The Cournot Model of Duopoly and Interval Matrix Games

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Abstract — *We reconsider the Cournot oligopoly problem in the light of the theory of Interval Matrix Games. Interval matrix games generelized matrix games have already been studied in recent years by various researchers (Collins and Hu, 2005:2008; Nayak and Pal, 2006). And there are also studies with regard to Cournot model of duopoly (Amir, 1993; Cunningham and et.al., 2002; Elsadany, 2016;). These games make possible matematically for taking a decision in uncertainties environment. In this paper, we discuss the Cournot Model of Duopoly on a competitive market modelled by means of interval matrix games. Several illustrative examples are provided.*

Keywords — *Interval numbers, Cournot-Nash equilibrium, model of duopoly.*

I. INTRODUCTION

Modern Game Theory has been developed by grant mathematician John von Neumann in the 1940's. The aim of this theory is to apprehend the general sense of interaction from war to competitive market in all matter. Von Neumann together with economist Oskar Morgenstern published "The Theory of Games and Economic Behavior" in 1944. John Nash's articles of the definition of equilibrium and existence in the early of 1950's has been founded modern non-cooperative games. In the meantime, it was achieved significant results in cooperative games. Game theory has been turned into basic way of thinking which consider structure of social relations beyond the property of effect to almost all area of economics and social life. It has many applications in broad areas from industrial organiziation to international trade, from labour market to politic economy. Further, the analytic tools of game theory has been used commonly in economic science, psychology and political science.

Interval matrix games may be described by taking interval numbers to replace the components of pay-off matrix in classical matrix games and it is the special case of fuzzy games. The most thorough study regarding interval numbers was composed by Moore (1979). In this study, ordering of interval numbers has importance. Moore (1979) firstly put forward an order relation which is similar to order relation of real number on ordering of the interval numbers. Then, Ishibuchi and Tanaka (1990) defined two order relations for non-discrete interval numbers. Also, Sengupta and Pal (1997), worked on ordering of interval numbers by means of a function which is called acceptability function.

Several types of oligopoly models have been studied by many researches. Main concern was the existence and uniqueness of the equilibrium of the different types of oligopoly. Furth (1986) has worked on stability and instability in oligopoly. Duopoly is a type of oligopoly that two firms competing have dominant influence and control over a market. To maximize the profit, each firm takes action depend on the reaction from its rival to compete with its rival. The introduction of complex behavior phenomena in Cournot oligopoly games is well documented in the matematical economics literature, starting with the studies (Ran, 1978; Dana and Montrucchio, 1986). In fact, the dynamical properties of Cournot duopoly games have been discussed by Puu (1996;2007). Elsadany (2017) considered the dynamic of a Cournot duopoly with relative profit maximization and costs function with externalities. Amir (1996) reviewed the Cournot oligopoly problem in terms of the theory of supermodular games and provides extensive and precise insight as to why decreasing best-responses are widely regarded as being "typical" for the Cournot model with production costs.

In this paper, we put forward a new approach as to the Cournot model of duopoly with intervals. The aim of the present paper is to investigate the extent to which the use of interval numbers may enrich the existing result on Cournot equilibrium.

II. MATERIALS AND METHODS

A. Interval Numbers

Interval numbers are closed subset of real numbers and represented as follows,

$$\tilde{a} = [a_L, a_R] = \{x \in \mathbb{R}: a_L \leq x \leq a_R\}$$

in which $a_L, a_R \in \mathbb{R}$ and $a_L \leq a_R$. If $a_L = a_R$, then $\tilde{a} = [a, a]$ is a real number.

Midpoint and radius of an interval number $\tilde{a}$ is defined as,

$$m(\tilde{a}) = \frac{a_L + a_R}{2}, \quad w(\tilde{a}) = \frac{a_R - a_L}{2}$$

Definition Let $\tilde{a} = [a_L, a_R]$ and $\tilde{b} = [b_L, b_R]$ be two interval numbers. The algebraic operations are defined as follow,

i. $\tilde{a} + \tilde{b} = [a_L + b_L, a_R + b_R]$

ii. $\tilde{a} - \tilde{b} = [a_L - b_R, a_R - b_L]$

iii. $\tilde{a}\tilde{b} = [minS, maxS], \ S = \{a_L b_L, a_L b_R, a_R b_L, a_R b_R\}$

$\frac{\tilde{a}}{\tilde{b}} = \tilde{a}.(1/\tilde{b})$, ($0 \notin \tilde{b}$ and $\frac{1}{\tilde{b}} = \{\tilde{b}: (\frac{1}{\tilde{b}}) \in \tilde{b}\} = [\frac{1}{b_R}, \frac{1}{b_L}]$)

$\frac{\tilde{a}}{\tilde{b}} = \tilde{a}.\left(\frac{1}{\tilde{b}}\right) = [a_L, a_R]\left[\frac{1}{b_R}, \frac{1}{b_L}\right] = \left[\min\left\{\frac{a_L}{b_R}, \frac{a_L}{b_L}, \frac{a_R}{b_R}, \frac{a_R}{b_L}\right\}, \max\left\{\frac{a_L}{b_R}, \frac{a_L}{b_L}, \frac{a_R}{b_R}, \frac{a_R}{b_L}\right\}\right]$

iv. $\alpha \in \mathbb{R}$ için $\alpha\tilde{a} = \alpha[a_L, a_R] = \begin{cases} \alpha[a_L, a_R], & \alpha \geq 0 \\ \alpha[a_R, a_L], & \alpha < 0 \end{cases}$

Proposition Let $\tilde{a} = [a_L, a_R]$ be a interval numbers then,

i) $\tilde{a} + (-\tilde{a}) \neq 0$

ii) $\tilde{a}.\frac{1}{\tilde{a}} \neq 1.$

Proof. i) If $\tilde{a} = [a_L, a_R]$ is an interval number then, $-\tilde{a} = [-a_R, -a_L]$ is also an interval number. In this case, $\tilde{a} + (-\tilde{a}) = [a_L - a_R, \ a_R - a_L] \neq 0 = [0, 0]$.

ii) If $\tilde{a} = [a_L, a_R]$ is an interval number then, $\frac{1}{\tilde{a}} = \left[\frac{1}{a_R}, \frac{1}{a_L}\right]$ is also an interval number and then $\tilde{a}.\left(\frac{1}{\tilde{a}}\right) = [a_L, a_R]\left[\frac{1}{a_R}, \frac{1}{a_L}\right] = \left[\min\left\{\frac{a_L}{a_R}, 1, \frac{a_R}{a_L}\right\}, \max\left\{\frac{a_L}{a_R}, 1, \frac{a_R}{a_L}\right\}\right]$, and so $a_R > a_L$ there is obtained that $\tilde{a}.\frac{1}{\tilde{a}} \neq 1.$

B. Comparison of Interval Numbers

We consider the order relation over intervals. Comparison of two intervals is very important concept in interval analysis. Moore (1979) put forward two transitive order relation. The first one is an extension of "<" on the real line as

$\tilde{a} < \tilde{b}$ iff $a_R < b_L$

The other one is

$\tilde{a} \subseteq \tilde{b}$ iff $a_L \geq b_L, a_R \leq b_R$.

Also, both of them don't order the interval numbers exactly. If interval numbers are overlapping, then we use the acceptability index idea suggested by Sengupta and Pal (1997).

Definition (Sengupta and Pal, 1997) Let I be the set of all interval numbers. The *acceptability function* is defined by

$$\phi: I \times I \rightarrow [0, \infty), \qquad \phi(\tilde{a} < \tilde{b}) = \frac{m(\tilde{b}) - m(\tilde{a})}{w(\tilde{b}) + w(\tilde{a})}$$

where $w(\tilde{b}) + w(\tilde{a}) \neq 0$ and $\phi(\tilde{a} < \tilde{b})$ can be signified as the grade of acceptability of the $\tilde{a}$ to be inferior to $\tilde{b}$.

On account of ϕ, for any interval numbers $\tilde{a}$ and $\tilde{b}$, we have

$i)\phi(\tilde{a} < \tilde{b}) \geq 1$ when $m(\tilde{b}) > m(\tilde{a})$ and $a_R \leq b_L$

$ii)$ $0 < \phi(\tilde{a} < \tilde{b}) < 1$ when $m(\tilde{b}) > m(\tilde{a})$ and $b_L < a_R$

$iii)$ $\phi(\tilde{a} < \tilde{b}) = 0$ when $m(\tilde{b}) = m(\tilde{a})$. In this case, for detailed information on comparison of $\tilde{a}$ and $\tilde{b}$ can be found in Sengupta et al. (2001).

C. Interval Matrix Games

Interval matrix games are described by a matrix whose entries are closed intervals as follows:

$$G = \begin{bmatrix} [a_{11L}, a_{11R}] & [a_{12L}, a_{12R}] & \cdots & [a_{1nL}, a_{1nR}] \\ [a_{21L}, a_{21R}] & [a_{22L}, a_{22R}] & \cdots & [a_{2nL}, a_{2nR}] \\ \vdots & \vdots & \cdots & \vdots \\ [a_{m1L}, a_{m1R}] & [a_{m2L}, a_{m2R}] & \cdots & [a_{mnL}, a_{mnR}] \end{bmatrix}$$

where the aim of player I is to maximize his profit and the aim of player II is to minimize his profit. When the player I, II chooses i, j pure strategies respectively, then $[a_{ij}, b_{ij}]$ is represented as profit of player I.

Definition The concept of saddle point is introduced by Neumann (1944). The (k, r) position of the pay-off matrix will be called a saddle point, if and only if,

$$[a_{kr}, b_{kr}] = \bigvee_i \left\{ \bigwedge_j [a_{ij}, b_{ij}] \right\} = \bigwedge_j \left\{ \bigvee_i [a_{ij}, b_{ij}] \right\}.$$

In this case, there is a saddle point in pure strategies of interval matrix game and $[a_{kr}, b_{kr}]$ is called a game value.

Example Let

$$G = \begin{matrix} & II_1 & II_2 & II_3 & II_4 \\ I_1 & [-12, -1] & [-10, 5] & [-6, -5] & [-8, 5] \\ I_2 & [-11, 6] & [-3, 5] & [1, 6] & [6, 7] \\ I_3 & [-5, -4] & [-3, 3] & [-4, 0] & [-5, 5] \end{matrix}$$

be interval matrix game. In this game, if player I plays second row $(x = (0,1,0))$ and player II plays third column $(y = (0,0,1,0))$, then player I receives and correspondingly, player II pays a payoff $g(I_2, II_3) \in [1, 6]$. (I_2, II_3) is saddle point and $[1, 6]$ is value of the game.

Definition (Owen, 1995) Let $\{I_1, I_2, \dots, I_m\}$ be set of pure strategies for player I and $\{II_1, II_2, \dots, II_n\}$ be set of pure strategies for player II. A mixed strategy for a player is probability distribution on the set of his pure strategies. Then the player has only finite number, m of pure strategies, a mixed strategy reduces to an m-vector $x = (x_1, x_2, \dots, x_m)$. It yields

$x_i \geq 0, \sum_{i=1}^{m} x_i = 1.$

Definition (Nayak and Pall, 2006) Let x, y be the mixed strategies for player I and II. Then the expected payoff for player I is defined by

$h(x, y) = \sum_{i=1}^{m} \sum_{j=1}^{n} x_i [a_{ijL}, a_{ijR}] y_j = xGy^T$.

Now, define the lower and upper value of interval matrix game, respectively as follows:

$V_L = \bigvee_{x \in X_m} \bigwedge_{y \in Y_m} h(x, y)$ and $V_U = \bigwedge_{y \in Y_m} \bigvee_{x \in X_m} h(x, y)$.

If $V_L = V_U = v$ then v is called the value of interval matrix game. The notation $\vee$ and $\wedge$ represents the maximum and minimum between the interval numbers, respectively.

Theorem (Nayak PK, Pal M, 2006, Theorem 2) Let $A = ([a_{ij}, b_{ij}]); i = 1, 2, \dots, m; j = 1, 2, \dots, n$ be an $m \times n$ payoff matrix for a interval matrix game in which $[a_{ij}, b_{ij}]$ are interval numbers. Then the following inequality is satisfied $\vee_i \{\wedge_j [a_{ij}, b_{ij}]\} \leq \wedge_j \{\vee_i [a_{ij}, b_{ij}]\}$.

D. The Cournot Model of Duopoly

The first theoretic work of oligopoly competition was introduced by Augustin Cournot in 1838 (Cournot, 1897).The model has become a classic in microeconomic oligopoly theory. Cournot games have some characteristics in common such as, homogenous product, nonstorable product, competition only in quantities and decision-making by players simultaneously. Firstly, Cournot (1838) examined interactive relation between the firms. The Cournot model remarks existence of output equilibrium due to competition between two firms. As though the opinion that this equilibrium belongs to Nash Equilibrium is not very old information, Nash is also not aware of Cournot. An article studied by Leonard (1944) connects with it (Leonard, 1994). And then, in time, the Cournot model has become an inseparable part of economic analysis. It is based on quantity competition. In this model, each firm chooses an output quantity to maximize profit. Firms are assumed to produce homogeneous goods that are nonstorable. So all quantities produced are instantly sold (Chuang and et.all, 2001). We consider a very simple version of Cournot's model here. There are two competing firms producing a single homogeneous product.

The cost of producing one unit of the good is a constant c the same for both firms. The cost function of each firm is a symmetric function. If a firm i produces the quantity q_i units of the good, then the cost of the firm i is cq_i, for $i = 1, 2$. If firm I produces q_1 and firm II produces q_2 for a total of

$Q = q_1 + q_2$ (1)

the price is

$$P(Q) = \begin{cases} a - Q, & for\ Q < a \\ 0, & for\ Q \geq a \end{cases} \quad (2)$$

for some constant a.

We assume that the firms must simultaneously choose their production quantities; no collusion is allowed. In the Cournot model, the strategies available to each firm are the different quantities it might produce. Each firm's pure strategy space can be represented as $S_i = [0, \infty]$, the nonnegative real numbers, in which case a typical strategy s_i is a quantity choice, $q_i \geq 0$. We assume that the firm's payoff is simply its profit.

In order to find the Nash equilibrium of the Cournot game, we need to find the best reaction functions of each firm. Thus, firstly we must find their profits. The profit of the firms for $i = 1,2$;

$$\pi_i(q_i, q_j) = q_i[P(q_i + q_j) - c] = q_i[a - (q_i + q_j) - c]. \quad (3)$$

In a two-player finite game, the strategy pair (s_1^*, s_2^*) is a Nash equilibrium if for each player i,

$$u_i(s_i^*, s_j^*) \geq u_i(s_i, s_j^*) \quad (4)$$

for every strategy s_i in S_i. Likewise, for each player i, s_i^* must solve the optimization problem

$$\max_{s_i \in S_i} u_i(s_i, s_j^*). \quad (5)$$

In the Cournot model of duopoly, the quantity pair (q_1^*, q_2^*) is a Nash equilibrium if, for each firm i, q_i^* solves

$$\max_{0 \leq q_i < \infty} \pi_i(q_i, q_j^*) = \max_{0 \leq q_i < \infty} q_i[a - (q_i + q_j^*) - c]. \quad (6)$$

Assuming $q_j^* < a - c$, the first-order condition for firm i's problem is both necessary and sufficient; it yields

$$q_i = \frac{1}{2}(a - q_j^* - c). \quad (7)$$

Thus, if the quantity pair (q_1^*, q_2^*) is a Nash equilibrium, the firms' quantity choices must satisfy

$$q_1^* = \frac{1}{2}(a - q_2^* - c) \quad (8)$$

and

$$q_2^* = \frac{1}{2}(a - q_1^* - c) \quad (9)$$

Solving this pair of equations yields

$$q_1^* = q_2^* = \frac{1}{3}(a - c), \quad (10)$$

which is indeed less than $a - c$, as assumed. Therefore, (q_1^*, q_2^*) is a Pure Strategy Equilibrium (PSE) for this problem. This is called Cournot-Nash Equilibrium. In this equilibrium, total quantity and market price level are respectively as follows:

$$Q^* = \frac{2}{3}(a - c), \quad (11)$$

$$p^*(Q^*) = \frac{1}{3}(a + 2c). \quad (12)$$

It is also possible to see that both equilibrium quantity Q^* and the market equilibrium price p^* increase as a increases, namely when the consumers are more willing to buy the product. On the other hand, when marginal costs increase, the market price rises while the quantity of each firm falls.

III.RESULTS AND DISCUSSION

A. The Cournot Duopoly Game with Interval Payoff

We reconsider the Cournot duopoly game in light of the theory of interval matrix games. In a market where two firms compete with each another, we assume that the quantity of each firm is interval numbers. This means that the output level to maximize profit for each firm is an interval number.

Let be now the quantity of the firms $\tilde{q}_1 = [q_{1L}, q_{1R}]$, $\tilde{q}_2 = [q_{2L}, q_{2R}]$ total amount $\tilde{Q} = \tilde{q}_1 + \tilde{q}_2$ and the cost $\tilde{c} = [c_L, c_R]$, a constant $\tilde{a} = [a_L, a_R]$. Then, interse-demand function is as follows:

$$P(\tilde{Q}) = \begin{cases} \tilde{a} - \tilde{Q}, & for\ \ 0 \leq \tilde{Q} \leq \tilde{a} \\ 0, & for\ \ \tilde{Q} \geq \tilde{a} \end{cases} \quad (11)$$

where $0 = [0,0]$ is interval number.

If unit cost of the product is $\tilde{c}$ then the cost function is $\tilde{c}_i(\tilde{q}) = \tilde{c}\tilde{q}$. In this case, the payoffs for two firms are the profits:

$$\tilde{\pi}_1(\tilde{q}_1, \tilde{q}_2; \tilde{c}) = \tilde{q}_1[(\tilde{a} - \tilde{q}_1 - \tilde{q}_2) - \tilde{c}] \quad (12)$$

$$\tilde{\pi}_2(\tilde{q}_1, \tilde{q}_2; \tilde{c}) = \tilde{q}_2[(\tilde{a} - \tilde{q}_1 - \tilde{q}_2) - \tilde{c}] \quad (13)$$

This defines the strategic form of the game. We assume that $\tilde{c} < \tilde{a}$, since otherwise the cost of production would be at least as great as any possible return. The optimization problems of each firm are respectively;

$$\vee_{q_1}[(\tilde{a} - \tilde{q}_1^* - \tilde{q}_2) - \tilde{c}]\tilde{q}_1 \quad (14)$$

$$\vee_{q_2}[(\tilde{a} - \tilde{q}_1 - \tilde{q}_2^*) - \tilde{c}]\tilde{q}_2 \quad (15)$$

and the reaction functions of the firms are respectively as follows;

$$\tilde{q}_1^* = \frac{1}{2}(\tilde{a} - \tilde{q}_2^* - \tilde{c}) \quad (16)$$

$\tilde{q}_2^* = \frac{1}{2}(\tilde{a} - \tilde{q}_1^* - \tilde{c})$ (17)

Solving these equations simultaneously and denoting the result by $\tilde{q}_1^*$ and $\tilde{q}_2^*$, we obtain

$\tilde{q}_1^* = \tilde{q}_2^* = \frac{(\tilde{a}-\tilde{c})}{3}$. (18)

Therefore, $(\tilde{q}_1^*, \tilde{q}_2^*)$ is a PSE for this problem. In the strategy equilibrium, the payoff each player receives from this SE is

$\tilde{\pi}_1^*(\tilde{q}_1^*, \tilde{q}_2^*) = \frac{(\tilde{a}-\tilde{c})^2}{9}$. (19)

Note that the total amount received by the firms in the equilibrium is $\frac{2}{9}(\tilde{a} - \tilde{c})^2$. Furthermore, the duopoly price is

$P(\tilde{q}_1^* + \tilde{q}_2^*) = \frac{(\tilde{a}+2\tilde{c})}{3}$. (20)

Example

Let's consider the market demand function;

$$P(\tilde{Q}) = \begin{cases} [5,11] - \tilde{Q}, & if\ \tilde{Q} < [5,11] \\ 0, & otherwise \end{cases}$$

Two firms on the market have unit costs $[2,2]$. Firms can choose any quantity.

1. Define the reaction functions of firm I and firm II;
2. Find the Cournot equilibrium;

Solution

1. In a Cournot duopoly the reaction function of firm I identifies its optimal response to any quantity produced by firm II. In the presence of private firms, the optimal quantity is the one that maximizes $\tilde{\pi}_I$, firm I's reaction function, where
   $\tilde{q}_1^* = \frac{1}{2}(\tilde{a} - \tilde{q}_2^* - \tilde{c}) = \frac{1}{2}([5,11] - \tilde{q}_2^* - [2,2]) = \frac{1}{2}([3,9] - \tilde{q}_2^*)$
   Similarly, Firm II's reaction function is as follows;
   $\tilde{q}_2^* = \frac{1}{2}(\tilde{a} - \tilde{q}_1^* - \tilde{c}) = \frac{1}{2}([5,11] - \tilde{q}_1^* - [2,2]) = \frac{1}{2}([3,9] - \tilde{q}_1^*)$

2. Cournot equilibrium is identified by the quantities that are mutually best responses for both firms; so they are obtained by the solution of the following equations:
   $$\begin{cases} \tilde{q}_1^* = \frac{[3,9]-\tilde{q}_2^*}{2} \\ \tilde{q}_2^* = \frac{[3,9]-\tilde{q}_1^*}{2} \end{cases}$$
   The equilibrium quantities are
   $\tilde{q}_1^* = \tilde{q}_2^* = [1,3]$
   or we can directly find as follows;
   $\tilde{q}_1^* = \tilde{q}_2^* = \frac{(\tilde{a}-\tilde{c})}{3} = [1,3]$.
   Total quantity is $\tilde{Q} = [2,6]$ and the equilibrium price is
   $P(\tilde{Q}) = [5,11] - [2,6] = [3,5]$
   and firms' profit are:
   $\tilde{\pi}_2^* = \tilde{\pi}_1^* = [3,5]\tilde{q}_1^* - [2,2]\tilde{q}_1^* = [1,9]$
   or we can obtain from $\tilde{\pi}_1^* = \frac{(\tilde{a}-\tilde{c})^2}{9} = \frac{[3,9]^2}{9} = \frac{[9,81]}{9} = [1,9]$.

IV. CONCLUSION

The Cournot model remarks the existence of output equilibrium due to competition between two firms in which the Cournot-Nash Equilibrium shows the level of output that maximizes profit. In this paper, we have submited a new approach for Cournot Model of Duopoly by means of which the quantities of each firm in the market can be derived as an interval number. The results show that the firms can behave loosely during production.

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