# Further Results on Minimum Dominating Color Energy of Graphs 

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#### Abstract

The energy of a graph $G$ is the sum of the absolute values of eigenvalues of the adjacency matrix of the graph. The color energy of the graph $G$ is the sum of the absolute values of the eigenvalues of the color matrix. Professor P. Siva Kota Reddy et al. introduced the concept of minimum dominating color energy of a graph. Motivated by this concept, in this paper, we compute the minimum dominating color energy for some families of graphs. We also find some color-hyper energetic families of graphs.


Keywords: Minimum dominating set, minimum dominating color eigenvalues, minimum dominating color energy of a graph.

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## I. INTRODUCTION

In early 1960's the mathematical study of domination in graphs started. Claude Berge[1] began the study of domination in graphs in 1958. He wrote a book known as the domination number of graphs in which he introduces the 'coefficient of external stability'. Oystein Ore[7] published his book in 1962 on graph theory where he introduces the concept of 'domination set' and 'domination number'.

Definition 1.1. A dominating set D in a graph G is a set of vertices such that each vertex of G is either in D or has atleast one neighbour in D . The minimum cardinality of the set is called domination number, denoted by $\gamma(\mathrm{G})$.

For various results in domination of graphs refer the book "Domination in Graphs"[5].
Another interesting research work in graph theory is about "Energy of graphs". The concept of energy of graph originated from chemistry to approximate the total $\pi$ electron energy of a molecule and these electrons are represented by a graph. The energy of a graph was first coined by Ivan Gutman in 1978 [3].

Definition 1.2. Let G be a graph with n vertices and m edges. Let the adjacency matrix of the graph G be $\mathrm{A}=\left[\mathrm{a}_{\mathrm{i} j}\right]$ and the eigenvalues of the adjacency matrix be $\lambda_{1}, \lambda_{2}, \ldots \lambda_{n}$. The energy $E(G)$ of a graph $G$ is defined as the sum of the absolute values of the eigenvalues of $\mathrm{A}(\mathrm{G})$. That is,

$$
\mathrm{E}(\mathrm{G})=\sum_{i=1}^{n}\left|\lambda_{i}\right|
$$

The concept of hyperenergetic graphs aroused because of the failure of the belief that the complete graph was the graph with largest energy which was proven to be false [6].

Definition 1.3 A graph G with n vertices is said to be hyperenergetic if $\mathrm{E}(\mathrm{G})>2 \mathrm{n}-2$.
Color energy of a graph was introduced by Chandrashekar Adiga et al. [2].The formal definition of color energy of a graph G is as follows:

Definition 1.4 Let $G$ be a vertex colored graph of order $n$. Then the color matrix of $G$ is the matrix $A_{c}(G)=\left[a_{i j}\right]_{n \times n}$ whose entries are given by
$a_{i j}=\left\{\begin{aligned} 1 & \text { if } v_{i} \text { and } v_{j} \text { are adjacent with } c\left(v_{i}\right) \neq c\left(v_{j}\right), \\ -1 & \text { if } v_{i} \text { and } v_{j} \text { are non adjacent with } c\left(v_{i}\right)=c\left(v_{j}\right), \\ 0 & \text { otherwise }\end{aligned}\right.$
where $c\left(v_{i}\right)$ is the color of a vertex $v_{i}$ in $G$
The eigenvalues of $A_{c}(G)$ are $\lambda_{1}, \lambda_{2}, \ldots \lambda_{\mathrm{n}}$ are called as color eigenvalues
Definition 1.5 The color energy $\mathrm{E}_{\mathrm{c}}(\mathrm{G})$ is defined as the sum of their absolute values of eigenvalues of the color matrix. Mathematically it can be expressed as

$$
\mathrm{E}_{\mathrm{c}}(\mathrm{G})=\sum_{i=1}^{n}\left|\lambda_{i}\right|
$$

The Minimum Dominating Energy of a graph was first defined by M. R. Rajesh Kanna et al [4].
Definition 1.6 A graph $G$ with $n$ vertices is said to be color-hyperenergetic if $\mathrm{E}_{\mathrm{c}}(\mathrm{G})>2 \mathrm{n}-2$.
Definition 1.7 Let $G$ be a simple graph of order $n$ with vertex set $V$, edge set $E$ and let $D$ be a minimum dominating set of a graph $G$. The minimum dominating matrix of $G$ is the $n \times n$ matrix defined by $A_{D}(G)=\left[a_{i j}\right]$ whose entries are given by
$\mathrm{a}_{\mathrm{ij}}=\left\{\begin{array}{l}1 \quad \text { if } \mathrm{v}_{\mathrm{i}} \text { and } \mathrm{v}_{\mathrm{j}} \text { are adjacent, } \\ 1 \quad \text { if } \mathrm{i}=\mathrm{j} \text { and } \mathrm{v}_{\mathrm{i}} \in \mathrm{D} \\ 0 \quad \text { otherwise }\end{array}\right.$

The concept of minimum dominating color energy of graphs was recently introduced by P. Shiva Kota Reddy et al. [8]
Definition 1.8 Let G be a simple graph of order n with vertex set V and edge set E . Let D be the minimum dominating set of a graph $G$. The minimum dominating color matrix of $G$ is the $n \times n$ matrix defined by $A_{c}^{D}(G)=\left[a_{i j}\right]$ where
$a_{i j}=\left\{\begin{array}{l}1 \text { if } v_{i} \text { and } v_{j} \text { are adjacent with } c\left(v_{i}\right) \neq c\left(v_{j}\right) \text { or if } i=j \text { and } v_{i} \in D, \\ -1 \text { if } v_{i} \text { and } v_{j} \text { are non adjacent with } c\left(v_{i}\right)=c\left(v_{j}\right), \\ 0 \quad \text { otherwise }\end{array}\right.$

Definition 1.9 The minimum dominating color eigenvalues of the graph G are the eigenvalues of $A_{c}^{D}(\mathrm{G})$. The characteristic polynomial of $A_{c}^{D}(\mathrm{G})$ is denoted by $f_{n}(\mathrm{G}, \lambda)=\operatorname{det}\left(\lambda \mathrm{I}-A_{c}^{D}(\mathrm{G})\right)$.

The minimum dominating color energy of $G$ is defined as $E_{c}^{D}=\sum_{i=1}^{n}\left|\lambda_{i}\right|$
If the color used is minimum then the energy is called minimum dominating chromatic energy and it is denoted by $\mathrm{E}_{\chi}^{\mathrm{D}}(\mathrm{G})$ [8].

## II. Minimum Dominating Color Energy of Some Standard Graphs

Definition 2.1 The Windmill graph $W_{n}^{m}$ is obtained by making $m$ copies of complete graph $K_{n}$ with a vertex in common.

In our study we will consider $W_{n}^{2}$.

Theorem -2.1 The minimum color dominating energy of Windmill graph $W_{n}^{2}$ is equal to $3 n-5+\sqrt{n^{2}+8}$ Proof:

Let $W_{n}^{2}$ be the windmill graph of order $n$ with $2 n-1$ vertices. The minimum color dominating set $D=\left\{v_{n}\right\}$

$$
A_{\chi}^{D}\left(W_{n}^{2}\right)=\left[\begin{array}{ccccccccc}
0 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 & -1 \\
1 & 0 & \cdots & 1 & 1 & 0 & \cdots & -1 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 1 & \cdots & 0 & 1 & -1 & \cdots & 0 & 0 \\
1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & 1 \\
0 & 0 & \cdots & -1 & 1 & 0 & \cdots & 1 & 1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & -1 & \cdots & 0 & 1 & 1 & \cdots & 0 & 1 \\
-1 & 0 & \cdots & 0 & 1 & 1 & \cdots & 1 & 0
\end{array}\right]
$$

The Characteristic Polynomial is
$(-\lambda)^{n-2}(\lambda+2)^{n-2}(\lambda-(n-1))\left(\lambda^{2}-(n-2) \lambda-(n+1)\right)=0$

The minimum dominating color eigen values are
$\operatorname{Spec}_{\mathrm{D}}\left(\mathrm{W}_{\mathrm{n}}^{2}\right)=\left(\begin{array}{lll}\mathrm{n}^{-2} & -\mathrm{n}+1 & \frac{\mathrm{n}-2+\sqrt{\mathrm{n}^{2}+8} \mathrm{n}-2-\sqrt{\mathrm{n}^{2}+8}}{\mathrm{n}^{2}} \\ \mathrm{n}^{2} & 1 & 1\end{array}\right)$

The minimum dominating color energy of
$\mathrm{E}_{\chi}^{\mathrm{D}}\left(\mathrm{W}_{\mathrm{n}}^{2}\right)=|2|(\mathrm{n}-2)+|\mathrm{n}+1|(1)+\left|\frac{\mathrm{n}-2+\sqrt{\mathrm{n}^{2}+8}}{2}\right|+\left|\frac{\mathrm{n}-2-\sqrt{\mathrm{n}^{2}+8}}{2}\right|$
$=2 n-4+n-1+\sqrt{n^{2}+8}$
$E_{\chi}^{D}\left(W_{n}^{2}\right)=3 n-5+\sqrt{n^{2}+8}$

Definition 2.2The double star graph $\left(B_{n, m}\right)$ is the graph obtained by joining the center of two stars $S_{r}$ and $S_{t}$ with an edge.

In our study we consider $B_{n, n}$ double star graph.

Theorem -2.2 The minimum color domination of double star graph $B_{n, n}$ is equal to $3 n-4+\sqrt{n^{2}+12 n-12}$
Proof:
Let $B_{n, n}$ be the double star graph of order $n$ with vertex set $\left\{u_{1}, u_{2}, \ldots u_{n}, v_{1}, v_{2} \ldots v_{n}\right\}$
The minimum dominating set $\mathrm{D}=\left\{\mathrm{u}_{\mathrm{n}}, \mathrm{v}_{\mathrm{n}}\right\}$.

$$
A_{\chi}^{\mathrm{D}}\left(\mathrm{~B}_{\mathrm{n}, \mathrm{n}}\right)=\left[\begin{array}{ccccccccc}
0 & -1 & \cdots & -1 & 1 & -1 & 0 & \cdots & 0 \\
-1 & 0 & \cdots & -1 & 1 & -1 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & \cdots & 0 & 1 & -1 & 0 & \cdots & 0 \\
1 & 1 & \cdots & 1 & 1 & 1 & -1 & \cdots & -1 \\
0 & 0 & \cdots & 0 & -1 & 1 & 0 & \cdots & -1 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & -1 & 1 & -1 & \cdots & 0
\end{array}\right]
$$

The characteristic polynomial is
$\left(\lambda+(n-2)(\lambda-1)^{2 n-4}(\lambda-2)\left(\lambda^{2}+(n-2) \lambda-(4 n-4)\right)=0\right.$

Minimum dominating color eigen values are
$\operatorname{Spec}_{\mathrm{D}}\left(\mathrm{B}_{\mathrm{n}, \mathrm{n}}\right)=\left(\begin{array}{cccc}-n+2 & 1 & 2 & \frac{-(n-2)+\sqrt{n^{2}+12 n-12}}{2} \frac{-(n-2)-\sqrt{n^{2}+12 n-12}}{2} \\ 1 & 2 n-4 & 1 & 1\end{array}\right)$

Minimum dominating color energy of
$E_{\chi}^{D}\left(B_{n, n}\right)=|-n+2|(1)+|1|(2 n-4)+|2|(1)+(1)\left|\frac{-(n-2)+\sqrt{n^{2}+12 n-12}}{2}\right|+(1)\left|\frac{-(n-2)+\sqrt{n^{2}+12 n-12}}{2}\right|$
$E_{\chi}^{D}\left(B_{n, n}\right)=3 n-4+\sqrt{n^{2}+12 n-12}$

Observation 2.1 The Double star graph $\mathrm{B}_{\mathrm{n}, \mathrm{n}}$ is a color-hyper energetic graph.

THEOREM 2.3 If $K_{m, n}$ is a bipartite graph then minimum dominating color energy is equal to $n+(m-3)+2+\sqrt{n^{2}+m^{2}+2 n m+2 n+2 m-7}$

## Proof:

Let $\mathrm{K}_{\mathrm{m}, \mathrm{n}}$ be the bipartite graph with vertex set $\mathrm{V}=\left\{\mathrm{u}_{1}, \mathrm{u}_{2}, \ldots \mathrm{u}_{\mathrm{n}}, \mathrm{v}_{1}, \mathrm{v}_{2} \ldots \mathrm{v}_{\mathrm{n}}\right\}$.

The minimum dominating set $\mathrm{D}=\left\{\mathrm{u}_{1}, \mathrm{v}_{1}\right\}$

$$
\mathrm{A}_{\chi}^{\mathrm{D}}\left(\mathrm{~K}_{\mathrm{m}, \mathrm{n}}\right)=\left[\begin{array}{ccccccccc}
1 & -1 & -1 & \cdots & -1 & 1 & 1 & \cdots & 1 \\
-1 & 0 & -1 & \cdots & -1 & 1 & 1 & \cdots & 1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \cdots & 0 & 1 & 1 & \cdots & 1 \\
1 & 1 & 1 & \cdots & 1 & 1 & -1 & \cdots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & 1 & \cdots & 1 & -1 & -1 & \cdots & 0
\end{array}\right]
$$

The characteristic polynomial is
$(\lambda-1)^{n+(m-3)}(\lambda-2)\left(\lambda^{2}+(n+(m-3)) \lambda-(2 n-(2 m-4))=0\right.$

The minimum dominating color eigen values are $\operatorname{spec}_{\chi}^{D}\left(K_{m, n}\right)=$
$\left(\begin{array}{ccc}1 & 2 \frac{-(n+(m-3))+\sqrt{n^{2}+m^{2}+2 n m+2 n+2 m-7}}{2}-(n+(m-3))-\sqrt{n^{2}+m^{2}+2 n m+2 n+2 m-7} \\ n+(m-3) & 1\end{array}\right)$

The minimum dominating color energy of

$$
\begin{aligned}
& E_{\chi}^{D}\left(K_{m, n}\right)=|1|(n+(m-3))+|2|(1)+\left|\frac{-(n+(m-3))+\sqrt{n^{2}+m^{2}+2 n m+2 n+2 m-7}}{2}\right|(1)+\left|\frac{-(n+(m-3))-\sqrt{n^{2}+m^{2}+2 n m+2 n+2 m-7}}{2}\right|(1) \\
& =(n+(m-3))+2 \sqrt{n^{2}+m^{2}+2 n m+2 n+2 m-7} \\
& E_{\chi}^{D}\left(K_{m, n}\right)=(n+(m-3))+2 \sqrt{n^{2}+m^{2}+2 n m+2 n+2 m-7}
\end{aligned}
$$

Observation 2.2 The Bipartite graph $\mathrm{K}_{\mathrm{m}, \mathrm{n}}$ is a color-hyper energetic graph.

THEOREM 2.4 If $\bar{K}_{n}$ is the complement of the complete graph with $n$ vertices then $E_{\chi}^{D}(G)\left(\bar{K}_{n}\right)=3 n-4$

## Proof:

Let $\overline{\mathrm{K}}_{\mathrm{n}}$ be the complement of the complete graph with vertex set $\mathrm{V}=\left\{\mathrm{v}_{1}, \mathrm{v}_{2} \ldots \mathrm{v}_{\mathrm{n}}\right\}$ and the minimum dominating set $\mathrm{D}=\left\{\mathrm{v}_{1}, \mathrm{v}_{2} \ldots \mathrm{v}_{\mathrm{n}}\right\}$

$$
\mathrm{A}_{c}^{\mathrm{D}}\left(\overline{\mathrm{~K}}_{\mathrm{n}}\right)=\left[\begin{array}{ccccc}
1 & -1 & -1 & \cdots & -1 \\
-1 & 1 & -1 & \cdots & -1 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \cdots & -1 \\
-1 & -1 & -1 & \cdots & 1
\end{array}\right]
$$

The characteristic polynomial is
$(-1)^{n}(\lambda+(n-2))(\lambda-2)^{n-1}=0$

The minimum dominating color eigen values are
$\operatorname{spec}_{\chi}^{D}\left(\bar{K}_{n}\right)=\left(\begin{array}{lr}2-n & 2 \\ 1 & n-1\end{array}\right)$

The minimum dominating color energy of
$\mathrm{E}_{\chi}^{\mathrm{D}}\left(\overline{\mathrm{K}}_{\mathrm{n}}\right)=|2-\mathrm{n}|(1)+|2|(\mathrm{n}-1)$
$=3 n-4$
$\mathrm{E}_{\chi}^{\mathrm{D}}\left(\overline{\mathrm{K}}_{\mathrm{n}}\right)=3 \mathrm{n}-4$

THEOREM 2.5 The complement of the complete graph $\overline{\mathrm{K}}_{\mathrm{i}}$ is a color-hyper energetic graph
Proof: We know that $E_{\chi}^{D}\left(\bar{K}_{n}\right) 3 n-4$,
For a color-hyper energetic graph $\mathrm{E}_{\mathrm{c}}(\mathrm{G})>2(\mathrm{n}-1)$
i.e., $3 n-4>2 n-2$

We will prove this theorem by contradiction method.

Suppose,
$3 n-4<2 n-2$
$\Rightarrow 3 \mathrm{n}-2 \mathrm{n}<4-2$
$\Rightarrow \mathrm{n}<2$
$\Rightarrow \mathrm{n}-2<0$
This is contradiction $\forall \mathrm{n} \geq 3$
Hence $\overline{\mathrm{K}}_{\mathrm{n}}$ is a color-hyper energetic graph.

Given below is a table of color-hyperenergy of some families of graphs.

| Name of the Graph | Total number of vertices | Hyper energy |
| :---: | :---: | :---: |
| Complete Bipartite | 6 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{K}_{3,3}\right)=11.4031 \geq 10$ |
|  | 8 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{K}_{4,4}\right)=15.5444 \geq 14$ |
|  | 10 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{K}_{5,5}\right)=19.6301 \geq 18$ |
|  | 12 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{K}_{6,6}\right)=23.6886 \geq 22$ |
| Double star | 6 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{B}_{3,3}\right)=10.7446 \geq 10$ |
|  | 8 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{B}_{4,4}\right)=15.2111 \geq 14$ |
|  | 10 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{B}_{5,5}\right)=19.544 \geq 18$ |
| Bipartite | 4 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{K}_{2,2}\right)=7.1231 \geq 6$ |
|  | 5 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{K}_{2,3}\right)=9.2915 \geq 8$ |
|  | 6 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{K}_{2,4}\right)=11.4031 \geq 10$ |
|  | 7 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{K}_{2,5}\right)=13.4833 \geq 12$ |
| Pan | 4 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{P}_{4}\right)=6.1537 \geq 6$ |
|  | 5 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{P}_{5}\right)=8.6252 \geq 8$ |
|  | 6 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{P}_{6}\right)=10.2574 \geq 10$ |
| Fan | 4 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{F}_{2,2}\right)=6.3382 \geq 6$ |
|  | 5 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{F}_{3,2}\right)=8.6402 \geq 8$ |
|  | 6 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{F}_{4,2}\right)=10.8471 \geq 10$ |
| Friendship | 5 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{F}_{2}\right)=8.1231 \geq 8$ |
|  | 7 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{F}_{3}\right)=12.2915 \geq 12$ |
|  | 9 | $\mathrm{E}_{\mathrm{c}}\left(\mathrm{F}_{4}\right)=16.1580 \geq 16$ |
| Complement of Complete | 3 | $\mathrm{E}_{\mathrm{c}}\left(\overline{\mathrm{K}}_{\mathrm{n}}\right)=5 \geq 4$ |
|  | 4 | $\mathrm{E}_{\mathrm{c}}\left(\overline{\mathrm{K}}_{\mathrm{n}}\right)=8 \geq 6$ |
|  | 5 | $\mathrm{E}_{\mathrm{c}}\left(\overline{\mathrm{K}}_{\mathrm{n}}\right)=11 \geq 8$ |

## III. CONCLUSION

Graph theory has undergone so much of advancement in the last few years. Now it has its application in almost every real world problem. Application of graph theory has made our life much easier when compared to before.
'Minimum Dominating Color Energy of Graph' is a new concept introduced by Professor P. Siva Kota Reddy et al. Motivated by this concept, in this project we have computed the minimum dominating color energy for Windmill graph, Double star graph, Bipartite graph and complement of the complete graph. We have also found out some hyper energetic families of graphs.

With the increasing importance of Coloring in graph theory the study can be extended for many other types of graphs and algorithm can also be developed for the same.

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