Stability And Hopf Branch of A Predator-prey Model with Two Time Delays And Refuges Effect

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Abstract* — *This paper mainly investigated a Predator-prey Model with two time delays and refuges effect .By analyzing the characteristic equations,we discussed the local stability of equilibrium point of the system and the sufficient condition for the existence of Hopf branch.By choosing the delay as a bifurcation parameter,we can determine the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions by using the centermanifold theorem and normal form theory.At last, some numerical simulation results are confirmed that the feasibility of the theoretical analysis.

Keywords — *Hopf bifurcation,Stability,Two time delays ,Center manifold theorem,Predator-prey Model*

I. INTRODUCTION

In nature, predator and prey are the most common interactions among populations. In recent years, the dynamic analysis of predator-prey model has attracted extensive research as an important subject and has made great progress.The development trend of a population depends not only on the current, but also on the state in a certain period of time in the past.Faria(2001)[1]for more detail about the biological interpretation of the parameters. Celik (2008)[2]studied a ratio-dependent predator-prey model for a class of predator populations with time delays. Celik(2009)[3]studied a proportionately dependent predator-prey model for a prey population with time delay. In these two papers, the local stability of equilibrium point of the system and the sufficient condition for the existence of Hopf branch were analyzed with the time delay τ as a branching parameter. Yuan and Song (2009) [4]studied the Leslie-Gower predator-prey model with time delay, and analyzed the stability of its positive equilibrium and the existence of the Hopf branch. Ma(2012)[5]studied a predator-prey model in which both predator and prey populations have time delay, the stability and the Hopf bifurcation in this system are exploited successively. Chen and Zhang (2013)[6]proposed a delayed predator－prey model with predator migration to describe biological control. They studied the existence and stability of equilibrium point. The specific model is as follows：

$$\begin{cases} \dot{x}(t) = x(t)[r - ay(t)], \\ \dot{y}(t) = y(t)[-d + bx(t-\tau) - cy(t)] + m[x(t) - py(t)]. \end{cases} \quad (1)$$

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where $x(t)$ and $y(t)$ denote the population of the prey and the predator at time t,respectively. r is the intrinsic growth rate of the prey; $ay(t)$ is the hunting term in the presence of predators; m is the migration rate of predators; p is the consumption rate of the predator on prey per predator per unit time; c is the self-limitation constant of the predator; d is the death rate of the predator ; the positive feedback $bx(t-\tau)$ has a positive delay τ , which is the time due to converting prey biomass into predator biomass.All parameters are positive except d .

For individual predators, predators do not have the ability to catch at birth, they need time to grow from childhood to adulthood, the impact of time on the ability to catch can not be ignored.We need to consider the time delay effect of predator maturation.Zhang 和 Zhou(2019) investigated a predator-prey model with time.The local asymptotic stability of the positive equilibrium and the existence of the Hopf branch is demonstrated by using linearization. The following model is proposed.

$$\begin{cases} \dot{x}(t) = x(t)[r - ay(t)], \\ \dot{y}(t) = y(t)[-d + bx(t) - cy(t-\tau)] + m[x(t) - py(t)]. \end{cases} \quad (2)$$

Based on models(1)and(2), this paper introduces time delay for both predator and prey, and obtains a predator-prey model with two time delays. The specific model is as follows:

$$\begin{cases} \dot{x}(t) = x(t)[r - ay(t)], \\ \dot{y}(t) = y(t)[-d + bx(t-\tau_1) - cy(t-\tau_2)] + m[x(t) - py(t)]. \end{cases} \quad (3)$$

where τ_1 is the time due to converting prey biomass into predator biomass, τ_2 is growth time delay of predator.

In order to reduce the number of parameters, let $u(t) = \frac{cx(\frac{t}{\tau})}{pr}$, $v(t) = \frac{cy(\frac{t}{\tau})}{r}$,we get

$$\begin{cases} \frac{du(t)}{dt} = u(t)[1 - \frac{a}{c}v(t)], \\ \frac{dv(t)}{dt} = v(t)[-\frac{d}{r} + \frac{bp}{c}u(t-r\tau_1) - v(t-r\tau_2)] + \frac{pm}{r}[u(t) - v(t)]. \end{cases} \quad (4)$$

Substitute a , d , b , τ_1 , τ_2 and m respectively for $\frac{a}{c}$, $\frac{d}{r}$, $\frac{bp}{c}$, $r\tau_1$, $r\tau_2$ and $\frac{pm}{r}$,we can get

$$\begin{cases} \frac{du(t)}{dt} = u(t) - au(t)v(t)], \\ \frac{dv(t)}{dt} = v(t)[-d + bu(t-\tau_1) - v(t-\tau_2)] + m[u(t) - v(t)]. \end{cases} \quad (5)$$

First, let the equilibrium point of model (5) be $E(u_0, v_0)$, so it satisfies the following equation:

$$u_0 - au_0v_0 = 0 \quad ; v_0[-d + bu_0 - v_0] + m[u_0 - v_0] = 0$$

That is

$$u_0 = \frac{1 + a(m+d)}{a(b+ma)}, \qquad v_0 = \frac{1}{a}.$$

Theorem 1. For system (5), if $(H_1)\ -d - \frac{1}{a} < m$ so, the system has a unique positive equilibrium point.

II. STABILITY AND LOCAL HOPF BIFURCATION ANALYSIS

In this section, we focus on the problems of the Hopf bifurcation and stability for the system(5).

Let $x(t) = u(t) - u_0$, $y(t) = v(t) - v_0$,then the linearized approximation equation corresponding to model (5) at the equilibrium point $E(u_0, v_0)$ is:

$$\begin{cases} \dfrac{dx(t)}{dt} = (1 - av_0)x(t) - au_0y(t)], \\ \dfrac{dv(t)}{dt} = mx(t) + bv_0x(t-\tau_1) - (d - bu_0 + v_0 + m)y(t) - v_0y(t-\tau_2). \end{cases} \tag{6}$$

The correspoding characteristic equation of system (5) is as follows.

$$\lambda^2 + P\lambda + Qe^{-\lambda\tau_1} + R\lambda e^{-\lambda\tau_2} + S = 0. \tag{7}$$

Where

$$P = \frac{mda + m^2a + m}{b + ma} = \frac{m[1 + a(m+d)]}{b + ma} > 0,$$
$$Q = \frac{b[1 + a(m+d)]}{a(b+ma)} > 0,$$
$$R = \frac{1}{a} > 0,$$
$$S = \frac{m[1 + a(m+d)]}{b + ma} > 0.$$

In order to study the stability and branching of the equilibrium point E of the system, we only need to discuss the distribution of the roots of the characteristic equation (7).If all the roots of equation (7) have negative real parts, the equilibrium point E is asymptotically stable.If one root of the equation contains positive real parts, the equilibrium point E is unstable.Since the dynamic properties of differential equations with multiple delays are very complex, we discuss the two delay τ_1 and τ_2 of system (5) in three cases.

Case 1: $\tau_1 = \tau_2 = 0$ we

Theorem 1. For system (5),when $\tau_1 = \tau_2 = 0$,the equilibrium E is stable.

Proof. When $\tau_1 = \tau_2 = 0$,the characteristic equation of system (5) becomes

$$\lambda^2 + (P+R)\lambda + Q + S = 0. \tag{8}$$

Since $P+R>0$， $Q+S>0$,the two roots of equation (8) always have negative real parts.So when $\tau_1 = \tau_2 = 0$, the equilibrium point of system (5) is asymptotically stable.

Case 2: $\tau_1 = 0,\quad \tau_2 > 0$

Lemma 1. For the system (5), assume that (H_1) is satisfied.

（i）When $P^2 - 2(S+Q) - R^2 > 0$， so

(H_2) $(\frac{mda+m^2a+m}{b+ma})^2 - 2\frac{1+a(m+d)}{a} - \frac{1}{a^2} > 0$ is satisfied，there is no pure imaginary root in equation (7);

（ii）When $P^2 - 2(S+Q) - R^2 < 0$ and $\left[P^2 - 2(S+Q) - R^2\right]^2 > 4(S+Q)^2$， so

(H_3) $(\frac{mda+m^2a+m}{b+ma})^2 - 2\frac{1+a(m+d)}{a} - \frac{1}{a^2} < 0$ and

$$\left[(\frac{mda+m^2a+m}{b+ma})^2 - 2\frac{1+a(m+d)}{a} - \frac{1}{a^2}\right]^2 > 4\left[\frac{1+a(m+d)}{a}\right]^2$$

are satisfied，Then equation(7) has two pairs of purely imaginary roots $\pm i\omega_{2\pm}$ when $\tau_2 = \tau_{2\pm}^k$, where

$$\omega_{2\pm} = \sqrt{\frac{2(S+Q)+R^2-P^2 \pm \sqrt{\left[P^2-2(S+Q)-R^2\right]^2 - 4(S+Q)^2}}{2}} \tag{9}$$

$$\tau_{2\pm}^k = \frac{1}{\omega_{2\pm}}[\arccos(\frac{P}{R}) + 2k\pi], k = 0,1,2,\cdots \tag{10}$$

Proof. When $\tau_1 = 0,\quad \tau_2 > 0$,the characteristic equation of system (5) becomes

$$\lambda^2 + P\lambda + R\lambda e^{-\lambda\tau_2} + Q + S = 0. \tag{11}$$

First, we assume that $i\omega_2$ ($\omega_2 > 0$)is a root of the characteristic equation (11), then it satisfies the following equation

$$-\omega_2{}^2 + i\omega_2 P + iR\omega_2 e^{-i\omega_2\tau_2} + S + Q = 0. \tag{12}$$

That is $-\omega_2{}^2 + i\omega_2 P + i\omega_2 R(\cos\omega_2\tau_2 - i\sin\omega_2\tau_2) + S + Q = 0.$ (13)

The separation of the real and imaginary parts, it follows

$$\begin{cases} -\omega_2{}^2 + \omega_2 R\sin\omega_2\tau_2 + S + Q = 0, \\ P\omega_2 + \omega_2 R\cos\omega_2\tau_2 = 0. \end{cases} \tag{14}$$

From (14) we obtain

$$\omega_2{}^4 + [P^2 - 2(S+Q) - R^2]\omega_2{}^2 + (S+Q)^2 = 0. \tag{15}$$

When $P^2 - 2(S+Q) - R^2 > 0$,the equation has no positive root, so there is no pure imaginary root;

When $P^2 - 2(S+Q) - R^2 < 0$ and $\left[P^2 - 2(S+Q) - R^2\right]^2 > 4(S+Q)^2$, the equation has two positive roots ω_{2+} and ω_{2-} ,

$$\omega_{2+} = \sqrt{\frac{2(S+Q) + R^2 - P^2 + \sqrt{\left[P^2 - 2(S+Q) - R^2\right]^2 - 4(S+Q)^2}}{2}}$$

$$\omega_{2-} = \sqrt{\frac{2(S+Q) + R^2 - P^2 - \sqrt{\left[P^2 - 2(S+Q) - R^2\right]^2 - 4(S+Q)^2}}{2}}$$

From(14), we can get

$$\tau_{2\pm}^k = \frac{1}{\omega_{2\pm}}[\arccos(\frac{P}{R}) + 2k\pi], k = 0,1,2,\cdots$$

This completes the proof.

Lemma 2.Let $\tau_{20} = min\{\tau_{2+}^0, \tau_{2-}^0\} = \tau_{2+}^0$, and let the corresponding ω be ω_{20} .

Let $\lambda(\tau_2) = \alpha(\tau_2) + i\omega(\tau_2)$ be a root of the characteristicequation (7),which satisfies $\alpha(\tau_{20}) = 0$ and

$\omega(\tau_{20}) = \omega_{20}$,then we have the following transversality condition $\mathrm{Re}(\frac{d\lambda}{d\tau_2})^{-1}\Big|_{\tau_2=\tau_{20}} > 0$ is satisfied.

Proof. By differentiating both sides of equation (11) with regard to τ_2 and applying the implicit function theorem, we have :

$$\frac{d\lambda}{d\tau_2} = \frac{R\lambda^2 e^{-\lambda\tau_2}}{2\lambda + P + Re^{-\lambda\tau_2} - \tau_2 R\lambda e^{-\lambda\tau_2}},$$

Therefore,

$$(\frac{d\lambda}{d\tau_2})^{-1} = \frac{2\lambda + P}{R\lambda^2 e^{-\lambda\tau_2}} + \frac{1}{\lambda^2} - \frac{\tau_2}{\lambda},$$

So

$$\begin{aligned}\mathrm{Re}(\frac{d\lambda}{d\tau_2})^{-1}|_{\tau_2=\tau_{20}} &= \frac{-2\omega_{20}\sin\omega_{20}\tau_{20} + P\cos\omega_{20}\tau_{20}}{-R\omega_{20}^2} - \frac{1}{\omega_{20}^2} \\ &= \frac{2\omega_{20}^2 - 2Q + P^2 - R^2}{R^2\omega_{20}^2}\end{aligned}$$

Obviously, $\mathrm{Re}(\frac{d\lambda}{d\tau_2})^{-1}|_{\tau_2=\tau_{20}} = \frac{\sqrt{\Delta}}{R^2\omega_{20}^2} > 0$,the proof is completed.

Theorem 2. For system (5), suppose that (H_1) is true.When $\tau_1 = 0$, $\tau_2 > 0$,the following conclusions are true:

(a)When (H_2) is satisfied, so for all $\tau_2 > 0$, the equilibrium point E of system (5)is asymptotically uniformly stable;

(b)When (H_3) is satisfied, if $\tau_2 \in [0, \tau_{20})$, the equilibrium point E is asymptotically uniformly stable.If $\tau_2 = \tau_{20}$,model (5) generates Hopf branch at the equilibrium point E .If $\tau_2 > \tau_{20}$,model (5) is unstable at the equilibrium point E .

Case 3: $\tau_1 > 0$, $\tau_2 \in [0, \tau_{20})$

Lemma 3. Let $g(\omega_1) = (\omega_1^2 - S)^2 + 2(S - \omega_1^2)R\omega_1\sin\omega_1\tau_2 + 2PR\omega_1^2\cos\omega_1\tau_2 + P^2\omega_1^2 + R^2\omega_1^2 - Q^2$. Suppose that (H_4) is true,so $g(\omega_1)$ has a finite number of positive roots $\{\omega_{11}, \omega_{12}, \omega_{13}, \cdots, \omega_{1s}\}$ .Then equation(7) has a pair of purely imaginary roots $\pm i\omega_{1j}$ when $\tau_1 = \tau_{1j}^k$, where

$$\tau_{1j}^{k}=\frac{1}{\omega_{1j}}[\arccos(\frac{\omega_{1j}^{2}-R\omega_{1j}\sin\omega_{1j}\tau_{2}-S}{Q})+2k\pi],k=0,1,2,\cdots j=1,2,\cdots,s \quad (16)$$

Proof. If $i\omega_1(\omega_1>0)$ is a root of the equation (7), then it satisfies the following equation

$$-\omega_1{}^2+i\omega_1P+Qe^{-i\omega_1\tau_1}+i\omega_1\mathrm{Re}^{-i\omega_1\tau_2}+S=0. \quad (17)$$

The separation of the real and imaginary parts, it follows

$$\begin{cases}-\omega_1{}^2+Q\cos\omega_1\tau_1+\omega_1R\sin\omega_1\tau_2+S=0,\\ P\omega_1-Q\sin\omega_1\tau_1+\omega_1R\cos\omega_1\tau_2=0.\end{cases} \quad (18)$$

From (18) we obtain

$$(\omega_1^2-S)^2+2(S-\omega_1^2)R\omega_1\sin\omega_1\tau_2+2PR\omega_1^2\cos\omega_1\tau_2+P^2\omega_1^2+R^2\omega_1^2-Q^2=0 \quad (19)$$

The positive roots of the equation are assumed to be $\{\omega_{11},\omega_{12},\omega_{13},\cdots,\omega_{1s}\}$,

From (18) we get that

$$\tau_{1j}^{k}=\frac{1}{\omega_{1j}}[\arccos(\frac{\omega_{1j}^{2}-R\omega_{1j}\sin\omega_{1j}\tau_{2}-S}{Q})+2k\pi],k=0,1,2,\cdots \ j=1,2,\cdots,s$$

So equation(7) has a pair of purely imaginary roots $\pm i\omega_{1j}$ when $\tau_1=\tau_{1j}^k$. This completes the proof.

Lemma 4. Let $\tau_{10}=min\{\tau_{1j}^k,\ \ k=0,1,2,\cdots j=1,2,\cdots,s\}$ and let the corresponding ω_{1j} be ω_{10}.

Let $\lambda(\tau_1)=\alpha(\tau_1)+i\omega(\tau_1)$ be a root of the characteristicequation (7),which satisfies $\alpha(\tau_{10})=0$ and $\omega(\tau_{10})=\omega_{10}$,then we have the following transversality condition $\mathrm{Re}(\frac{d\lambda}{d\tau_1})^{-1}\Big|_{\substack{\lambda=i\omega_{10}\\ \tau_1=\tau_{10}}}\neq 0$ is satisfied.

Proof. By differentiating both sides of equation (7) with regard to τ_1 and applying the implicit function theorem, we have :

$$\frac{d\lambda}{d\tau_1}=\frac{Q\lambda e^{-\lambda\tau_1}}{2\lambda+P-Q\tau_1e^{-\lambda\tau_1}+Re^{-\lambda\tau_2}-\tau_2R\lambda e^{-\lambda\tau_2}},$$

Therefore,

$$(\frac{d\lambda}{d\tau_1})^{-1}=\frac{2\lambda+P+Re^{-\lambda\tau_2}-\tau_2R\lambda e^{-\lambda\tau_2}}{Q\lambda e^{-\lambda\tau_1}}-\frac{\tau_1}{\lambda},$$

So

$$\mathrm{Re}(\frac{d\lambda}{d\tau_1})^{-1}|_{\tau_1=\tau_{10}}^{\lambda=i\omega_{10}}=\frac{R\sin(\omega_{10}\tau_2-\omega_{10}\tau_{10})+R\omega_{10}\tau_2\cos(\omega_{10}\tau_2-\omega_{10}\tau_{10})-P\sin\omega_{10}\tau_{10}-2\omega_{10}\cos\omega_{10}\tau_{10}}{-Q\omega_{10}}$$

Obviously, $\mathrm{Re}(\frac{d\lambda}{d\tau_1})^{-1}|_{\tau_1=\tau_{10}}^{\lambda=i\omega_{10}}\neq 0$. The proof is completed.

According to the above analysis and Hopf branching theory $[7]$, we can obtain the following theorem.

Theorem 3. For system (5), when $\tau_1=0$, $\tau_2>0$,assume that (H_1) and (H_4) are true.the following conclusions are true:

If $\tau_1\in[0,\tau_{10})$, the equilibrium point E is asymptotically uniformly stable;

If $\tau_1=\tau_{10}$,model (5) generates Hopf branch at the equilibrium point E ;

If $\tau_1>\tau_{10}$,model (5) is unstable at the equilibrium point E .

III. DIRECTION AND STABILITY OF THE HOPF BIFURCATION

In the analysis in the above section, we have obtained the conditions for the system to generate Hopf branch. In thissection, we shall study the direction and stability of the bifurcating periodic solutions by applying the normal form theoryand center manifold theorem introduced by Hassard et al.$[7]$.

Let $x(t)=u(t)-u_0$, $y(t)=v(t)-v_0$,we consider the Taylor expansion of model (5) at the equilibrium point E ,

$$\begin{cases}\dot{x}(t)=(1-av_0)x(t)-au_0y(t)-ax(t)y(t),\\ \dot{y}(t)=mx(t)-(d+m-bu_0+v_0)y(t)+bv_0x(t-\tau_1)-v_0y(t-\tau_2)+\\ \qquad bx(t-\tau_1)y(t)-y(t)y(t-\tau_2).\end{cases}\quad(20)$$

Without loss of generality, we assume that $\tau_{10}>\tau_2^*$, $\tau_2^*\in[0,\tau_{20})$, $t\to(t/\tau_{10})$.For the sake of research, let $\tau_1=\tau_{10}+\mu$,then $\mu=0$ represents the Hopf branch parameter of system (5).Then the model (5) is equivalent to the following Functional Differential Equation (FDE) system

$$\dot{u}(t)=L_\mu u_t+F(u_t,\mu).\quad(21)$$

$$L_\mu(\phi) = (\tau_{10} + \mu)\left(B_1\phi(0) + B_2\phi(-\frac{\tau_2^*}{\tau_{10}}) + B_3\phi(-1) \right), \quad (22)$$

and

$$F(\phi, \mu) = (\tau_{10} + \mu)\begin{pmatrix} -a\phi_1(0)\phi_2(0) \\ b\phi_1(-1)\phi_2(0) - \phi_2(0)\phi_2(-\frac{\tau_2^*}{\tau_{10}}) \end{pmatrix}. \quad (23)$$

Where

$$B_1 = \begin{pmatrix} 1-av_0 & -au_0 \\ m & -(d+m-bu_0+v_0) \end{pmatrix} = \begin{pmatrix} 0 & -\dfrac{1+a(m+d)}{b+ma} \\ m & -\dfrac{m(da+ma+1)}{b+ma} \end{pmatrix},$$

$$B_2 = \begin{pmatrix} 0 & 0 \\ 0 & -v_0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -\dfrac{1}{a} \end{pmatrix},$$

$$B_3 = \begin{pmatrix} 0 & 0 \\ bv_0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \dfrac{b}{a} & 0 \end{pmatrix}.$$

Bythe Riesz representation theorem, there exists a bounded variation function $\eta(\theta, \mu)$, $\theta \in [-1,0]$, such that

$$L_\mu\phi = \int_{-1}^{0} d\eta(\theta, \mu)\phi(\theta), \quad \phi \in C. \quad (24)$$

In fact, we can choose

$$\eta(\theta, \mu) = \begin{cases} (\tau_{10} + \mu)(B_1 + B_2 + B_3), & \theta = 0, \\ (\tau_{10} + \mu)(B_2 + B_3), & \theta \in [-\dfrac{\tau_2^*}{\tau_{10}}, 0), \\ (\tau_{10} + \mu)B_3, & \theta \in (-1, -\dfrac{\tau_2^*}{\tau_{10}}), \\ 0, & \theta = -1. \end{cases} \quad (25)$$

For $\phi \in C'([-1,0], R^2)$, the operators A and R are defined as follow

$$A(\mu)\phi(\theta)=\begin{cases}\dfrac{d(\phi(\theta))}{d\theta}, & \theta\in[-1,0),\\ \int_{-1}^{0} d(\eta(t,\mu)\phi(t)), & \theta=0.\end{cases} \tag{26}$$

$$R(\mu)\phi(\theta)=\begin{cases}0, & \theta\in[-1,0),\\ F(\mu,\theta), & \theta=0.\end{cases} \tag{27}$$

Hence the system (5) can be written as the following form:

$$\dot{u}_t = A(\mu)u_t + R(\mu)u_t \tag{28}$$

Where $u_t = u(t+\theta), \theta\in[-1,0)$.

For $\psi\in C^{'}[0,1]$, we define the adjoint operator $A^{*}(0)$ of $A(0)$ as

$$A^{*}\psi(s)=\begin{cases}-\dfrac{d\psi(s)}{ds}, & s\in(0,1],\\ \int_{-1}^{0} d(\eta^{T}(s,0)\psi(-s)), s=0.\end{cases} \tag{29}$$

For $\phi\in C^{'}([-1,0],R^{2})$ and $\psi\in C^{'}[0,1]$, we define a Bilinear form

$$<\psi,\phi>=\overline{\psi}(0)\phi(0)-\int_{-1}^{0}\int_{\varepsilon=0}^{\theta}\overline{\psi}(\xi-\theta)d\eta(\theta)\phi(\xi)d\xi. \tag{30}$$

where $\eta(\theta)=\eta(\theta,0)$

Lemma 5.The eigenvectors $q(\theta)=(\rho_1,1)^{T}e^{i\omega_{10}\tau_{10}\theta}$ and $q^{*}(s)=D(\rho_1^{*},1)^{T}e^{i\omega_{10}\tau_{10}s}$ are respectively the eigenvectors corresponding to the eigen values $i\omega_{10}\tau_{10}$ and $-i\omega_{10}\tau_{10}$ of $A(0)$ and $A^{*}(0)$, and

$$<q^{*},q>=1,\quad <q^{*},\overline{q}>=0,$$

where

$$(\rho_1,1)^{T}=(\frac{i[1+a(m+d)]}{\omega_{10}(b+ma)},1)^{T},(\rho_1^{*},1)^{T}=(\frac{i(ma+be^{i\omega_{10}\tau_{10}})}{a\omega_{10}},1)^{T}$$

$$.\ \bar{D}=(1+\rho_1\bar{\rho}_1^{*}+\frac{b}{a}\tau_{10}e^{-i\omega_{10}\tau_{10}}\bar{\rho}_1^{*}+\frac{1}{a}\tau_2^{*}e^{-i\omega_{10}\tau_{20}}\rho_1\bar{\rho}_1^{*})^{-1}$$

proof. $\pm i\omega_{10}\tau_{10}$ are the eigen values of $A(0)$, so they are also the eigen values of $A^{*}(0)$.In order to determine the standard form of the operator $A(0)$,we assume that the eigenvectors $q(\theta)$ and $q_1^*(s)$ are respectively the eigenvectors correspondingto the eigenvalues $i\omega_{10}\tau_{10}$ and $-i\omega_{10}\tau_{10}$ of $A(0)$ and $A^*(0)$.We can obtain

$$\begin{cases} A(0)q(\theta) = i\omega_{10}\tau_{10}q(\theta) \\ A^*(0)q_1^*(s) = -i\omega_{10}\tau_{10}q_1^*(s) \end{cases} \tag{31}$$

From (24) and (26), (31) can be written as

$$\begin{aligned} &\frac{dq(\theta)}{d\theta} = i\omega_{10}\tau_{10}q(\theta), \qquad \theta \in [-1,0). \\ &L_0 q(0) = i\omega_{10}\tau_{10}q(0), \qquad \theta = 0. \end{aligned} \tag{32}$$

therefore $$q(\theta) = q(0)e^{i\omega_{10}\tau_{10}\theta}, \qquad \theta \in [-1,0].$$

Where $q(0) = (q_1(0), q_2(0))^T \in C^2$ is a constant vector, obtained from (22), (23)

$$[B_1 + B_2 e^{-i\omega_{10}\tau_2^*} + B_3 e^{-i\omega_{10}\tau_{10}}]q(0) = i\omega_{10}Iq(0)$$

By direct calculate, we get $q(0) = \begin{pmatrix} \rho_1 \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{i[1+a(m+d)]}{\omega_{10}(b+ma)} \\ 1 \end{pmatrix}$

then $q(\theta) = (\rho_1, 1)^T e^{i\omega_{10}\tau_{10}\theta}$

For non-zero vectors $q_1^*(s), s \in [0,1]$, we have $[B_1^T + B_2^T e^{i\omega_{10}\tau_2^*} + B_3^T e^{i\omega_{10}\tau_{10}}]q_1^*(0) = -i\omega_{10}Iq_1^*(0)$

Similarly $q_1^*(0) = \begin{pmatrix} \rho_1^* \\ 1 \end{pmatrix} = \begin{pmatrix} \dfrac{i(ma + be^{i\omega_{10}\tau_{10}})}{a\omega_{10}} \\ 1 \end{pmatrix}$,

then $q_1^*(s) = (\rho_1^*, 1)^T e^{i\omega_{10}\tau_{10}s}$,we make $q^*(s) = D(\rho_1^*, 1)^T e^{i\omega_{10}\tau_{10}s}$,

Now let's prove that $<q^*, q>= 1$ and $<q^*, q>= 1$, from equation (30), we get

$$\left\langle q^*, q \right\rangle$$
$$= \bar{q}^*(0)^T q(0) - \int_{\theta=-1}^{0} \int_{\xi=0}^{\theta} \bar{q}^{*T}(\xi-\theta) d\eta(\theta) q(\xi) d\xi.$$

$$= \bar{D}[(\bar{\rho}_1^*, 1)(\rho_1, 1)^T - \int_{\theta=-1}^{0} \int_{\xi=0}^{\theta} (\bar{\rho}_1^*, 1) e^{i\omega_{10}\tau_{10}(\theta-\xi)} d\eta(\theta)(\rho_1, 1)^T e^{i\omega_{10}\tau_{10}\xi} d\xi\}$$

$$= \bar{D}[1+\rho_1\bar{\rho}_1^* - (\bar{\rho}_1^*, 1)\int_{-1}^{0} \theta e^{i\omega_{10}\tau_{10}\theta} d\eta(\theta)(\rho_1, 1)^T]$$

$$= \bar{D}(1+\rho_1\bar{\rho}_1^* + \frac{b}{a}\tau_{10} e^{-i\omega_{10}\tau_{10}} \bar{\rho}_1^* + \frac{1}{a}\tau_2^* e^{-i\omega_{10}\tau_{20}} \rho_1\bar{\rho}_1^*) \quad (33)$$

Let $\bar{D} = (1+\rho_1\bar{\rho}_1^* + \frac{b}{a}\tau_{10} e^{-i\omega_{10}\tau_{10}} \bar{\rho}_1^* + \frac{1}{a}\tau_2^* e^{-i\omega_{10}\tau_{20}} \rho_1\bar{\rho}_1^*)^{-1}$, we can get $< q^*, q >= 1$.By

$\left\langle \psi, A\varphi \right\rangle = \left\langle A^*\psi, \varphi \right\rangle$, we obtain

$$-i\omega_{10}\tau_{10} \left\langle q^*, \bar{q} \right\rangle = \left\langle q^*, A\bar{q} \right\rangle = \left\langle A^* q^*, \bar{q} \right\rangle = \left\langle -i\omega_0\tau_{10} q^*, \bar{q} \right\rangle = i\omega_0\tau_{10} \left\langle q^*, \bar{q} \right\rangle. \quad (34)$$

So $< q^*, \bar{q} >= 0$. The proof is completed.

Next, we will use the same notations as in Hassard et al., we firstcompute the coordinates to describe the center manifold C_0 at $\mu = 0$. Define

$$z(t) =< q^*, u_t > \quad (35)$$

and $$W(t,\theta) = u_t(\theta) - 2\,\mathrm{Re}\{z(t)q(\theta)\}. \quad (36)$$

On the center manifold C_0, we have

$$W(t,\theta) = W(z(t), \bar{z}(t), \theta) \quad (37)$$

Where $W(z, \bar{z}, \theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta) z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \cdots$.

For the central epidemic C_0, z and $\bar{z}$ respectively represent the local coordinates of the central epidemic in the direction of q and q^*. Note that W is real if u_t is real, therefore we onlyreal solutions. Since $\mu = 0$, it is easy to see that

$$\dot{z}(t) = i\omega_{10}\tau_{10}z(\mathrm{t}) + \overline{q}^{*}(0)f(\mathrm{W}(\mathrm{z},\overline{\mathrm{z}},0) + 2\,\mathrm{Re}\{\mathrm{z(t)\,q(0)}\})$$

$$\stackrel{\Delta}{=} i\omega_{10}\tau_{10}z(\mathrm{t}) + \overline{q}^{*T} f_0(z,\overline{z}). \tag{38}$$

Let

$$\dot{z}(t) = i\omega_{10}\tau_{10}z + g(z,\overline{z}), \tag{39}$$

Where

$$g(z,\overline{z}) = g_{20}\frac{z^2}{2} + g_{11}z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + \cdots, \tag{40}$$

from (28) and (40), we have

$$\dot{W} = \dot{u}_t - \dot{z}q - \dot{\overline{z}}\dot{q} = \begin{cases} AW - 2\,\mathrm{Re}\,\overline{q}^{*T}(0)f_0(z,\overline{z})q(\theta), & \theta \in [-1,0), \\ AW - 2\,\mathrm{Re}\{\overline{q}^{*T}(0)f_0(z,\overline{z})q(\theta)\} + f_0(z,\overline{z}), & \theta = 0. \end{cases} \tag{41}$$

Which can be rewritten as

$$\dot{W} = AW + H(z,\overline{z},\theta) \tag{42}$$

where

$$H(z,\overline{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta)z\overline{z} + H_{02}(\theta)\frac{\overline{z}^2}{2} + \cdots \tag{43}$$

On the other hand, on C_0,

$$\dot{W} = W_z\dot{z} + W_{\overline{z}}\dot{\overline{z}} \tag{44}$$

Using (38) and (40) to replace W_z and $\dot{z}$ and their conjugates by their power series expansions, we obtain

$$\dot{W} = i\omega_{10}\tau_{10}W_{20}(\theta)z^2 - i\omega_{10}\tau_{10}W_{02}(\theta)\overline{z}^2 + \cdots. \tag{45}$$

Comparing the coefficients of the above equation with those of (43) and (45), we get

$$\begin{cases} (A - 2i\omega_{10}\tau_{10}I)W_{20}(\theta) = -H_{20}(\theta), \\ AW_{11}(\theta) = -H_{11}(\theta), \\ (A + 2i\omega_{10}\tau_{10}I)W_{02}(\theta) = -H_{02}(\theta). \end{cases} \tag{46}$$

Notice that $u_t(\theta) = W(z(t), \bar{z}(t), \theta) + zq + \overline{zq}$ and $q(\theta) = (\rho_1, 1)^T e^{i\omega_{10}\tau_{10}\theta}$, we get

$$u_t(\theta) = \begin{pmatrix} W^{(1)}(z,\bar{z},\theta) \\ W^{(2)}(z,\bar{z},\theta) \end{pmatrix} + z\begin{pmatrix} \rho_1 \\ 1 \end{pmatrix} e^{i\omega_{10}\tau_{10}\theta} + \bar{z}\begin{pmatrix} \bar{\rho}_1 \\ 1 \end{pmatrix} e^{-i\omega_{10}\tau_{10}\theta}. \quad (47)$$

so

$$\phi_1(0) = z\rho_1 + \bar{z}\bar{\rho}_1 + W_{20}^{(1)}(0)\frac{z^2}{2} + W_{11}^{(1)}(0)z\bar{z} + W_{02}^{(1)}(0)\frac{\bar{z}^2}{2} + \cdots$$

$$\phi_2(0) = z + \bar{z} + W_{20}^{(2)}(0)\frac{z^2}{2} + W_{11}^{(2)}(0)z\bar{z} + W_{02}^{(2)}(0)\frac{\bar{z}^2}{2} + \cdots$$

$$\phi_1(-1) = z\rho_1 \mathrm{e}^{-i\omega_{10}\tau_{10}} + \bar{z}\bar{\rho}_1 \mathrm{e}^{i\omega_{10}\tau_{10}} + W_{20}^{(1)}(-1)\frac{z^2}{2} + W_{11}^{(1)}(-1)z\bar{z} + W_{02}^{(1)}(-1)\frac{\bar{z}^2}{2} + \cdots$$

$$\phi_2(-\frac{\tau_2^*}{\tau_{10}}) = z\,\mathrm{e}^{-i\omega_{10}\tau_2^*} + \bar{z}\,\mathrm{e}^{i\omega_{10}\tau_2^*} + W_{20}^{(2)}(-\frac{\tau_2^*}{\tau_{10}})\frac{z^2}{2} + W_{11}^{(2)}(-\frac{\tau_2^*}{\tau_{10}})z\bar{z} + W_{02}^{(2)}(-\frac{\tau_2^*}{\tau_{10}})\frac{\bar{z}^2}{2} + \cdots$$

From (23), we obtain

$$f_0(z,\bar{z}) = \tau_{10}\begin{pmatrix} K_1 z^2 + K_2 z\bar{z} + K_3\bar{z}^2 + K_4 z^2\bar{z} \\ K_5 z^2 + K_6 z\bar{z} + K_7\bar{z}^2 + K_8 z^2\bar{z} \end{pmatrix}$$

where

$$K_1 = -a\rho_1, \quad K_2 = -a(\rho_1 + \bar{\rho}_1), \quad K_3 = -a\bar{\rho}_1,$$

$$K_4 = -a[\rho_1 W_{11}{}^{(2)}(0) + W_{11}{}^{(1)}(0) + \frac{1}{2}\bar{\rho}_1 W_{20}{}^{(2)}(0) + \frac{1}{2}W_{20}{}^{(1)}(0)],$$

$$K_5 = b\rho_1 e^{-i\omega_{10}\tau_{10}} - \rho_1 e^{-i\omega_{10}\tau_2^*},$$

$$K_6 = b(\rho_1 e^{-i\omega_{10}\tau_{10}} + \bar{\rho}_1 e^{i\omega_{10}\tau_{10}}) - (e^{-i\omega_{10}\tau_2^*} + e^{i\omega_{10}\tau_2^*}),$$

$$K_7 = b\bar{\rho}_1 e^{i\omega_{10}\tau_{10}} - e^{i\omega_{10}\tau_2^*},$$

$$K_8 = b[\rho_1 e^{-i\omega_{10}\tau_{10}} W_{11}{}^{(2)}(0) + \frac{1}{2}\bar{\rho}_1 e^{i\omega_{10}\tau_{10}} W_{20}{}^{(2)}(0) + \frac{1}{2} W_{20}{}^{(1)}(-1) + W_{11}{}^{(1)}(-1)] -$$
$$[W_{11}{}^{(2)}(-\frac{\tau_2{}^*}{\tau_{10}}) + \frac{1}{2} e^{i\omega_{10}\tau_2{}^*} W_{20}{}^{(2)}(0) + e^{-i\omega_0\tau_2{}^*} W_{11}{}^{(2)}(0) + \frac{1}{2} W_{20}{}^{(2)}(-\frac{\tau_2{}^*}{\tau_{10}})]$$

From $\bar{q}^{*T}(0) = \bar{D}(\bar{\rho}_1{}^*, 1)$, we obtain

$$\begin{aligned} g(z,\bar{z}) &= \bar{q}^{*T}(0) f_0(z,\bar{z}) \\ &= \tau_{10}\bar{D}(\bar{\rho}_1{}^*, 1)\begin{pmatrix} K_1 z^2 + K_2 z\bar{z} + K_3 \bar{z}^2 + K_4 z^2\bar{z} \\ K_5 z^2 + K_6 z\bar{z} + K_7 \bar{z}^2 + K_8 z^2\bar{z} \end{pmatrix} \\ &= \tau_{10}\bar{D}[(\bar{\rho}_1{}^* K_1 + K_5) z^2 + (\bar{\rho}_1{}^* K_2 + K_6) z\bar{z} + (\bar{\rho}_1{}^* K_3 + K_7)\bar{z}^2 + (\bar{\rho}_1{}^* K_4 + K_8) z^2\bar{z}] \end{aligned}$$

Comparing the cofficients of the above equation with those in (40), we get

$$\begin{aligned} g_{20} &= 2\tau_{10}\bar{D}(\bar{\rho}_1{}^* K_1 + K_5), g_{11} = \tau_{10}\bar{D}(\bar{\rho}_1{}^* K_2 + K_6), \\ g_{02} &= 2\tau_{10}\bar{D}(\bar{\rho}_1{}^* K_3 + K_7), g_{20} = 2\tau_{10}\bar{D}(\bar{\rho}_1{}^* K_4 + K_8). \end{aligned} \quad (48)$$

In order to determine the value of g_{21}, we also need to compute the values of $W_{20}(\theta)$ and $W_{11}(\theta)$,from $\theta \in [-1, 0)$,we obtain

$$\begin{aligned} H(z,\bar{z},\theta) &= -2\,\mathrm{Re}[\bar{q}^{*T}(0) f_0(z,\bar{z}) q(\theta)] \\ &= -(g_{20}(\theta)\frac{z^2}{2} + g_{11} z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots) q(\theta) \\ &\quad -(\bar{g}_{20}(\theta)\frac{z^2}{2} + \bar{g}_{11} z\bar{z} + \bar{g}_{02}\frac{\bar{z}^2}{2} + \cdots)\bar{q}(\theta). \end{aligned} \quad (49)$$

Comparing the coefficients with (43), we gives that

$$\begin{aligned} H_{20}(\theta) &= -g_{20} q(\theta) - \bar{g}_{02}\bar{q}(\theta), \\ H_{11}(\theta) &= -g_{11} q(\theta) - \bar{g}_{11}\bar{q}(\theta). \end{aligned} \quad (50)$$

When $\theta = 0$, we have

$$\begin{aligned} H(z,\bar{z},0) &= -2\,\mathrm{Re}[\bar{q}^{*T}(0) f_0(z,\bar{z}) q(0)] + f_0(z,\bar{z}) \\ &= -(g_{20}\frac{z^2}{2} + g_{11} z\bar{z} + g_{02}\frac{\bar{z}^2}{2} + \cdots) q(0) \\ &\quad -(\bar{g}_{20}(\theta)\frac{z^2}{2} + \bar{g}_{11} z\bar{z} + \bar{g}_{02}\frac{\bar{z}^2}{2} + \cdots)\bar{q}(0) + \tau_{10}\begin{pmatrix} K_1 z^2 + K_2 z\bar{z} + K_3 \bar{z}^2 + K_4 z^2\bar{z} \\ K_5 z^2 + K_6 z\bar{z} + K_7 \bar{z}^2 + K_8 z^2\bar{z} \end{pmatrix}. \end{aligned}$$

Comparing the coefficients with (43), we have

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau_{10}\begin{pmatrix} K_1 \\ K_5 \end{pmatrix},$$
$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + \tau_{10}\begin{pmatrix} K_2 \\ K_6 \end{pmatrix}. \quad (51)$$

Using (46), (50), we obtain

$$W_{20}(\theta) = \frac{ig_{20}}{\omega_{10}\tau_{10}}q(0)e^{i\omega_{10}\tau_{10}\theta} + \frac{i\bar{g}_{02}}{3\omega_{10}\tau_{10}}\bar{q}(0)e^{-i\omega_{10}\tau_{10}\theta} + E_1 e^{2i\omega_{10}\tau_{10}\theta},$$
$$W_{11}(\theta) = -\frac{ig_{11}}{\omega_{10}\tau_{10}}q(0)e^{i\omega_{10}\tau_{10}\theta} + \frac{i\bar{g}_{11}}{\omega_{10}\tau_{10}}\bar{q}(0)e^{-i\omega_{10}\tau_{10}\theta} + E_2. \quad (52)$$

Where $E_1 \in R^2, E_2 \in R^2$ are two two-dimensional vectors.

From the definition of $A(0)$ and (46) , we have

$$\int_{-1}^{0} d\eta(\theta)W_{20}(\theta) = 2i\omega_{10}\tau_{10}W_{20}(0) - H_{20}(0)$$

$$\int_{-1}^{0} d\eta(\theta)W_{11}(\theta) = -H_{11}(0)$$

and

$$(i\omega_{10}\tau_{10}I - \int_{-1}^{0} e^{i\omega_{10}\tau_{10}\theta} d\eta(\theta))q(0) = 0$$

$$(-i\omega_{10}\tau_{10}I - \int_{-1}^{0} e^{-i\omega_{10}\tau_{10}\theta} d\eta(\theta))\bar{q}(0) = 0.$$

Hence, we can get

$$(2i\omega_{10}\tau_{10}I - \int_{-1}^{0} e^{2i\omega_{10}\tau_{10}\theta} d\eta(\theta))E_1 = 2\tau_{10}\begin{pmatrix} K_1 \\ K_5 \end{pmatrix}$$

$$(\int_{-1}^{0} d\eta(\theta))E_2 = -\tau_{10}\begin{pmatrix} K_2 \\ K_6 \end{pmatrix}$$

Therefore, we have

$$\begin{cases} \begin{pmatrix} 2i\omega_{10} & \dfrac{1+a(m+d)}{b+ma} \\ -m-\dfrac{b}{a}e^{-2i\omega_{10}\tau_{10}} & 2i\omega_{10}+\dfrac{m(da+ma+1)}{b+ma}+\dfrac{1}{a}e^{-2i\omega_{10}\tau_2^*} \end{pmatrix} E_1 = 2\begin{pmatrix} K_1 \\ K_5 \end{pmatrix} \\ \begin{pmatrix} 0 & -\dfrac{1+a(m+d)}{b+ma} \\ m+\dfrac{b}{a} & -\dfrac{m(da+ma+1)}{b+ma}-\dfrac{1}{a} \end{pmatrix} E_2 = -\begin{pmatrix} K_2 \\ K_6 \end{pmatrix} \end{cases} \quad (53)$$

By calculation we can get

$$\begin{cases} \begin{pmatrix} 2i\omega_{10} & \dfrac{1+a(m+d)}{b+ma} \\ -m-\dfrac{b}{a}e^{-2i\omega_{10}\tau_{10}} & 2i\omega_{10}+\dfrac{m(da+ma+1)}{b+ma}+\dfrac{1}{a}e^{-2i\omega_{10}\tau_2^*} \end{pmatrix} E_1 = 2\begin{pmatrix} K_1 \\ K_5 \end{pmatrix} \\ \begin{pmatrix} 0 & -\dfrac{1+a(m+d)}{b+ma} \\ m+\dfrac{b}{a} & -\dfrac{m(da+ma+1)}{b+ma}-\dfrac{1}{a} \end{pmatrix} E_2 = -\begin{pmatrix} K_2 \\ K_6 \end{pmatrix} \end{cases} \quad (54)$$

Based on the above analysis, we can get the following parameter values:

$$C_1(0) = \frac{i}{2\omega_{10}\tau_{10}}\left(g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2\right) + \frac{g_{21}}{2},$$

$$\mu_2 = -\frac{Re\{C_1(0)\}}{Re\{\lambda'(\tau_{10})\}},$$

$$\beta_2 = 2Re\{C_1(0)\},$$

$$T_2 = -\frac{Im\{C_1(0)\} + \mu_2(Im\{\lambda'(\tau_{10})\})}{\omega_{10}\tau_{10}}. \quad (55)$$

Theorem 4. In the case of system (5), the conclusion holds

a) The direction of the Hopf bifurcation is determined by the parameter μ_2. If $\mu_2 > 0$, the Hopf bifurcation is supercritical . If $\mu_2 < 0$, the Hopf bifurcation is subcritical .

b) β_2 determines the stability of the bifurcating periodic solution. If $\beta_2 < 0$, the bifurcating periodic solutions is stable; if $\beta_2 > 0$, the bifurcating periodic solutions is unstable.

c) The period of the bifurcating periodic solution is decided by the parameter T_2. If $T_2 > 0(< 0)$, the period increases(decreases).

IV. NUMERICAL SIMULATION

In this section, we present numerical results to confirm the analytical predictions obtained in the previous section.

For system (5) , We take the parameters: $a = 10, b = 0.1, m = 1, d = 2$ .According to the previous analysis, we get thatthe equilibrium point of the system (5) is $E = (\frac{31}{101}, \frac{1}{10})$.And these coefficients satisfy conditions (H_1) and (H_2) .When $\tau_1 = 0$, $\tau_2 > 0$,for all $\tau_2 > 0$, the equilibrium point $E = (\frac{31}{101}, \frac{1}{10})$ of system (5)is asymptotically uniformly stable(see Fig1).

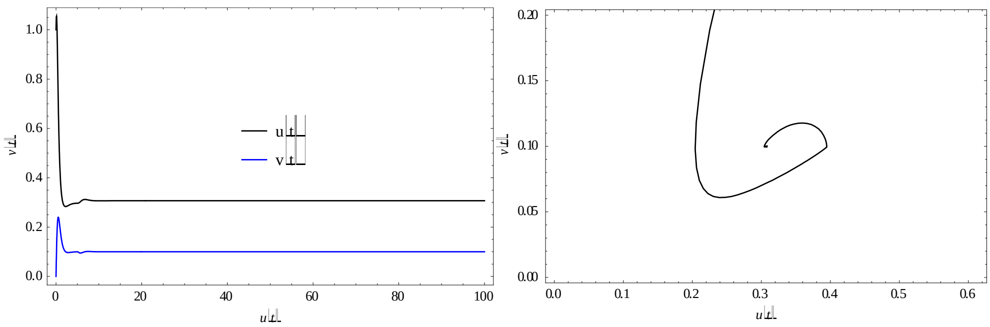

Figure 1 the equilibrium point $E = (\frac{31}{101}, \frac{1}{10})$ is asymptotically stable with $\tau_2 = 5$

For system (5) , We take the parameters: $a = 4, b = 2, m = 1, d = -1$.According to the previous analysis, we get thatthe equilibrium point of the system (5) is $E = (\frac{1}{24}, \frac{1}{4})$ .And these coefficients satisfy conditions (H_1) and (H_3) .When $\tau_1 = 0$, $\tau_2 > 0$,we can get $\omega_{20} = 0.18$, $\tau_{20} = 4.666$ .When $\tau_2 < \tau_{20}$,the equilibrium point $E = (\frac{1}{24}, \frac{1}{4})$ of system (5)is asymptotically uniformly stable(see Fig2).When $\tau_2 = \tau_{20}$,a stable periodic solution branches off from the equilibrium $E = (\frac{1}{24}, \frac{1}{4})$ (see Fig3).When $\tau_2 > \tau_{20}$,the equilibrium point $E = (\frac{1}{24}, \frac{1}{4})$ of the system (5) loses its stability and the system is unstable(see Fig 4).

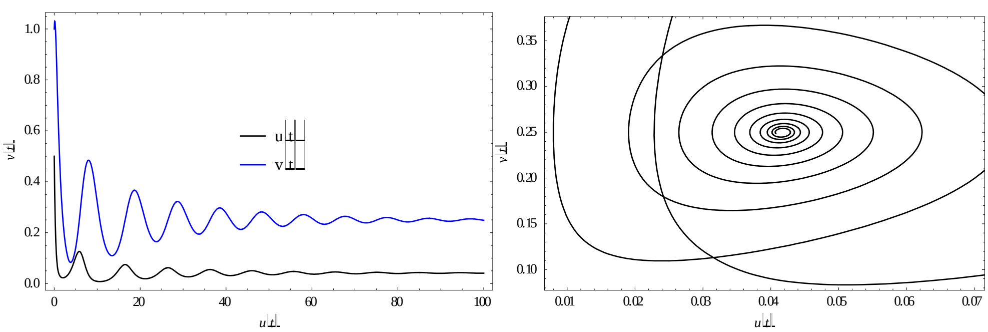

Figure 2 the equilibrium point $E=(\frac{1}{24},\frac{1}{4})$ is asymptotically stable with $\tau_2=3$

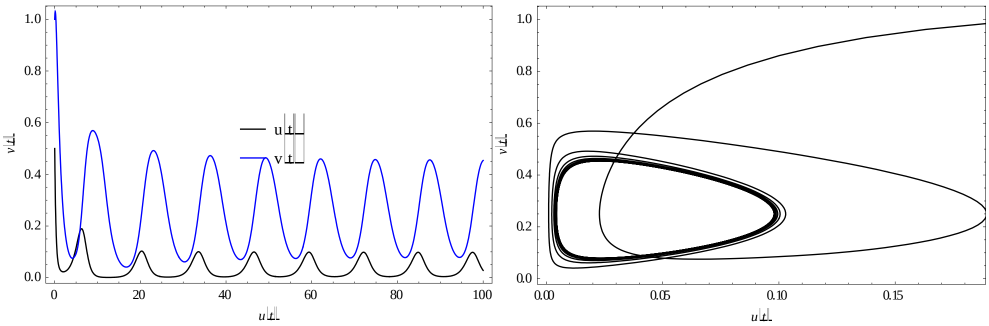

Figure 3 a stable periodic solution appears when $\tau_2=4.666$

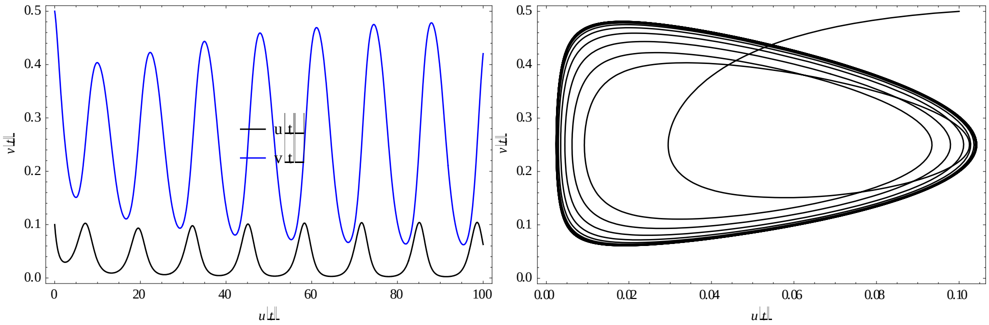

Figure 4 an unstable periodic solution appears when $\tau_2=5$

When $\tau_1>0$, $\tau_2\in[0,\tau_{20})$,let $\tau_2=3\in[0,\tau_{20})$,we can get $\tau_{10}=0.996$.When $\tau_1<\tau_{10}$, the equilibrium point $E=(\frac{1}{24},\frac{1}{4})$ of system (5)is asymptotically uniformly stable(see Fig5).When $\tau_1>\tau_{10}$,the equilibrium point $E=(\frac{1}{24},\frac{1}{4})$ of the system (5) loses its stability and the system is unstable(see Fig 6).

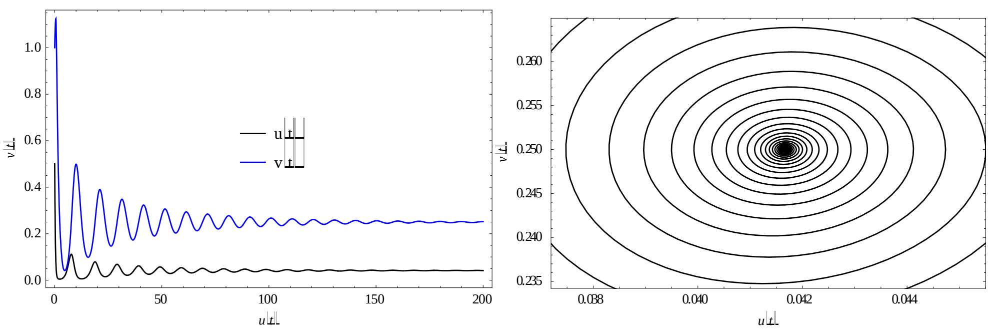

Figure 5 the equilibrium point $E=(\frac{1}{24},\frac{1}{4})$ is asymptotically stable with $\tau_1=0.5$

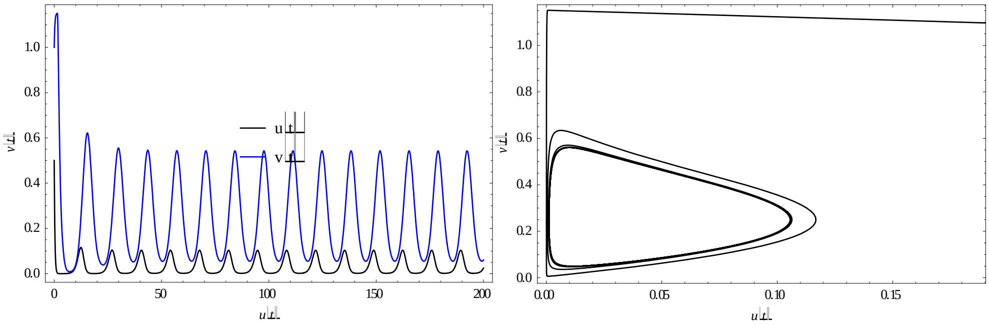

Figure 6 an unstable periodic solution appears when $\tau_1 = 2$

V. CONCLUSIONS

In this paper,a Predator-prey Model with two time delays and refuges effect is studied. Firstly,if $\tau_1 = 0$,when (H_1) is satisfied, that is, there is a unique positive equilibrium point in model (5).If the coefficients in model (5) satisfy condition (H_2),the delay will not affect the stability of the model positive equilibrium;If the coefficients in model (5) satisfy the condition (H_3), the sufficient conditions for the equilibrium stability of the system are obtained, and the existence conditions of the linear stability region and Hopf branch of the system are given. Then, if $\tau_1 \neq 0$,fixed τ_2, (H_4) is satisfied, when the hysteresis τ_1 reaches a certain critical value, it will affect the stability of the equilibrium point of the model and lead to the emergence of branches.Next,we can determine the direction of the Hopf bifurcation and the stability of the bifurcating periodic solutions by using the center manifold theorem and normal form theory.Finally, some numerical simulation results are confirmed that the feasibility of the theoretical analysis.The asymptotic stability of the equilibrium point and the existence of the periodic orbit can be determined by using the obtained fundamental theorem.

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