PBIB Designs Constructed Through Edges and Paths of Graphs

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Abstract -- *In this paper, partially balanced incomplete block (PBIB) designs with two and higher associate classes have been constructed by using edges of incidence vertices and closed paths in various graphs. Forthese formulations, we have considered pentagon graph, pappus graph and Peterson graph for construction offive, three and two associateclassPBIB designsalong with their association schemes respectively. Various kinds of efficiencies of these constructed PBIB designs are also computed for the purpose of comparisons.*

Keywords -- *Partially balanced incomplete block design, path, vertices, edge, pentagon graph, regular graph*

Mathematical Subject Code -- Primary 62K10, Secondary 62K99

I. Introduction

In available literature, Alwardi &Soner[1]constructed partially balanced incomplete block (PBIB) designs by arisingarelation between minimum dominating sets ofstrongly regulargraphs. Kumaret.al.[4]established a link between PBIB designs and graphs with minimum perfect dominating sets of Clebeschgraph.Later on,Shailaja .et.al. [5] revised the construction of partially balanced incomplete block design arising from minimum total dominating sets in a graph.In this direction,Sharma and Garg [3] constructed some higher associate class PBIB designs by using some sets of initial blocks.

Here, we have studied PBIB designs with two and three associate classes through chosen lines and Triangles of Graphs constructed by Garg and Syed [2]and modified these designs based on incidence relation of edges and closed paths in given graphs.

In this paper, we have constructed five associate classes PBIB design by using pentagon graph, three associate classes PBIB design by using pappus graph and two associate class PBIB design by using peterson graphalongwiththeir respectiveassociation schemes by taking particular parametric combination of PBIB designs.

II. Basic Definitions and Preliminary

Definition 2.1: Consider a set of symbols S= {1,2,3 . . . v} and an association scheme with m classes, we have a partially balanced incomplete block designs (PBIB) if v symbols are arranged in b blocks of size k(<v) such that

Every symbol occurs exactly in rblocks.

Every symbol occurs at most once in ablock.

Symbols say α and β are i^{th}associates if they occur together λ_i blocks and is being independent number.

Definition 2.2: G= (V, E) is a graph with vertex set V and edge set E, stands for a finite connected undirected graph, two vertices are adjacent if there is an edge between them otherwise non**-** adjacent.

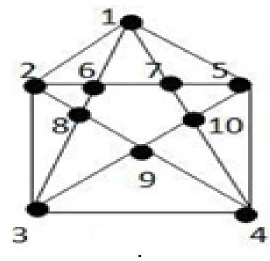
Definition 2.3*:* A Pentagon graph $G_1$is called a PBIB Graph if edges of incidence vertices of G_1form blocks of PBIB designs with five association scheme.

Definition 2.4: A Pappus graph G_2 is called a PBIB Graph if the chosen edges inside the graph G_2 forms blocks of PBIB designs with three association scheme.

Definition 2.5: A Peterson graph G_3 is called a PBIB Graph if finite closed paths of length '5' of all vertices in G_3 forms blocks of PBIB designs with two association scheme.

III. Construction Methodology

Theorem 3.1: The set of all incidence vertices of Pentagon graph G_1 form the blocks of a PBIB design with five associate class association scheme.



G_1: Pentagon Graph

Proof: By considering the vertices of G_1 as treatments and edges of incidence vertices of the graph G_1 as the set of blocks of a PBIB design are

(1,2,6,7,5) (2,1,6,8,3) (3,2,4,8,9) (4,3,5,9,10) (5,1,4,7,10)

(6,2,7,1,8) (7,1,10,5,6) (8,2,3,6,9) (9,8,3,4,10) (10,4,5,7,9)

Weobservethat in edges of incidence vertices,each vertex oftheG$_1$occurexactly in rsets andverifytheconditionsvr=bk, asv =10, b=10, r=5, k=5.
Byconsideringthese sets of incidence verticesasblocks,vertices as treatments, number of vertices in an edge as block size, then the given graph is describe aPBIBdesignswithfiveassociateclassassociationscheme. This association scheme is defined asunder:

Let us considera particular treatment say 'θ', then treatments which occur three times with treatment 'θ' in three different blocks are 1st associates of 'θ', treatments which occur four times with treatment 'θ' in four different blocks are 2nd associates of 'θ', treatments which occur two times with treatment 'θ' in two different blocks are3rd associates of 'θ' , treatments which occur only once with treatment 'θ' in two different blocks are 4th associates of 'θ' and treatment not commonly occurs with 'θ' is called 5th associate of 'θ'.Total existing pairs of a particular treatment 'θ' are $n_1 = 2$, $n_2 = 2$, $n_3 = 2$, $n_4 = 2$ and $n_5 = 1$.

Following table expressed the five associates of all treatments:

Vertex	1st Associates	2nd Associates	3rd Associates	4th Associates	5th Associates
1	2,5	6,7	8,10	3,4	9
2	1,3	6,8	7,9	5,4	10
3	2,4	8,9	6,10	1,5	7
4	3,5	9,10	8,7	2,1	6
5	1,4	10,7	6,9	2,3	8
6	7,8	1,2	5,3	9,10	4
7	6,10	1,5	2,4	8,9	3
8	6,9	2,3	1,4	7,10	5
9	8,10	3,4	2,5	6,7	1
10	9,7	4,5	3,1	6,8	2

Theparametersoffirstkindarev=10,b=10,r=5,k=5,λ_1=3,λ_2=4, λ_3=2, λ_4=1, λ_5=0 and parametersofsecondkind are n_1=2,n_2=2, n_3=2, n_4=2 and n_5=1.

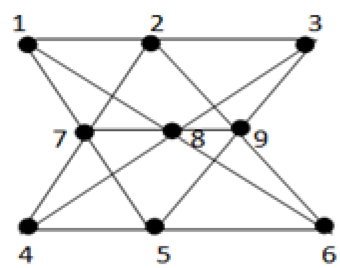
P-matrices of the association scheme are

$$P_1 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} P_2 = \begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} P_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} P_4 = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \quad P_5 = \begin{bmatrix} 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Various kinds of efficiencies and overall efficiency factor are

E_1=89.03%, E_2=94.97%, %, E_3=83.36%, %, E_4=78.75%, %, E_5= 75.60% and E=84. 79%.

Theorem 3.2: The set of all collinear vertices in pappus graph G_2 and three possible triangles forms blocks of PBIB designs with three associate class association scheme.



G_2: Pappus Graph

Proof: Firstly, sets are constructed by considering the combinations of collinear points one by one. Secondly, construct triangles by considering incidence vertices of one segment with incidence vertex of next segment.
For instance,
In 1st segment: vertices are 1,2 and 3.Here, V_1 and V_2 are incidence by an edge and both incidence with V_7 and make a triangle. In 2nd segment: vertices are 7,8 and 9. Here, V_7 is already utilized and V_8 and V_9 are incidence by an edge and both incidence with V_6 of next segment and make a triangle.
Lastly, in 3rd segment: vertices are 4,5 and 6. Here, V_4 and V_5 are incidence by an edge and both incidence with balanced vertex V_3 of first segment and make a triangle.
By taking points of the pappus graph G_2 as the treatments and blocks of the designs are constructed from above procedure in a pappus graph G_2.The blocks of the graph G_2 are given as

(1,2,3) (1,7,5) (1,8,6) **(1,2,7)** (4,5,6) (2,7,4)
(2,9,6) **(4,5,3)** (7,8,9) (3,8,4) (3,9,5) **(8,9,6)**

We observe each treatment occurs exactly in r sets and also verify the conditions vr=bk, as v=9, b=12, r=4, k=3. By considering all possible collinear points and triangles as blocks of a PBIB designs, the resultant is a PBIB design with three associate class association scheme is defined as below.
Let us consider a particular treatment say 'θ', then, treatment pair say 'θ' and 'φ' occur together twice in blocks are 1st associates to each other, treatment pair say 'θ' and 'δ' occurs once in blocks is2nd associates to each other and treatments pair says 'θ' and 'ε' do not occur in any block are 3rd associates to each other. Thetotal existing pairs of a particular treatment 'θ' are n_1 = 2, n_2 = 4 and n_3 = 2.

The following table expressed three associates of all treatments:

Symbols	*Ist associates*	*2nd associates*	*3rd associates*
1.	2,7	3,5,6,8	4,9
2	1,7	3,4,6,9	5,8
3.	4,5	1,2,8,9	6,7
4.	3,5	2,6,7,8	1,9

5.	3,4	1,6,7,9	2,8
6.	8,9	1,2,4,5	3,7
7.	1,2	4,5,8,9	3,6
8.	6,9	1,3,4,7	2,5
9.	6,8	2,3,5,7	1,4

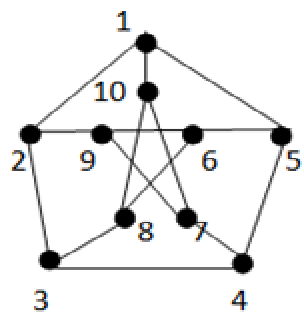
Hence the graph G_2 is a PBIB-graph with the parameters of first kind are v=9, b=12, r=4, k=3, λ_1 =2, λ_2=1, λ_3 =0 and parameters of second kind are n_1=2, n_2=4, n_3=2 .

P-matrices of the association scheme are

$$P_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 2 \\ 0 & 2 & 0 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad P_3 = \begin{bmatrix} 0 & 2 & 0 \\ 2 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Various kinds of efficiencies and overall efficiency factor are E_1=81.81%, E_2=69.23%, E_3=64.28%, E=70.58%.

Theorem 3.3: The set of finite closed paths of length '5' of all vertices in a Petersen graph G_3 forms blocks of PBIB designs with two association scheme.



G_3: Peterson Graph

Proof: Firstly, by choosing an vertex of the graph G_5 and draw a closed path of length '5' with internal and external edges under a condition that a closed path contain exactly 3 external and 2 internal vertices. Draw all possible closed paths for each external vertex of Petersen Graph one by one. Secondly, draw a closed path by using internal vertices only.

By considering every closed path treated as a block of PBIB design and following are the possible blocks.

(1,10,8,3,2) (1,10,7,4,5) (2,1,5,6,9) (2,9,7,4,3) (3,8,6,5,4) **(6,7,8,9,10)**

Results to be noted that all vertices in drawing closed paths of given graph occurs equally and satisfied the sufficient condition vr=bkfor v=10, b=6, r=3, k=5.

By considering vertices as treatments, sets of all closed paths as blocks, length of closed path as block size, thenthe graph G_3 be formulated a PBIB designs with two associate class association scheme defined as below:

Treatment pair says u and v together occur either λ_1 times or λ_2 times.Those treatment pairs which occur twice in blocks are first associates andremainingtreatment pairs occur only once are called 2^{nd}associatesofeach other. The total existing pairs of a particular treatment 'θ' are n_1 = 3and n_2 = 6.

The following table shows two associates for all treatments with two associate classes:

Symbols	*I^{st} associates*	*2^{nd} associates*
1.	2,5,10	3,4,6,7,8,9
2.	1,3,9	4,5,6,7,8,10
3.	2,4,8	1,5,6,7,9,10
4.	3,5,7	1,2,6,8,9,10

5.	1,4,6	2,3,7,8,9,10
6.	5,8,9	1,2,3,4,7,10
7	4,9,10	1,2,3,5,6,8
8	3,6,10	1,2,4,5,7,9
9	2,6,7	1,3,4,5,8,10
10	1,7,8	2,3,4,5,6,9

Thefirst kind parametric values arev=10,b=6,r=3,k=5,λ_1=2,λ_2=1andsecondkindparameters values are n_1=3,n_2=6.
P-matrices of the association scheme are

$$P_1 = \begin{bmatrix} 0 & 2 \\ 2 & 4 \end{bmatrix} \qquad P_2 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

Various kind of efficiencies and overall efficiency factor are E_1=92.30%, E_2=85.71% and E=87. 80%.

V. Conclusion

In this paper, we have formulated three configurations through edges, incidence and closed path of graphs to construct PBIB designs having two and higher associate classes PBIB designs. Various kind of efficiencies as well as overall efficiency factors of these designs are at significant levels for the purpose of comparison of existing designs with same parameters as listed in the tables of Clatworthy [6].

Construction methodologies of these designs are easy to understand because it based on the concept graph theory and generated a link of graphs with PBIB designs, which is not widely used in available literature.

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