $x \in V(G \circ H) \setminus S$. If $S_G = S$, then $S \subset V(G)$. This implies that S cannot be a dominating set of $G \circ H$ contrary to our assumption. Thus, S_G must be a proper subset of S, that is, $S_G \subset S$. Let $z \in S \setminus S_G$. Then $z \in V(H^x)$for some $x \in V(G) \setminus S_G$. Suppose that there exists $y \in \bigcup_{x \in V(G)\setminus S_G} V(H^x)$such that $y \notin S$. Let $yz \in E(G \circ H)$for some $z \in S \cap (V(H^x) \setminus \{y\})$. Then $N_{G \circ H}(z) \cap (V(G \circ H) \setminus S) = \{x, y\}$ contradict to our assumption that S is a super dominating set of $G \circ H$. Thus, there does not exist $y \in \bigcup_{x \in V(G)\setminus S_G} V(H^x)$ such that $y \notin S$. This implies that for every $y \in \bigcup_{x \in V(G)\setminus S_G} V(H^x)$, $y \in S$, that is, $\bigcup_{x \in V(G)\setminus S_G} V(H^x) \subseteq S$. Thus, $S_G \cup \left(\bigcup_{v \in V(G)} V(H^v)\right) \subseteq S$. Since $S_G \subseteq V(G) \cap S$ and $S_G \subset V(G)$, it follows that $(V(G) \setminus S_G) \cap S = \emptyset$. Thus, $S \subseteq V(G \circ H) \setminus (V(G) \setminus S_G) = S_G \cup \left(V(G \circ H) \setminus V(G)\right) = S_G \cup (\bigcup_{v \in V(G)} V(H^v))$. Hence, $S = S_G \cup (\bigcup_{v \in V(G)} V(H^v))$ by principle of set equality. Now, suppose that S_G is not a fair dominating set of G. Then there exist $u, v \in V(G) \setminus S_G$ such that $|N_G(u) \cap S_G| \neq |N_G(v) \cap S_G|$. Thus, there exist $u, v \in V(G \circ H) \setminus S$ such that $|N_{G \circ H}(u) \cap S| \neq |N_{G \circ H}(v) \cap S|$. Consequently, S is not a fair dominating set of $G \circ H$ contrary to our assumption that S is a super fair dominating set of $G \circ H$. Thus, S_G is must be a fair dominating set of G. Similarly, if S_G is not a super dominating set, then S is not a super dominating set of $G \circ H$ contrary to our assumption that S is a super fair dominating set of $G \circ H$. Hence, S_G must be a super dominating set of G. Therefore S_G is a super fair dominating set of G. This proves statement (ii).

Subcase 2. Suppose that $S_G = V(G)$. If $S_G = S$, then S cannot be a super dominating set of $G \circ H$ since H is a nontrivial connected graph. Thus, S_G must be a proper subset of S, that is, $S_G = V(G) \subset S$. *Let* $x \in S \setminus V(G)$. Then $x \in S \cap V(H^v)$ for some $v \in V(G)$. Set $S_v = S \cap V(H^v)$. Then $x \in \bigcup_{v\in V(G)} S_v$. Thus, $S \setminus V(G) \subseteq \bigcup_{v\in V(G)} S_v$, that is, $S \subseteq V(G) \cup (\bigcup_{v\in V(G)} S_v)$. Now, let $S_v \subset S$ for all $v \in V(G)$. Then $\bigcup_{v\in V(G)} S_v \subset S$. Since $V(G) \subset S$, it follows that $V(G) \cup (\bigcup_{v\in V(G)} S_v) \subseteq S$, that is, $S = V(G) \cup (\bigcup_{v\in V(G)} S_v)$. Next, suppose that S_v is not a fair dominating set of H^v. Then there exist $x, y \in V(H^v)$ such that $|N_{H^v}(x) \cap S_v| \neq |N_{H^v}(y) \cap S_v|$. Thus, there exist $x, y \in V(G \circ H) \setminus S$ such that $|N_{G\circ H}(x) \cap S| \neq |N_{G\circ H}(y) \cap S|$. Consequently, S is not a fair dominating set of $G \circ H$ contrary to our assumption that S is a super fair dominating set of $G \circ H$. Thus, S_v must be a fair dominating set of H^v for all $v \in V(G)$. Similarly, if S_v is not a super dominating set, then S is not a super dominating set of $G \circ H$ contrary to our assumption. Thus, S_v must be a super dominating set of H^v for all $v \in V(G)$. Hence, S_v is a super fair dominating set of H^v for all $v \in V(G)$. This proves statement (iii).

Finally, to show statement (iv), let $X \subset V(G)$. Then $(V(G) \setminus X) \subset V(G)$. In view of statement (iii),

$$V(G) \cup (\bigcup_{v\in V(G)} S_v) = V(G) \cup (\bigcup_{v\in X} S_v) \cup \left(\bigcup_{v\in V(G)\setminus X} S_v\right) \subseteq V(G) \cup (\bigcup_{x\in X} S_v) \cup (\bigcup_{v\in V(G)\setminus X} V(H^v)).$$

where S_v is a super fair dominating set of H^v for each $v \in X$ and $X \subseteq V(G)$. Clearly, the $V(G) \cup (\bigcup_{v\in X} S_v) \cup (\bigcup_{v\in V(G)\setminus X} V(H^v))$ is also a super fair dominating set of $G \circ H$. Set $S = V(G) \cup (\bigcup_{v\in X} S_v) \cup (\bigcup_{v\in V(G)\setminus X} V(H^v))$. This proves statement (iv).

For the converse, suppose that statement (i) is satisfied. Then $S = \bigcup_{v\setminus in V(G)} V(H^v)$. Let $u \in V(G \circ H) \setminus S = V(G)$. Then there exists $x \in S$ such that $ux \in E(G \circ H)$. Hence, S is a dominating set of $G \circ H$. Now, for each $u, v \in V(G)$, $|N_{G\circ H}(u)| = |V(H^u)| = |V(H^v)| = |N_{G\circ H}(v)|$. Hence, S is a fair dominating set of $G \circ H$. Further, for every $u \in V(G)$, there exists $x \in S$, say $x \in V(H^u)$, such that $N_{G\circ H}(x) \cap (V(G \circ H) \setminus V(H^u)) = \{u\}$. Hence, S is a super dominating set of $G \circ H$. Accordingly, S is a super fair dominating set of $G \circ H$.

Suppose that statement (ii) is satisfied. Then $S = S_G \cup (\bigcup_{v\setminus in V(G)} V(H^v))$.where $S_G \subset V(G)$ is a fair dominating set of G. In view of statement (i), S is a dominating set of $G \circ H$. Now, for each $u, v \in V(G \circ H) \setminus S = V(G) \setminus S_G$, $|N_{G\circ H}(u) \cap S| = |V(H^u)| + |N_G(u) \cap S_G|$ and $|N_{G\circ H}(v) \cap S| = |V(H^v)| + |N_G(v) \cap S_G|$. Since S_G is a fair dominating set of G, it follows that $|N_G(u) \cap S_G| = |N_G(v) \cap S_G|$. Thus, $|N_{G\circ H}(u) \cap S| = |N_{G\circ H}(v) \cap S|$. Hence S is a fair dominating set of $G \circ H$. Further, for every $x \in V(G \circ H) \setminus S = V(G) \setminus S_G$, there exists $y \in S$, say $y \in V(H^x)$ such that $N_{G\circ H}(y) \cap (V(G) \setminus S_G) = \{x\}$. Thus, S is a super dominating set of $G \circ H$. Accordingly, S is a super fair dominating set of $G \circ H$.

Suppose that statement (iii) is satisfied. Then $S = V(G) \cup (\bigcup_{v\setminus in V(G)} V(S^v))$ where S_v is a super fair dominating set of H^v for all $v \in V(G)$. Since $V(G) \subset S$, it follows that S is a dominating set of $G \circ H$. Now, for each $x, y \in V(G \circ H) \setminus S = \bigcup_{v\in V(G)} (V(H^v) \setminus S_v)$, $|N_{G\circ H}(x) \cap S| = |\{v\} \cup N_{H^v}(x)| = |\{v'\} \cup N_{H^v}(y)| = |N_{G\circ H}(y) \cap S|$. Hence, S is a fair dominating set of $G \circ H$. Further, for every $u \in V(G \circ H) \setminus S = \bigcup_{v\in V(G)} (V(H^v) \setminus S_v)$, there exists $z \in S$, say $y \in S_v$ such that $N_{G\circ H}(z) \cap (V(G) \setminus S_v) = \{x\}$ since S_v is a super dominating set of H^v where $v \in V(G)$, that is, $N_{G\circ H}(z) \cap (V(G) \setminus S) = \{x\}$. Thus, S is a super dominating set of $G \circ H$. Accordingly, S is a super fair dominating set of $G \circ H$. Following similar arguments used in proving statement (iii), if statement (iv) is satisfied, then S is a super fair dominating set of $G \circ H$. ■

The following result is an immediate consequence of Theorem 2.3.

Corollary 2.4 *Let G and H be a nontrivial connected graphs of order m and n respectively. Then*

$$\gamma_{sfd}(G \circ H) = min\{mn, m(\gamma_{sfd}(H) + 1)\}.$$

Proof: Suppose that S is a super fair dominating set of $G \circ H$. Then $\gamma_{sfd}(G \circ H) \leq |S|$. Further, S satisfies one of the statements in Theorem 2.3.

If $S = \bigcup_{v \in V(G)} V(H^v)$. Then $\gamma_{sfd}(G \circ H) \leq |S| = |\bigcup_{v \in V(G)} V(H^v)|$

$$= \sum_{v \in V(G)} |V(H^v)|$$

$$= |V(G)||V(H)| = mn.$$

If $S = S_G \cup (\bigcup_{v \in V(G)} V(H^v))$ where $S_G \subset V(G)$ is a fair dominating set of G. Then

$$\gamma_{sfd}(G \circ H) \leq |S| = |S_G| + |\bigcup_{v \in V(G)} V(H^v)|$$

$$= |S_G| + \sum_{v \in V(G)} |V(H^v)|$$

$$= |S_G| + |V(G)||V(H)| = |S_G| + mn.$$

Clearly, $mn < |S_G| + mn$ for all $|S_G| \geq 1$.

If $S = V(G) \cup (\bigcup_{v \in V(G)} S_v)$ where S_v is a super fair dominating set of H^v for all $v \in V(G)$. Then

$$\gamma_{sfd}(G \circ H) \leq |S|$$

$$= |V(G) \cup (\bigcup_{v \in V(G)} S_v)|$$

$$= |V(G)| + \sum_{v \in V(G)} |S_v|$$

$$= |V(G)| + |V(G)||S_v|$$

$$= |V(G)|(1 + |S_v|) = m(|S_v| + 1)$$

for all super fair dominating set S_v of H^v for each $v \in V(G)$. Thus, $\gamma_{sfd}(G \circ H) \leq m(1 + \gamma_{sfd}(H))$.

If $S = V(G) \cup (\bigcup_{v \in X} S_v) \cup (\bigcup_{v \in V(G) \setminus X} V(H^v))$ where S_v is a super fair dominating set of H^v for each $v \in X$ and $X \subseteq V(G)$, then

$$\gamma_{sfd}(G \circ H) \leq |S| = |V(G)| + |(\bigcup_{v \in X} S_v)| + |(\bigcup_{v \in V(G) \setminus X} V(H^v))|$$

$$= m + \sum_{v \in X} |S_v| + \sum_{v \in V(G)} |V(H^v)|$$

$$= m + |X||S_v| + (|V(G)| - |X|)|V(H^v)|$$

$$= m + |X||S_v| + (m - |X|)n$$

$$= m(n + 1) - |X|(n - |S_v|).$$

Since

$$m(|S_v| + 1) = m(n - n + |S_v| + 1)$$

$$= mn - m(n - |S_v| - 1)$$

$$= mn + m - m(n - |S_v|)$$

$$\leq mn + m - |X|(n - |S_v|)$$

$$= m(n + 1) - |X|(n - |S_v|)$$

for all $X \subseteq V(G)$, $m(|S_v| + 1) \leq m(n + 1) - |X|(n - |S_v|)$ for all $|X| \leq m$. Since $\gamma_{sfd}(G \circ H) \leq mn < |S_G| + mn$ for $|S_G| \geq 1$ and $\gamma_{sfd}(G \circ H) \leq m(|S_v| + 1) \leq m(n + 1) - |X|(n - |S_v|)$ for all $|X| \leq m$, it follows that either mn or $m(\gamma_{fd}(H) + 1)$ is a super fair domination number of $G \circ H$. Hence, $\gamma_{sfd}(G \circ H) = min\{mn, m(\gamma_{sfd}(H) + 1)\}$. ■

The lexicographic product of two graphs G and H is the graph $G[H]$ with vertex-set $V(G[H]) = V(G) \times V(H)$ and edge-set $E(G[H])$ satisfying the following conditions: $(x, u)(y, v) \in E(G[H])$ if and only if either $xy \in E(G)$ or $x = y$ and $uv \in E(H)$.

Theorem 2.5 *Let* $G = P_n = [x_1, x_2, \ldots, x_n]$, $n \geq 3$ *and* $H = K_2 = [y_1, y_2]$ *and* $S_G \subset V(G)$. *A proper subset* $S = (V(G) \times \{y_1\}) \cup (S_G \times \{y_2\})$ *of* $V(G[H])$ *is a super fair dominating set if one of the following statement is satisfied.*

(i) $S_G = \{x_1\} \cup \{x_{2k+1}: k = 1,2, \ldots, \frac{n-1}{2}\}$ *where* $n = 1(mod\ 2)$.

(ii) $S_G = \{x_{4k-3}, x_{4k}: k = 1,2, \ldots, \frac{n}{4}\}$ *where* $n = 0(mod\ 4)$.

(iii) $S_G = \{x_1\} \cup \left\{x_{4k}, x_{4k+1}: k = 1,2, \ldots, \frac{n-2}{4}\right\} \cup \{x_n\}$ *where* $n = 2(mod\ 4), n \neq 4$.

Proof: Let $G = P\_n = [x\_1, x\_2, \ldots, x\_n]$, $n \geq 3$ and $H = K_2 = [y_1, y_2]$. Suppose that statement (i) is satisfied. Then $S_G = \{x_1\} \cup \{x_{2k+1}: k = 1,2, \ldots, \frac{n-1}{2}\}$ where $n = 1(mod\ 2)$. Let $x_i, x_j \in V(G) \setminus S_G$ with $i \neq j$. Then $N_G(x_i) = \{x_{i-1}, x_{i+1}\}$ and $N_G(x_j) = \{x_{j-1}, x_{j+1}\}$. Since $x_i, x_j \notin S_G$, it follows that $x_{i-1}, x_{i+1} \in S_G$ and $x_{j-1}, x_{j+1} \in S_G$. Thus, $|N_G(x_i) \cap S_G| = 2 = |N_G(x_j) \cap S_G|$ for $i \neq j$. This implies that S_G is a fair dominating set of G. Let $S = (V(G) \times \{y_1\}) \cup (S_G \times \{y_2\})$. Then $(x_i, y_2), (x_j, y_2) \in V(G[H]) \setminus S$ for some $i \neq j$. Thus, the

$$N_{G[H]}((x_i, y_2)) = \{(x_{i-1}, y_2), (x_{i+1}, y_2), (x_{i-1}, y_1), (x_i, y_1), (x_{i+1}, y_1)\} \text{ and}$$

$$N_{G[H]}((x_j, y_2)) = \{(x_{j-1}, y_2), (x_{j+1}, y_2), (x_{j-1}, y_1), (x_j, y_1), (x_{j+1}, y_1)\}.$$

$$\text{Since } (x_{i-1}, y_2), (x_{i+1}, y_2), (x_{i-1}, y_1), (x_i, y_1), (x_{i+1}, y_1) \in S$$

$$\text{and } (x_{j-1}, y_2), (x_{j+1}, y_2), (x_{j-1}, y_1), (x_j, y_1), (x_{j+1}, y_1) \in S,$$

it follows that $|N_{G[H]}((x_i, y_2))| = 5 = |N_{G[H]}((x_j, y_2))|$. This implies that S is a fair dominating set of $G[H]$. Now, for every $(x_k, y_2) \in V(G[H]) \setminus S$, there exists $(x_k, y_1) \in S$ for some k such that

$$N_{G[H]}((x_k, y_1)) \cap (V(G[H]) \setminus S) = \{(x_k, y_2)\}.$$

This implies that S is a super dominating set of $G[H]$. Accordingly, S is a super fair dominating set of $G[H]$.

Suppose that statement (ii) is satisfied. Then $S_G = \{x_{4k-3}, x_{4k} : k = 1,2,\ldots,\frac{n}{4}\}$ where $n = 0(mod\ 4)$. Let $x_i, x_{i+1} \in V(G) \setminus S_G$. Then $N_G(x_i) = \{x_{i-1}\}$ and $N_G(x_{i+1}) = \{x_{i+2}\}$. Since $x_i, x_{i+1} \notin \setminus S_G$, it follows that $x_{i-1} \in S_G$ and $x_{i+2} \in S_G$. Thus, $|N_G(x_i) \cap S_G| = 1 = |N_G(x_{i+1}) \cap S_G|$. This implies that S_G is a fair dominating set of G. Let $S = (V(G) \times \{y_1\}) \cup (S_G \times \{y_2\})$. Then $(x_i, y_2), (x_{i+1}, y_2) \in V(G[H]) \setminus S$. Thus, the

$$N_{G[H]}\{G[H]\}((x_i, y_2)) = \{(x_{i-1}, y_2), (x_{i+1}, y_2), (x_{i-1}, y_1), (x_i, y_1), (x_{i+1}, y_1)\} \text{ and}$$

$$N_{G[H]}((x_{i+1}, y_2)) = \{(x_i, y_2), (x_{i+2}, y_2), (x_{i+2}, y_1), (x_{i+1}, y_1), (x_i, y_1)\}.$$

Since $(x_{i-1}, y_2), (x_{i-1}, y_1), (x_i, y_1), (x_{i+1}, y_1) \in S$ and $(x_{i+2}, y_2), (x_{i+2}, y_1), (x_{i+1}, y_1), (x_i, y_1) \in S$, it follows that $|N_{G[H]}((x_i, y_2))| = 4 = |N_{G[H]}((x_{i+1}, y_2))|$. This implies that S is a fair dominating set of $G[H]$. Now, for every $(x_k, y_2)(x_{k+1}, y_2) \in V(G[H]) \setminus S$, there exist $(x_{k-1}, y_2), (x_{k+2}, y_2) \in S$ for some k such that $N_{G[H]}\big((x_{k-1}, y_2)\big) \cap (V(G[H]) \setminus S) = \{(x_k, y_2)\}$ and $N_{G[H]}\big((x_{k+2}, y_2)\big) \cap (V(G[H]) \setminus S) = \{(x_{k+1}, y_2)\}$. This implies that S is a super dominating set of $G[H]$. Accordingly, S is a super fair dominating set of $G[H]$.

Suppose that statement (iii) is satisfied. Then $S_G = \{x_1\} \cup \left\{x_{4k}, x_{4k+1} : k = 1,2,\ldots,\frac{n-2}{4}\right\} \cup \{x_n\}$ where $n = 2(mod\ 4)$, $n \neq 4$. To show that S is a super fair dominating set of $G[H]$, the proof is similar to the proof of (ii). ■

The following result is an immediate consequence of Theorem 2.5.

Corollary 2.6 *Let $G = P_n = [x_1, x_2, \ldots, x_n]$, $n \geq 3$ and $H = K_2 = [y_1, y_2]$ and $S_G \subset V(G)$. Then*

$$\gamma_{sfd}(G[H]) \leq \begin{cases} \dfrac{3n+1}{2} & if \quad n \equiv 1(mod\ 2) \\ \dfrac{3n}{2} & if \quad n \equiv 0(mod\ 4) \\ \dfrac{3n+2}{2} & if \quad n \equiv 2(mod\ 4), n \neq 4 \end{cases}$$

Proof: Let $G = P_n = [x_1, x_2, \ldots, x_n]$, $n \geq 3$ and $H = K_2 = [y_1, y_2]$ and $S_G \subset V(G)$. Suppose that $S_G = \{x_1\} \cup \{x_{2k+1} : k = 1,2,\ldots,\frac{n-1}{2}\}$ where $n = 1(mod\ 2)$. Then $S = (V(G) \times \{y_1\}) \cup (S_G \times \{y_2\})$ of $V(G[H])$ is a super fair dominating set by Theorem 2.5. Thus,

$$\begin{aligned} \gamma_{sfd}(G[H]) &\leq |S| \\ &= |((V(G) \times \{y_1\}) \cup (S_G \times \{y_2\})| \\ &= |(V(G) \times \{y_1\})| + |(S_G \times \{y_2\})| \\ &= |(V(G)||\{y_1\}| + |S_G||\{y_2\})| \\ &= n \cdot 1 + \left(1 + \frac{n-1}{2}\right) \cdot 1 \\ &= n + (1 + \tfrac{n-1}{2}) = \tfrac{3n+1}{2}. \end{aligned}$$

Suppose that $S_G = \{x_{4k-3}, x_{4k} : k = 1,2,\ldots,\frac{n}{4}\}$ where $n = 0(mod\ 4)$. Then $S = (V(G) \times \{y_1\}) \cup (S_G \times \{y_2\})$ of $V(G[H])$ is a super fair dominating set by Theorem 2.5. Thus,

$$\begin{aligned}\gamma_{sfd}(G[H]) &\leq |S| \\ &= |((V(G) \times \{y_1\}) \cup (S_G \times \{y_2\})| \\ &= |(V(G) \times \{y_1\})| + |(S_G \times \{y_2\})| \\ &= |(V(G)||\{y_1\}| + |S_G||\{y_2\})| \\ &= n \cdot 1 + \left(2 \cdot \frac{n}{4}\right) \cdot 1 \\ &= n + \frac{n}{2} = \frac{3n}{2}.\end{aligned}$$

Suppose that $S_G = \{x_1\} \cup \left\{x_{4k}, x_{4k+1} : k = 1,2,\ldots,\frac{n-2}{4}\right\} \cup \{x_n\}$ where $n = 2(mod\ 4)$, $n \neq 4$. Then $S = (V(G) \times \{y_1\}) \cup (S_G \times \{y_2\})$ of $V(G[H])$ is a super fair dominating set by Theorem 2,5. Thus,

$$\begin{aligned}\gamma_{sfd}(G[H]) &\leq |S| \\ &= |((V(G) \times \{y_1\}) \cup (S_G \times \{y_2\})| \\ &= |(V(G) \times \{y_1\})| + |(S_G \times \{y_2\})| \\ &= |(V(G)||\{y_1\}| + |S_G||\{y_2\})| \\ &= n \cdot 1 + \left(1 + 2 \cdot \frac{n-2}{4} + 1\right) \cdot 1 \\ &= n + (2 + \frac{n-2}{2}) = \frac{3n+2}{2}. \blacksquare\end{aligned}$$

CONCLUSION

In this work, we extend the concept of the super fair domination in graphs. The super fair domination in the corona and lexicographic product of two connected graphs were characterized. The exact super fair domination number resulting from the corona and lexicographic product of two connected graphs were computed. This study will motivate research enthusiasts to work on super fair dominating set of other binary operation such as the Cartesian product of two graphs. Other parameters involving super fair domination in graphs may also be explored. Finally, the characterization of a super fair domination in graphs and its bounds is a promising extension of this study.

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