Complementary Non-negative Signed Domination Number

S.Anandha Prabhavathy

Department of Mathematics, Velammal College Of Engineering and Technology, Madurai,Tamil Nadu, INDIA

***Abstract** —A function $f: V \rightarrow \{-1, +1\}$ is called a Complementary Non-negativeSigned Dominating Function (CNSDF) of G if $\sum_{u \notin N[v]} f(u) \geq 0$ for every$v \in V(G)$ with $deg\ v \neq n-1$. The Complementary Non-negative Signed Domination Number of G is denoted by $\gamma_{CNN}(G)$ and is defined as $\gamma_{CNN}(G) = min\{w(f) \mid f\ is\ a\ minimal\ CNSDF\ of\ G\}$. In this paper, we initiate the study of complementary Non-negative Signed Domination number in graphs.*

Keywords — *Complementary Non-negative signed dominating function, Complementary Non-negative signed Domination number.*

I. INTRODUCTION

By a graph $G = (V, E)$, we mean a finite, non-trivial, connected, and undirected graph with neither loops nor multiple edges. The order and size of G are denoted by n and m, respectively. For graph theoretic terminology we refer to Chartand and Lesniak [1].

The study of domination is one of the fastest growing areas within graph theory. A subset D of vertices is said to be a $dominatingset$ of G if every vertex in V either belongs to D or is adjacent to a vertex in D. The $dominationnumber\gamma(G)$ is the minimum cardinality of a dominating set of G. Survey of several advanced topics on domination is given in the book edited by Haynes et al. [2].

For a real valued function $f: V \rightarrow R$ on V, *weight of* f is defined to be $w(f) = \sum_{v \in V} f(v)$ and also for a subset $S \subseteq V$, we define $f(S) = \sum_{v \in S} f(v)$. Therefore, $w(f) = f(V)$. Further, for a vertex $v \in V$, let $f[v] = f(N[v])$ for notation convenience. A function $f: V \rightarrow \{-1, +1\}$ is called a Non-negative signed dominating function (NNSDF) of G if $f(N[v]) \geq 0$ for all vertices in G. The *Non-negative Signed Domination Number* of G is denoted by $\gamma_{NN}(G)$ and is defined as $\gamma_{NN}(G) = min\{w(f) \mid\ f\ is\ an\ NNSDF\ of\ G\}$. The parameter $\gamma_{NN}(G)$ was investigated in [5].

A function $f: V \rightarrow \{-1, +1\}$ is called a Complementary Signed Dominating Function of G if $\sum_{u \notin N[v]} f(u) \geq 1$ for every $v \in V(G)$ with $deg\ v \neq n-1$. The Complementary Signed Domination Numberof G is denoted by $\gamma_{CS}(G)$ and is defined as

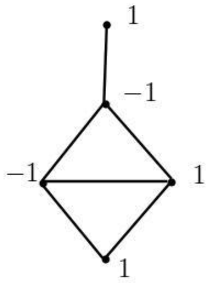
$$\gamma_{CS}(G) = \min\{w(f) \mid \text{f is a minimal complementary signed dominating function of G}\}$$

The parameter $\gamma_{CS}(G)$ was first investigated in [6].

II. DEFINITION OF $\gamma_{CNN}(G)$

Definition 2.1 A function $f: V \rightarrow \{-1, +1\}$ is called a Complementary Non-negative Signed Dominating Function (CNSDF) of G if $\sum_{u \notin N[v]} f(u) \geq 0$ for all vertices $v \in V(G)$ with $deg\ v \neq n-1$. The Complementary Non-negative Signed Domination Number of G is denoted by $\gamma_{CNN}(G)$ and is defined as $\gamma_{CNN}(G) = \min\{w(f) \mid f \text{ is a minimal CNSDF of G}\}$.

Example 2.2 Now consider the graph G as follows



By the way of assigninig -1 and $+1$ to the vertices of G, we obtain $\sum_{u \notin N[v]} f(u) \geq 0$ for all the vertices of G. Hence $\gamma_{CNN}(G) = 1$.

Remark 2.3 If f is a complementary non-negative signed dominating function of a graph G, we define the sets P_f and M_f as follows.

(i) $P_f(G) = \{v \in V(G): f(v) = +1\}$
(ii) $M_f(G) = \{v \in V(G): f(v) = -1\}$

Remark 2.4 If f is any complementary non-negative signed dominating function of a graph G of order n, then it is obvious that $|P_f| + |M_f| = n$ and $\gamma_{CNN}(G) = |P_f| - |M_f|$.

III. SOME RESULTS ON $\gamma_{CNN}(G)$

Theorem 3.1 If G is any connected graph of order n and $H = G \circ K_1$, then $\gamma_{CNN}(H) = 0$

Proof. Let $H = G \circ K_1$. Now suppose g be a complementary non-negative signed dominating function of H with $\gamma_{CNN}(H) = g(V)$.Let v_j be a pendant vertex of H. Then $\sum_{u \notin N[v_j]} g(u) \geq 0$. If all the pendant vertices are assigned with -1, then for a vertex $v_i \in V(G)$ with deg $v_i = 2$, we have $\sum_{u \notin N[v_i]} g(u) \geq -1(n-1) + 1(n-2) = -1$. Hence there exists a pendant vertex v_j with $g(v_j) = +1$. Therefore

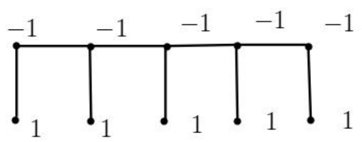
$$\sum_{v \in V(H)} g(v) = \sum_{u \notin N[v_j]} g(u) + g(v_j) + g(N(v_j)) \geq 0 + 1 - 1 = 0.$$

Now define f: $V(H) \longrightarrow \{-1, +1\}$ by

$$f(v) = \begin{cases} -1 & if \quad v \in V(G) \\ +1 & otherwise \end{cases}$$

Then it is easy to verify that f is a complementary non-negative signed dominating function of H. Hence $\gamma_{CNN}(H) \leq 0$. Consequently the result follows.

Example 3.2 Now consider the graph $H = P_5 \circ K_1$ as follows



Theorem 3.3 Let $G = K_{2n} - M$; where M is a perfect matching in the complete graph K_{2n} .Then $\gamma_{CNN}(G) = n$.

Proof. Let $V(K_{2n}) = \{v_1, v_2, ..., v_{2n}\}$ and $M = \{v_1v_2, v_3v_4, v_5v_6, ..., v_{2n-1}v_{2n}\}$. Now define a function f: V(G) $\longrightarrow \{-1, +1\}$ by $f(v_i) = +1$ for all the vertices of G. Then it is easy to verify that f is a complementary non-negative signed dominating function of G. Hence $\gamma_{CNN}(G) \le n$.

Now suppose g be a complementary non-negative signed dominating function of G with $\gamma_{CNN}(G) = g(V)$. Since every vertex of G is non-adjacent to exactly one vertex in G, we have $\gamma_{CNN}(G) \ge n$. Consequently the result follows.

Corollary 3.4 If every vertex of G is non-adjacent to exactly one vertex in G, then $\gamma_{CS}(G) = \gamma_{CNN}(G) = n$.

Theorem 3.5 For any complete bipartite graph $(K_{m,n})$

$$\gamma_{CNN}(K_{m,n}) = \begin{cases} 2 & if\ m\ and\ n\ are\ odd, \\ 3 & if\ m\ is\ odd\ and\ n\ is\ even, \\ 3 & if\ m\ is\ even\ and\ n\ is\ odd, \\ 4 & if\ m\ and\ n\ are\ even. \end{cases}$$

Proof. Let (U, W) be the bipartition of $K_{m,n}$ with $|U| = m$ and $|W| = n$. Let $U = \{u_1, u_2, \dots, u_m\}$ and $W = \{w_1, w_2, \dots, w_n\}$.Now suppose f be a minimum complementary non-negative signed dominating function of $K_{m,n}$ with $\gamma_{CNN}(K_{m,n}) = f(V)$. If all the vertices in U are assigned with -1, then $\sum_{v \notin N[u_i]} f(v) < 0$, for all $1 \le i \le m$. This argument is similar for the vertices in the partite set W. Therefore there exists vertices $u_i \in U$ and $w_j \in W$ such that $f(u_i) = f(w_j) = +1$. Hence $\sum_{v \notin N[u_i]} f(v) \ge 0$ if m is odd and $\sum_{v \notin N[u_i]} f(v) \ge 1$ if m is even. Similarly $\sum_{v \notin N[w_j]} f(v) \ge 0$ if n is odd and $\sum_{v \notin N[w_j]} f(v) \ge 1$ if n is even. Hence the lower bound follows.

On the other hand define a function $g: V \longrightarrow \{-1, +1\}$ by

$$g(v) = \begin{cases} +1 & for\ any \left\lfloor \frac{m}{2} \right\rfloor + 1 vertices\ in\ U\ and\ \ for\ any \left\lfloor \frac{n}{2} \right\rfloor + 1 vertices\ in\ W \\ -1 & otherwise \end{cases}$$

Then it is easy to verify that g is a complementary non-negative signed dominating function of $K_{m,n}$. Hence

$$\gamma_{CNN}(K_{m,n}) \le \begin{cases} 2 & \text{if m and n are odd} \\ 3 & \text{if m is odd and n is even} \\ 3 & \text{if m is even and n is odd} \\ 4 & \text{if m and n are even} \end{cases}$$

Consequently the result follows.

Theorem 3.6 Let $G = P_n \circ K_2$. Then $\gamma_{CNN}(P_n \circ K_2) = 2$

Proof. Let $V_1 = \{v_1, v_2, \dots, v_n\}$ and $V_2 = \{u_1, u_2, \dots, u_n\}$ be the vertex sets of the two copies of P_n in G and let $v_iu_i \in E(G)$.

Case 1.n is odd.

Define a function $f: V \longrightarrow \{-1, +1\}$ by

$$f(v) = \begin{cases} +1 & if \quad v = v_i, u_i, (i\ is\ odd) \\ -1 & otherwise \end{cases}$$

Claim:f is a complementary non-negative signed dominating function.

For $x \in v_i, u_i,\ (i = 1, n)$

$\sum_{u\notin N[x]} f(u) = (+1)\left[2\left\lceil\frac{n-2}{2}\right\rceil\right] + (-1)\left[2\left\lfloor\frac{n-2}{2}\right\rfloor + 1\right] = 2 - 1 = 1.$

For $x \in v_i, u_i$, $(i \text{ is even})$

$\sum_{u\notin N[x]} f(u) = (+1)\left[2\left(\frac{n-3}{2}\right)\right] + (-1)\left[2\left(\frac{n-3}{2}\right)\right] + 2 = 2.$

For $x \in v_i, u_i$, $(i \text{ is odd})$

$\sum_{u\notin N[x]} f(u) = (+1)\left[2\left(\frac{n-3}{2} + 1\right)\right] + (-1)\left[2\left(\frac{n-3}{2} - 1\right)\right] - 2 = 2.$

Therefore f is a complementary non-negative signed dominating function. Since $\sum_{u\notin N[x]} f(u) = 1$ for $x \in v_i, u_i$, $(i = 1, n)$, the labelling is minimum with respect to all the vertices in G. If any one of $f(v_i) = f(u_i) = -1$, for $(2 \le i \le n-1)$, i is odd, then $\sum_{u\notin N[x]} f(u) = -1$ for $x \in \{v_1, u_1, v_n, u_n\}$. Also if $f(v_1) = -1$, then $\sum_{u\notin N[x]} f(u) = -1$ for $x \in \{v_n, u_n\}$. This argument is similar for the vertices u_1,u_n and v_n. It is easy to observe that

$$\sum_{v\in V(P_n \circ K_2)} f(v) = (+1)\left[2\left\lceil\frac{n}{2}\right\rceil\right] + (-1)\left[2\left\lfloor\frac{n}{2}\right\rfloor\right] = 2$$

Hence $\gamma_{CNN}(P_n \circ K_2) = 2$ if n is odd.

Case 2.n is even.

Define a function $f: V \longrightarrow \{-1, +1\}$ by

$$f(v) = \begin{cases} +1 & \text{if } \quad v = v_i, u_i, u_n, (i \text{ is odd}) \\ -1 & \text{otherwise} \end{cases}$$

Claim:f is a complementary non-negative signed dominating function.

For $x \in v_i, u_i (i = 1, n)$

$\sum_{u\notin N[x]} f(u) = (+1)\left[2\left(\frac{n-2}{2}\right) + 1\right] + (-1)\left[2\left(\frac{n-2}{2}\right)\right] = 1.$

For $x \in v_i, u_i (i \neq 1, n)$

$\sum_{u\notin N[x]} f(u) = (+1)\left[2\left\lfloor\frac{n-3}{2}\right\rfloor + 3\right] + (-1)\left[2\left\lfloor\frac{n-3}{2}\right\rfloor + 1\right] = 2.$

Therefore f is a complementary non-negative signed dominating function. Since $\sum_{u\notin N[x]} f(u) = 1$ for $x \in v_i, u_i$, $(i = 1, n)$, the labelling is minimum with respect to all the vertices in G. If any one of $f(v_i)$=$f(u_i)$=-1, for $(2 \le i \le n-1)$, i is odd, then $\sum_{u\notin N[x]} f(u) = -1$ for $x \in \{v_1, u_1, v_n, u_n\}$. Also if $f(v_1) = -1$, then $\sum_{u\notin N[x]} f(u) = -1$ for $x \in \{v_n, u_n\}$. This argument is similar for the vertices u_1,u_n and v_n. It is easy to observe that

$$\sum_{v\in V(P_n \circ K_2)} f(v) = (+1)\left[2\left(\frac{n}{2}\right) + 1\right] + (-1)\left[2\left(\frac{n}{2}\right) - 1\right] = 2$$

Hence $\gamma_{CNN}(P_n \diamond K_2) = 2$ if n is even.

Theorem 3.7 Let G denote the friendship graph with t-triangles.Then $\gamma_{CNN}(G) = -1$.

Proof. Let u be the central vertex of G and let $(v_1, v_2, \ldots, v_{2t})$ be the vertices in the triangles. Define $f: V \longrightarrow \{-1, +1\}$ by $f(u) = -1$ and

$$f(v_i) = \begin{cases} +1 & if \quad i \text{ is odd}, \\ -1 & otherwise \end{cases}$$

Then it is to verify that $\sum_{u \notin N[v_i]} f(u) \geq 0$ for all i. Hence f is a complementary non-negative signed dominating function of G. Therefore $\gamma_{CNN}(G) \leq -1(t+1) + 1(t) = -1$.

Now suppose g be a complementary non-negative signed dominating function of G such that $\gamma_{CNN}(G) = g(V)$. Then $|P_g| \geq 1$. Since $deg\ u = n-1$, $u \notin P_g$.Therefore $v_i \in P_g$, for some i. Hence

$$\gamma_{CNN}(G) = \sum_{u \notin N[v_i]} g(u) + g(v_i) + g(u) + g(v_{i+1})$$

$\geq$ 0+1-1-1
$= -1$.
Consequently the result follows.

Theorem 3.8 Let $G = K_n \diamond K_2$. Then

$$\gamma_{CNN}(K_n \diamond K_2) = \begin{cases} 2 & if \quad n \text{ is odd}, \\ 4 & if \quad n \text{ is even}. \end{cases}$$

Proof. Let $V_1 = \{v_1, v_2, \ldots, v_n\}$ and $V_2 = \{u_1, u_2, \ldots, u_n\}$ be the vertex sets of the two copies of K_n in G and let $v_i u_i \in E(G)$.
Case 1. n is odd.

Define a function $f: V \longrightarrow \{-1, +1\}$ by

$$f(v) = \begin{cases} -1 & if \quad v = v_i, u_i, (1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor) \\ +1 & otherwise \end{cases}$$

Claim: f is a complementary non-negative signed dominating function.

For $x \in v_i, u_i$, $\left(1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor\right)$

$$\sum_{u \notin N[x]} f(u) = (+1)\left[\left\lceil \frac{n}{2} \right\rceil\right] + (-1)\left[\left\lfloor \frac{n}{2} \right\rfloor - 1\right] = 1 + 1 = 2.$$

For $x \in v_i, u_i$, $\left(\left\lceil \frac{n}{2} \right\rceil \leq i \leq n\right)$

$$\sum_{u \notin N[x]} f(u) = (+1)\left[\left\lfloor \frac{n}{2} \right\rfloor\right] + (-1)\left[\left\lfloor \frac{n}{2} \right\rfloor\right] = 0.$$

Therefore f is a complementary non-negative signed dominating function. Hence $\gamma_{CNN}(K_n \diamond K_2) \leq 2$, if n is odd.

Now suppose g be complementary non-negative signed dominating function with $\gamma_{CNN}(K_n \diamond K_2) =$

$g(V)$. Let $v_i \in V(G)$. Then $\sum_{u \notin N[v_i]} g(u) \geq 0$. Hence there exists some vertex $u_j \in P_g$. Then for the corresponding vertex v_j, $\sum_{u \notin N[v_j]} g(u) \geq 0$. This argument is similar for some vertex $u_i \in V(G)$. Therefore $|P_g| \geq 2\left(\left\lfloor\frac{n}{2}\right\rfloor + 1\right)$. Hence $\gamma_{CNN}(K_n \diamond K_2) \geq 2$, if n is odd.
Thus $\gamma_{CNN}(K_n \diamond K_2) = 2$ if n is odd.

Case 2. n is even.

Define a function $f: V \longrightarrow \{-1, +1\}$ by

$$f(v) = \begin{cases} -1 & \text{if } v = v_i, u_i, (1 \leq i \leq \frac{n}{2} - 1) \\ +1 & \text{otherwise} \end{cases}$$

Claim: f is a complementary non-negative signed dominating function.

For $x \in v_i, u_i$, $\left(1 \leq i \leq \frac{n}{2} - 1\right)$

$$\sum_{u \notin N[x]} f(u) = (+1)\left[\frac{n}{2} + 1\right] + (-1)\left[\frac{n}{2} - 2\right] = 1 + 2 = 3.$$

For $x \in v_i, u_i$, $\left(\frac{n}{2} \leq i \leq n\right)$

$$\sum_{u \notin N[x]} f(u) = (+1)\left[\frac{n}{2}\right] + (-1)\left[\frac{n}{2} - 1\right] = 1.$$

Therefore f is a complementary non-negative signed dominating function. Hence $\gamma_{CNN}(K_n \diamond K_2) \leq 4$, if n is even.

Now suppose g be complementary non-negative signed dominating function with $\gamma_{CNN}(K_n \diamond K_2) = g(V)$. Let $v_i \in V(G)$. Then $\sum_{u \notin N[v_i]} g(u) \geq 1$. Hence there exists some vertex $u_j \in P_g$. Then for the corresponding vertex v_j, $\sum_{u \notin N[v_j]} g(u) \geq 1$. This argument is similar for some vertex $u_i \in V(G)$. Therefore $|P_g| \geq 2\left(\frac{n}{2} + 1\right)$. Hence $\gamma_{CNN}(K_n \diamond K_2) \geq 4$, if n is odd.
Thus $\gamma_{CNN}(K_n \diamond K_2) = 4$ if n is odd.

IV. OPEN PROBLEMS

(i) Given any integer k, does there exist graph G such that $\gamma_{CNN}(G) - \gamma_{Cmaj}(G) = k$.
(ii) Characterization of graphs G for which $\gamma_{Cmaj}(G) = \gamma_{CNN}(G)$.
(iii) Find both lower and upper bound for γ_{CNN}.

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