On an initial value problem of delay-refereed differential Equation

EL-Sayed A.M.A[#1], Ebead H.R[*2]
[1,2]*Faculty of Science, Alexandria University, Alexandria, Egypt*

Abstract — *In this paper we study the existence of positive solutions for an initial value problem of a delay-refereed differential equation. The continuous dependence of the unique solution on the initial data and the delay-refereed function will be proved. Some especial cases and examples will be given.*

Keywords — *Delay-refereed differential equation, existence of solutions, continuous dependence, Arzela-Ascoli Theorem, Schauder fixed point Theorem.*

I. INTRODUCTION

Many authors studied the differential and integral equations with deviating arguments only in the time itself, however, the case of the deviating arguments depend on both the state variable x and the time t is important in theory and practice, see for example [1]-[4], [7], [8], [9], [11]-[17].

In [4], the author studied the existence of a unique solution $x \in C[a,b]$ and its continuous dependence on the initial data of the initial value problem of the self-refereed differential equation

$$\frac{d}{dt}x(t) = f(t, x(x(t))), \quad t \in (0,T] \ \ and\ x(0) = x_o$$

where $f \in (C[a,b], C[a,b])$. Here we relax the assumptions of [4] and generalize the results.
Let $C[0,T]$ be the class of continuous functions defined on $[0,T]$ with norm

$$||x|| = \sup_{t\in[0,T]} |x(t)|, \quad x \in C[0,T].$$

Let g be a delay-refereed function defined such that
$g: [0,T] \times R^+ \to [0,T]$ be continuous and $g(t, x(t)) \le t$.

Consider the initial value problem of the delay-refereed differential equation

$$\frac{d}{dt}x(t) = f\left(t, x\left(g\left(t, x(t)\right)\right)\right), \ \ a.e., \quad t \in (0,T] \quad (1)$$

$$x(0) = x_0 \in [0,T]. \quad (2)$$

Our aim in this work is to prove the existence of positive solutions $x \in C[0,T]$ of the initial value problem (1)-(2).

The continuous dependence of the unique solution on the initial data x_0 and the delay-refereed function g will be studied.

II. Existence of solutions

Consider now, the initial value problem (1)-(2) under the following assumptions:
(1) $f: [0,T] \times R^+ \to R^+$ satisfies Carathéodory condition i.e. f is measurable in t for all $x \in C[0,T]$ and continuous in x for almost all $t \in [0,T]$.
(2)There exists a bounded measurable function $m: [0,T] \to R^+$,
$M = \sup_{t\in[0,T]} |m(t)|$ and a constant $b \ge 0$ such that

$$|f(t,x)| \le |m(t)| + b|x|.$$

(3) $g: [0,T] \times R^+ \to [0,T]$ is continuous and $g(t, x(t)) \le t$.
(4) $bT < 1$.
Define the set S_L by

$$S_L = \{x \in C[0,T]: |x(t_2) - x(t_1)| \leq L|t_2 - t_1|\} \subset C[0,T],\ L = \frac{M + b|x_0|}{1 - bT}.$$

It clear that S_L is nonempty, closed, bounded and convex subset of $C[0,T]$.
Now we have the following existence theorem.

Theorem 1 *Let the assumptions* $(1) - (4)$ *be satisfied, then the initial value problem (1), (2) has at least one positive solution* $x \in S_L \subset C[0,T]$.

Proof. Let$x$be a soltion of the problem (1)-(2). Integrating the differential equation (1) we obtain the corresponding integral equation

$$x(t) = x_0 + \int_0^t f(s, x(g(t, x(s)))\, ds > 0,\ \ t \in [0,T]. \tag{3}$$

Define the operator F associated with equation (3) by

$$Fx(t) = x_0 + \int_0^t f(s, x(g(t, x(s)))dst \in [0,T].$$

First, we prove that F is uniformly bounded.
Let $x \in C[0,T]$, then for $t \in [0,T]$ we have

$$\begin{aligned}
&|Fx(t)| \leq |x_0| + \int_0^t |f(s, x(g(s, x(s)))|ds \\
&\leq |x_0| + \int_0^t \{|m(s)| + b|x(g(s, x(s)))|\}ds \\
&\leq |x_0| + \int_0^t \{M + b|x(g(s, x(s)))|\}ds.
\end{aligned}$$

But

$$|x(g(t, x(t))))| - |x_0| \leq |x(g(t, x(t))) - x(0)| \leq L|g(t, x(t))|$$

and

$$|x(g(t, x(t)))| \leq L|g(t, x(t))| + |x_0|, \tag{4}$$

then

$$\begin{aligned}
&|Fx(t)| \leq |x_0| + \int_0^t \{M + b(L|g(s, x(s))| + |x_0|)\}ds \\
&\leq |x_0| + \int_0^t \{M + b\,(Ls + |x_0|)\}ds \\
&\leq |x_0| + (M + b(LT + |x_0|))t \\
&\leq LT + |x_0|.
\end{aligned}$$

This proves that the class functions $\{Fx\}$ is uniformly bounded on S_L.

Let $x \in S_L$ and $t_1, t_2 \in [0,T]$ with $t_1 < t_2$ such that $|t_2, -t_1| < \delta$, then

$$\begin{aligned}
&|Fx(t_2) - Fx(t_1)| = |\int_{t_1}^{t_2} f(s, x(g(s, x(s)))ds| \\
&\leq \int_{t_1}^{t_2} |f(s, x(g(s, x(s)))|ds \\
&\leq \int_{t_1}^{t_2} \{M + b|x(g(s, x(s)))|\}ds \\
&\leq \int_{t_1}^{t_2} \{M + b(L|g(t, x(s))| + |x_0|)\}ds \\
&\leq \int_{t_1}^{t_2} (M + b(Ls + |x_0|))ds \\
&\leq L|t_2 - t_1|.
\end{aligned}$$

This proves that $F: S_L \to S_L$ and the class of functions $\{Fx\}$ is equi-continuous on S_L. Now by Arzela-Ascoli Theorem [5] F is compact on S_L.

Finally, we will show that F is continuous.
Let $\{x_n\} \subset S_L,\ x_n \to x$ on $[0,T]$, i.e, $|x_n(t) - x(t)| \leq \epsilon_1$, this implies that $|x_n(g(t, x(t))) - x(g(t, x(t)))| \leq \epsilon_2$ and for arbitrary $\epsilon_1, \epsilon_2 \geq 0$, we can get

$$\begin{aligned}
&|x_n(g(t, x_n(t))) - x(g(t, x(t)))| = \\
&|x_n(g(t, x_n(t))) - x_n(g(t, x(t))) + x_n(g(t, x(t))) - x(g(t, x(t))| \\
&\leq |x_n(g(t, x_n(t))) - x_n(g(t, x(t)))| + |x_n(g(t, x(t))) - x(g(t, x(t)))| \\
&\leq L|g(t, x_n(t)) - g(t, x(t))| + |x_n(g(t, x(t))) - x(g(t, x(t)))| \\
&\leq \epsilon
\end{aligned}$$

and

$$x_n(g(t, x_n(t))) \to x(g(t, x(t)))\ in S_L.$$

From the continuity of the function f we obtain

$$f(t, x_n(g(t, x_n(t)))) \to f(t, x(g(t, x(t)))).$$

Then from the Lebesgues dominated convergence theorem [10] we have

$$\lim_{n\to\infty} x_n(t) = x_0 + \lim_{n\to\infty} \int_0^t f(s, x_n(g(s, x_n(s))))ds$$
$$= x_0 + \int_0^t f(s, x_o(g(s, x_o(s))))ds$$

and the operator F is continuous.

Now all conditions of Schauder fixed point Theorem [10] are satisfied, then the operator F has at least one fixed point $x \in S_L$. Consequently there exist at leat one positive solution $x \in C[0, T]$ of the integral equation (3).

To complete the proof, differentiating the integral equation (3) we obtain the differential equation (1). Also letting $t = 0$ in (3) we obtain the initial data (2).

This completes the proof of the equivalence between the initial value problem (1)-(2) and the integral equation (3). Hence the initial value problem (1)-(2) has at least one positive solution $x \in C[0, T]$.

III. Uniqueness of the solution

In this section we prove the uniqueness of the solution of the integral equation (3). For this aim we assume that:

$$(1')|f(t, x) - f(t, y)| \le b\,|x - y|.$$
$$(2')\ \sup_{t\in[0,T]}|f(t, 0)| \le M.$$

$(3')$ There exists a constant $k \in (0,1)$ such that $|g(t, x) - g(t, y)| \le k|x - y|$.

Theorem 2 *Let the assumptions (1),(3),(4) of Theorem 1 and* $(1'), (2')$ *and* $(3')$ *be satisfied, if* $bT(Lk + 1) < 1$, *then the solution of the integral equation (3) is unique.* **Proof.** Let $y = 0$ in $(1')$ we obtain

$$|f(t, x)| \le b\,|x| + |f(t, 0)|,$$

then we deduce that all assumptions of Theorem 1 are satisfied and the solution of equation (3) exists. Now let x, y be two solutions of the integral equation (3), then we obtain

$$|x(t) - y(t)| = |\int_0^t f(s, x(g(s, x(s)))ds - \int_0^t f(s, y(g(s, y(s)))ds|$$
$$\le \int_0^t |f(s, x(g(s, x(s))) - f(s, y(g(y, y(s)))|ds$$
$$\le b\int_0^t |x(g(s, x(s))) - y(g(s, y(s)))|ds$$
$$\le b\int_0^t |x(g(s, x(s))) - x(g(s, y(s)))|ds$$
$$+b\int_0^t |x(g(s, y(s))) - y(g(s, y(s)))|ds$$
$$\le bL\int_0^t |g(s, x(s)) - g(s, y(s))|ds$$
$$+b\int_0^t |x(g(s, y(s))) - y(g(s, y(s)))|ds$$
$$\le bLk\int_0^t |x(s) - y(s)|ds$$
$$+b\int_0^t |x(g(s, y(s))) - y(s, y(s)))|ds$$
$$\le bLkT \parallel x - y \parallel + bT \parallel x - y \parallel$$
$$= bT\,(Lk + 1) \parallel x - y \parallel$$

and

$$\parallel x - y \parallel \le bT\,(Lk + 1) \parallel x - y \parallel.$$

Since $bT\,(Lk + 1) < 1$, it follows that $x(t) = y(t),\ t \in C[0, T]$ and the solution of (3) is unique.

IV. Continuous dependence

Here we prove that the solution of integral equation (3) depends continuously of the initial data x_0 and the delay-refereed function g.

Definition 1 The solution of the integral equation (3) depends continuously on the initial data $x_0 if,\ \forall\epsilon >$

$0\ \exists\delta(\epsilon) > 0$ such that

$$|x_0 - x_0^*| \leq \delta \Rightarrow \| x - x^* \| \leq \epsilon \tag{5}$$

where x^* is the unique solution of the integral equation

$$x^*(t) = x_0^* + \int_0^t f(s, x^*(g(s,(x^*(s))))ds, \quad t \in [0,T]. \tag{6}$$

Theorem 3 *Let the assumptions of Theorem 2 be satisfied, then the solution of (3) depends continuously on the initial data* x_0.

Proof. Let x, x^* be the solution of the integral equation (3) and (6), then we have

$$\begin{aligned}
&|x(t) - x^*(t)| = |x_0 + \int_0^t f(s, x(g(s,x(s))))ds - x_0^* + \int_0^t f(s, x^*(g(s,x^*(s))))ds| \\
&\leq |x_0 - x_0^*| + \int_0^t |f(s, x(g(s,x(s))) - f(s, x^*(g(s,x^*(s))))|ds \\
&\leq |x_0 - x_0^*| + b\int_0^t |x(g(s,x(s))) - x^*(g(s,x^*(s)))|ds \\
&\leq |x_0 - x_0^*| + bL\int_0^t |g(s,x(s)) - g(s,x^*(s))|ds \\
&+b\int_0^t |x(g(s,x^*(s))) - x^*(g(s,x^*(s)))|ds \\
&\leq |x_0 - x_0^*| + bLk\int_0^t |x(s) - x^*(s)|ds \\
&+b\int_0^t |x(g(s,(x^*(s))) - x^*(g(s,x^*(s)))|ds \\
&\leq \delta + bLk \ \| x - x^* \| \ T + b \| x - x^* \| \ T
\end{aligned}$$

and

$$\| x - x^* \| \leq \delta + bT(Lk+1) \| x - x^* \|,$$

then

$$\| x - x^* \| \leq \frac{\delta}{(1 - bT(Lk+1))}.$$

Since $bT(Lk+1) < 1$ it follows that the solution of (3) depends continuously on the initial data x_0.

Definition 2 The solution of the integral equation (3) depends continuously on the function gif, $\forall\epsilon > 0\ \exists\delta(\epsilon) > 0$ such that

$$|g(t,x(t) - g^*(t,x(t)| \leq \delta \Rightarrow \| x - x^* \| \leq \epsilon \tag{7}$$

where x^* is the unique solution of the integral equation

$$x^*(t) = x_0 + \int_0^t f(s, x^*(g^*(s,(x^*(s))))ds, \quad t \in [0,T]. \tag{8}$$

Theorem 4 *Let the assumptions of Theorem 2 be satisfied, then the solution of (3) depends continuously on the function* g.

Proof. Let x, x^* be the solution of the integral equation (3) and (8), then we have

$$\begin{aligned}
&|x(t) - x^*(t)| = |x_0 + \int_0^t f(s, x(g(s,x(s))))ds - x_0 - \int_0^t f(s, x^*(g^*(s,x^*(s))))ds| \\
&\leq \int_0^t |f(s, x(g(s,x(s))) - f(s, x^*(g^*(s,x^*(s))))|ds \\
&\leq b\int_0^t |x(g(s,x(s))) - x^*(g^*(s,x^*(s)))|ds \\
&\leq b\int_0^t |x(g(s,x(s))) - x(g^*(s,x^*(s)))|ds \\
&+b\int_0^t |x(g^*(s,x^*(s))) - x^*(g^*(s,x^*(s)))|ds \\
&\leq bL\int_0^t |g(s,x(s)) - g^*(s,x^*(s))|ds + b \| x - x^* \| \ T \\
&\leq bL\int_0^t |g(s,x(s)) - g(s,x^*(s))|ds \\
&+bL\int_0^t |g(s,x^*(s)) - g^*(s,x^*(s))|ds + b \| x - x^* \| \ T \\
&\leq bLk\int_0^t |x(s) - x^*(s)|ds + \delta bLT + b \| x - x^* \| \ T \\
&\leq bLk \ \| x - x^* \| \ T + \delta bLT + b \| x - x^* \| \ T
\end{aligned}$$

and

$$\| x - x^* \| \leq bT\,(1 + Lk)\ \| x - x^* \| + \delta bLT,$$

then

$$\| x - x^* \| \leq \frac{\delta bLT}{(1 - bT\,(Lk+1))}.$$

Since $bT(Lk+1) < 1$ it follows that the solution of (3) depends continuously on the function g.

V. Examples

Example 1*Consider the nonlinear differential equation*

$$\frac{dx}{dt} = \frac{e^{-t}}{t+3} + \frac{1}{2}\ |x(\frac{t\sin^2 x(t)}{1+x^2(t)})|,\ \ t \in (0,1] \tag{9}$$

with the initial condition

$$x(0) = \frac{1}{3}. \tag{10}$$

Set

$$f(t, x(g(t, x(t))) = \frac{e^{-t}}{t+3} + \frac{1}{2}\ |x(\frac{t\sin^2 x(t)}{1+x^2(t)})|,$$

then

$$|f(t,x)| \leq \frac{e^{-t}}{t+3} + \frac{1}{2}\ |x|$$

and we have $m(t) = \frac{e^{-t}}{t+3}$, $M = 1/3$, $b = 1/2$, then $bT = 1/2 < 1$.

Applying to Theorem 1, then the initial value problem (9)-(10) has a positive continuous solution.

Example 2*Consider the nonlinear differential equation*

$$\frac{dx}{dt} = \frac{1}{2}\ln(1+t) + \frac{1}{6}\ |x(\frac{te^{-x^2(t)}}{1+|x(t)|})|,\ \ t \in (0,\frac{1}{2}], \tag{11}$$

with the initial condition

$$x(0) = 0.1. \tag{12}$$

Set

$$f(t, x(g(t, x(t))) = \frac{1}{2}\ln(1+t) + \frac{1}{6}\ |x(\frac{te^{-x^2(t)}}{1+|x(t)|})|,$$

then

$$|f(t,x)| \leq \frac{1}{2}|\ln(1+t)| + \frac{1}{6}\ |x|$$

and we have $m(t) = \frac{1}{2}\ln(1+t)$, $M = 1/4$ and $b = 1/6$, then $bT = 1/12 < 1$.

Applying to Theorem 1, then the initial value problem (11)-(12) has a continuous solution.

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