Vulnerability Parameter of Book graph

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Abstract : *Book graph B_m is cross product of star graph S_{m+1} and path graph P_2. Tenacity of an incomplete connected graph G is defined as $T(G) = min\left\{\frac{|S|+m(G-S)}{w(G-S)} : s\subset V(G), w(G-S) > 1\right\}$. Where $w(G-S)$ is the number of components in G - S. Rupture degree of graph is defined as $R(G) = max\,\{w(G-X) - |X| - m(G-X); X\subset V(G) and\ w(G-X) > 1\}$. Tenacity and Rupture degree are vulnerability majors of graph, which reflects on the difficulty in breaking down the network.*
In this paper we present results on Tenacity and Rupture degree of Book graph. We also present relation between tenacity and rupture degree with other parameters like integrity, toughness, scattering number of book graph.

Keywords:. *Book graph, Tenacity, Rupture degree, Vulnerability.*

I. INTRODUCTION

Graph theory has became one of the most powerful tool in the analysis of networks. The vulnerability of a communication network, composed of processing nodes and communication links, is of prime important to network designers. As the network begins losing links or nodes, eventually there is a loss in its effectiveness. Thus, communication networks must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network. The communication network often has as considerable an impact on network's performance as the processors themselves. Performance measures for communication networks are essential to guide the designers in choosing an appropriate topology[13].

The communication network can be represented as an undirected and unweighted graph, where a processor(station) is represented as a node and a communication link between processors(station) as an edge between corresponding nodes. If we use a graph to model a network , there are many graph theoretical parameters used to describe the vulnerability of communication networks.[4]

The basic terminology we refer to Chartrand and Lesnaik[3].The study of cross product of graph was initiated by Imrich [7]. For structure and recognition of cross product of graph we refer to Imrich[8].Cozzens, Moazzami and Stueckle introduced the concept of tenacity of a graph G, as a useful measure of the vulnerability of G.[12 & 13].The concept of rupture degree was first introduced by Li. Zhang and Li in [14].Rupture degrees parameter can be used to measure the vulnerability of networks. Book graph concept we refer to [10].

Graph toughness was first introduced by Vaclav Chvatal(1973). Since then there has been extensive work by other mathematicians on toughness; the recent survey by Bauer, Broersma and Schmeichel(2006) lists 99 theorem and 162 papers on the subject.[1 & 3]

Intergrity was introduced by Barefoot, Entringer and Swart[2] as an alternative measure of the vulnerability of graphs to disruption caused by the removal of vertices. The motivation was that, in some respect connectivity is oversensitive to local weakness and does not reflect the overall vulnerability. The scattering number of a graph was defined by H.A Jung.[7]

II. PRELIMINARIES

Definition 2.1 : In graph theory , the Cross product $G \times H$ of graphs G and H is a graph such that

- The vertex set of $G \times H$ is the Cartesian Product $V(G) \times V(H)$; and
- Two vertices (u, u') and (v, v') are adjacent in $G \times H$ if and only if either
  - $u = v$ and u' is adjacent to v' in H, or
  - $u' = v'$ and u is adjacent to v in G.

When a network becomes disconnected it is derivable to capture the extend of disruption by meaning the order and number of the reaming connected components denote $m(G-S)$ and $w(G-S)$ as more components more severe is the degradation than a system with smaller number of larger components.

For a sub graph S of an in complete connected graph G, Let $w(G-S)$ be the number of components in $G-S$ and $m(G-S)$ be the number of vertices with largest component including $G-S$.

Definition 2.2: The tenacity of an incomplete connected graph G are defined as

$$T(G) = min\left\{\frac{|S| + m(G-S)}{W(G-S)} : S \subset V(G), w(G-S) > 1\right\}$$

Where the minimum is taken over all vertex cut sets S of V(G).

Definition 2.3: The rupture degree for an incomplete connected graph G is defined by

$$r(G) = \max\{w(G-S) - |S| - m(G-S), S \subset V(G), w(G-S) > 1\}$$

Definition 2.4: The integrity of a graph G, denoted $I(G)$, is defined as

$$I(G) = \min\{|S| + m(G-S) : S \subset V(G)\}$$

An I-set of G is any set S for which the minimum is attained.

Definition 2.5: The toughness number of an incomplete connected graph G are defined as

$$t(G) = min\left\{\frac{|S|}{w(G-S)} : S \subset V(G), w(G-S) > 1\right\}$$

Definition 2.6 : The scattering number of as in complete connected graph G is defined by

$$s(G) = \max\{w(G-s) - |S| : S \subset V(G) \text{ and } w(G-S) \neq 1\},$$

Definition 2.7 : The connectivity of an incomplete graph is defined by

$$k(G) = min\{|S| : S \subset V(G), w(G-s) > 1\}$$

III. Vulnerability of Book Graph

Theorem 3.1: The tenacity of the book graph is the $T(B_m) = \frac{4}{m}$

Proof: The book graph B_m is cross product $S_m + 1$ and P_2. Suppose $V(S_{m+1}) = \{v, u_1, u_2, \dots \dots \dots \dots u_m\}$ with $d(v) = m$ and $d(u_i) = 1$. Let $V(P_2) = \{w_1, w_2\}$.

$V(B_m) = V(S_{m+1} \times P_2) = \{(v, w_1), (v, w_2), (u_i, w_1), (u_i, w_2)\}$ where $i = 1,2,3,\dots\dots\dots\dots\dots\dots$ with $|V(B_m)| = 2m + 2$.

To find the tenacity of the book graph B_m. Consider the subset S of $V(B_m)$ such that $B_m - S$ has more than one components. i.e. $w(B_m - S) > 1$, Consider $S_1 = \{(v, w_1), (v, w_2)\}$, then $B_m - S_1$ is disconnected graph with m components where each components is a path on two vertices. $\{(u_i, w_1), (u_i, w_2)\}$ for $i = 1,2,3, \dots \dots \dots m$. Therefore $w(B_m - S_1) = m$ and $m(B_m - S_1) = 2$.

Therefore

$$T_1 = \frac{|S_1| + m(B_m - S_1)}{w(B_m - S_1)} = \frac{2+2}{m} = \frac{4}{m}$$

The vertex set of book graph can be written as union of $S_1 \cup P \cup Q$,

Where $P = \{P_i = (u_i, w_1) \quad i = 1,2,3, \dots \dots \dots m\}$

$Q = \{q_i = (u_i, w_2) \quad i = 1,2,3, \dots \dots \dots m\}$

Suppose $S = S_1 \cup P' \cup Q'$ where $P' \subset Q'$ and $Q' \subset Q$ with $|P'| = a$ and $|Q'| = b$ giving $|S| = a + b + 2$

Suppose $w(B_m - S_1) = m - r$, Where r is the number vertices from P' which has an edge with vertex of Q'

$$r = |E(< P' \cup Q' >)|, \text{ So } \ r \le |\min(a, b)|$$

and $m(B_m - S_1) = 2$ as the largest components in $B_m - S$ is path P_2

$$\therefore t = \frac{|S| + m(B_m - S)}{w(B_m - S)}$$

$$= \frac{a + b + 2 + 2}{m - r}$$

Case 1:- Suppose $a + b > 0$ and $r = 0$ then

$$T = \frac{a+b+4}{m} > \frac{4}{m} \text{ as } a + b > 0.$$

Case 2:- Suppose $a + b > 0$ and $r > 0$ then

$$T = \frac{a+b+4}{m-r} > \frac{4}{m} \text{ as } 0 < r \le \min(a, b), \text{ and } a + b > 0.$$

From case 1 and case 2, it is evident that the minimum value $T_1 = min\left\{\frac{|S|+m(B_m-|S_1|)}{w(B_m-S_1)}\right\} = \frac{4}{m}$

$$\therefore T(B_m) = \frac{4}{m}.$$

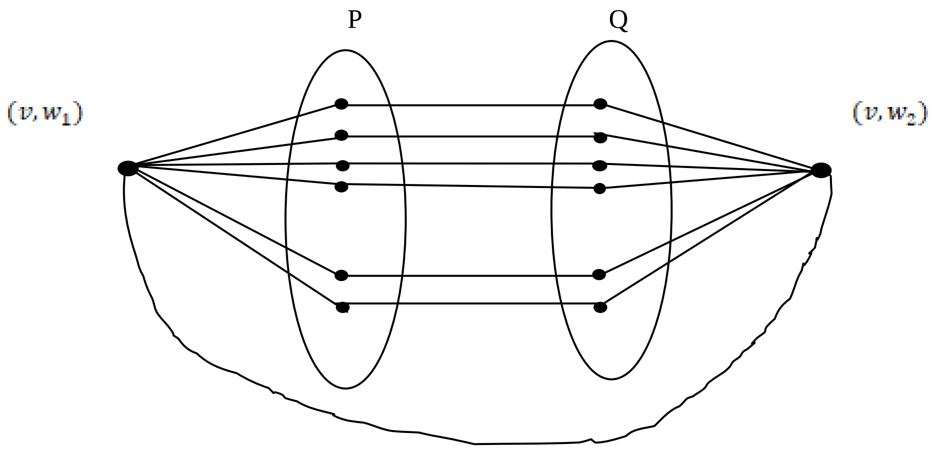

Fig 1: B_m

Theorem 3.2 : The Rupture degree of graph $R(B_m) = m - 4$

Proof: To find the Rupture degree of the book graph B_m, consider the subset S of $V(B_m)$ such that $w(B_m - S) > 1$. As in theorem 3.1, consider $S_1 = \{(v,w_1),(v,w_2)\}$. $B_m - S$ is disconnected with $w(B_m - S_1) = m$ and $m(B_m - S_1) = 2$. Therefore

$r_1(B_m) = w(B_m - S_1) - |S_1| - m(G - S_1)$;

$$r_1(B_m) = m - 2 - 2 = m - 4$$

Again as in theorem 3.1. Consider a cut set S of B_m such that $S = S_1 \cup P' \cup Q'$ with $|P'| = a$, $|Q'| = b$ and $w(B_m - S) = m - r,\ \ r \le \min(a,b)$

And $m(B_m - S) = 2$ as the largest components in $B_m - S$ is path P_2.

$\therefore r(G) = [(m - r) - (a + b + 2) - 2]$

Case 1: Suppose $a + b > 0,\ \ r = 0$

Then $r(B_m) = m - (a + b + 2) - 2$

$= m - a - b - 4$

$= m - (a + b + 4) < m - 4$

Case 2: Suppose $a + b > 0$ and $r > 0$ then,

$r(B_m) = (m - r) - (a + b + 2) - 2$

$= m - (r + a + b + 4) < m - 4$ as $0 < r \le \min(a,b) \quad a + b > 0$

From Case 1 and Case 2 it is evident that the maximum value is

$\max \{w(B_m - S) - |S| - m(B_m - S) = m - 4$

giving $r(B_m) = m - 4$.

n [13] Li has shown that rupture degree of a star graph is $r(S_{m+1}) = m - 2$. $\therefore$ Rupture degree of book graph is equal to rupture degree of star.

$$r(B_m) = r(S_{m+1}).$$

Theorem 3: The connectivity, toughness, scattering and integrity number of book graph is

(i) $\boldsymbol{k(B_m) = 2}$ (ii) $\boldsymbol{t(B_m) = \frac{2}{m}}$ (iii) $\boldsymbol{s(B_m) = m - 2}$ (iv) $\boldsymbol{I(B_m) = 4}$

Proof: From the discuss in theorem 1, We observer that $S_1 = \{(v,w_1),(v,w_2)\}$ is the minimum be subset of book graph B_m. $w(B_m - S_1) = m, m(B_m - S_1) = 2$.

(i). $\therefore \min\left\{\frac{|S|}{w(B_m - S)} > 1\right\} = |S_1| = 2$

(ii) $\min\left\{\frac{|S|}{w(B_m - S)}\right\} = \frac{|S_1|}{w(B_m - S_1)} = \frac{2}{m}$

(iii) $\max\{w(B_m - S) - |S|\} = m - 2$

(iv) $\max \{|S| + m(B_m - S) = 2 + 2 = 4$

Thus we get.

(i) $k(B_m) = 2$ (ii) $t(B_m) = \frac{2}{m}$ (iii) $s(B_m) = m - 2$ (iv) $I(B_m) = 4$

IV. RESULTS

Result 5.1: Vulnerability parameter of book graph

Sl no	Parameters	Book Graph B_m
1	Tenacity	$T(B_m) = \frac{4}{m}$
2	Rupture degree	$r(B_m) = m-4$
3	Toughness	$t(B_m) = \left(\frac{2}{m}\right)$
4	Scattering	$s(B_m)=m-2$
5	Integrity	$I(B_m) = 4$
6	Connectivity	$k(B_m) = 2$

V. CONCLUSION

1. It is found that the tenacity of book graph is inversely proportional to the number of pendants in the star graph. Therefore, higher the number of pages in the book graph lower is its tenacity.
2. Rupture degree and scattering number are directly proportional to number of pendants in the star and higher the number of pages in book graph higher the rupture degree and scattering number.
3. The tenacity of book graph is inversely proportional to the number of pendants in the star graph.
4. From the above results we find relationship between the vulnerability parameters of book graph as given in below table

Sl no.	Relation ship
i	$r(B_m) = s(B_m) - k(B_m)$
ii	$T(B_m) = k(B_m).t(B_m)$
iii	$I(B_m) = 2k(B_m)$

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