A Two-Stage Approach and Algorithm for Single, Bi and Multi-Objective Integer Linear Programming Problems

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Abstract: *The objective of this paper is to present a new exact approach for solving Single, Bi, and Multi-Objective Integer Linear Programming. The new approach employing two of the existing exact algorithms in the literature, including the approximation algorithms, interactive algorithms, balanced box and e-constraint methods, in two stages. A computationally study shows that the new approach has four desirable characteristics. (1) It solves less single-objective integer linear programming. (2) It solves less bi-objective integer linear programming. (3) Its solution time is significantly smaller. (4) It is competitive with two-stage algorithms proposed by Sylva, J. & Crema, A; in 2004.*

Keywords: *two-stage approach, Balanced Box Method, E-Constraint Method, Single, Bi and Multi-Objective Integer Linear Programming, approximation algorithms and interactive algorithms.*

1. INTRODUCTION

Single, Bi, and Multi-Objective Integer Programs (MOIPs) have many application areas in real life, such as facility location problems, scheduling problems, network design problems, routing problems, capital budgeting problems, and workforce planning problems. Since the Decision Maker (DM) has to deal with many conflicting criteria, MOIPs usually do not have a unique solution and are difficult to solve. Several approaches have been developed to generate all nondominated points for MOIPs (Ozlen and Azizoglu, 2009; Lokman and Koksalan, 2013; Kırlık and Sayın, 2014; Dachert and Klamroth, 2015). Those methods work in a similar way and partition the solution space into a set of regions using bounds on the objectives. They show that there exists a linear bound on the number of sub-models to be solved for the three-criteria case. Although the recently developed algorithms work efficiently for medium-sized problems, generating all nondominated points is not practical for many problems. The number of nondominated points increases substantially with the problem size (Ehrgott and Gandibleux, 2000) and even if all those points are generated, the difficulty of comparing and choosing among a large number of points remains. An integer linear programming can be formulated as many problems in different fields such as scheduling, transportation, and production planning. However, these problems often involve multiple conflicting objectives in which there exists no feasible solution that simultaneously optimizes all objectives. Consequently, in practice, decision makers want to understand the trade of between the objectives for these problems before choosing a suitable solution. Thus, generating many or all efficient solutions, i.e., solutions in which it is impossible to improve the value of one objective without a deterioration in the value of at least one others objective, is the primary goal in Multi-Objective Integer Linear Programming this work focuses on developing an exact algorithm for Multi-Objective Integer Linear Programming (MOILPs). The main contribution of our research is efficiently combining two of the fastest algorithms, including the Balanced Box Method (BBM) developed by Boland, N., *et al.* (2015), and the e-constraint method developed by Chankong and Haimes (1983), to take the main advantage of both of these

Balance Box Method (BBM) is a recently developed and extend algorithms can be viewed as an extension of the box algorithms Boland, N., *et al.* (2015), have numerically shown that BBM can compute the nondominated frontier, i.e., the set of point in the criterion space corresponding to the efficient solutions, faster than many (if not all) of the existing methods such as the E-Constraint Method, the augmented weighted Tchebycheff method and the perpendicular search method (1986). It is worth mentioning that if $Y_N \neq \emptyset$ denotes the set of nondominated points of a MOILP, then BBM solves $3|Y_N|$ feasible solution of Bi-Objective Integer Linear Programming's (BOILPs). On the other hand, the e-constraint method is perhaps the most well-known algorithm for computing the (entire) nondominated frontier of MOILPs because of its simplicity and its long history. Boland, N., *et al.* (2015). It has shown that this algorithm does not outperform BBM in terms of solution time mainly because in BBM high-quality feasible solutions are naturally available to be initialized in Bi-Objective Integer Linear Programming's (BOILPs). Note that in the e-constraint method, this may be done making additional computing efforts, e.g., developing a heuristic approach. However, the main advantage of the e-constraint method is the fact that it solves only $2|Y_N|+1$ feasible solution BOILPs. The main goal of this paper is to develop a combined approach that (1) is good than BBM and E-Constraint Method in terms of solution time, (2) is better than BBM and E-Constraint method in terms of solution time exact, and (3) needs to solve less BOILPs than BBM and it similar to the E-Constraint method. To achieve these properties at the same time, the proposed approach starts by employing the BBM and at some point, it switches to the E-Constraint method. Of course, the switching time is critical because if we switch too early the solution time would probably not be much different from the E-constraint method. Similarly, if it occurs too late, solving less BOILPs than BBM will not probably be achieved, and the solution time would probably not be much different from BBM. We develop a simple but effective mechanism for the switching that causes up to around 30% and 45% improvements in the solution time in comparison to the solution times of the original BBM and E-Constraint Method. We have proposed modification of the well known BBM and E-Constraint scalarization technique for multi objective programming with the modification we are able to prove result on paper efficiency of optimal solution presented a simple but effective two-stage approach for solving MOILP this method combines BBM and E-Constraint method to remedy their faster proposed method.

II. DEFINITION, PRELIMINARIES AND PROBLEM FORMULATION

In this section, we extend and introduce some necessary notation and concept related to MOILPs to facilitate presentation and discussion of other sections. Let c^1 and c^2 be n-vectors. A be an m×n matrix, and b be an m-vector, a MOILP can be started as follows:

$$\max_{x \in X}\{z^1(x), z^2(x) \ldots\ldots z^n(x)\},$$

Where: $\{x \in \mathbb{Z}^n_+ : Ax \le b\}$ represent the feasible set in the decision space, and $z_1(x) := c^1x$ and $z_2(x) := c^2x$ are two linear objective functions. Note that $\mathbb{Z}^n_+ := \{s \in \mathbb{Z}^n : s \ge 0\}$. The image Y of X under vector-valued function $z = (z_1, z_2)$ represent the feasible set in the objective / criterion space, i.e., $Y := z(X) := \{y \in \mathbb{R}^2 : y = z(x) \text{ for some } x \in X\}$. It is assumed that X is bounded, and all coefficients / parameters are integer, i.e., $A \in \mathbb{Z}^{m\times n}$, $b \in \mathbb{Z}^m$. $c^i \in \mathbb{Z}^n$ for i = 1,2…n.

2.1 DEFINITION: A feasible solution $x \in X$ is called efficient or Pareto optimal, if there is no other $x' \in X$ such that $z_k(x') \le z_k(x)$ for k = 1,2…n and $z(x') \ne z(x)$. It x is efficient, then $z(x)$ is called nondominated point. The set of all efficient solution is denoted by X_E. The set of nondominated points is denoted by Y_N and referred to as the nondominated frontier. Overall, Bi-objective optimization is concerned with finding all nondominated points. Since by assumption X is bounded, the set of nondominated points of a MOILP, i.e., Y_N, is finite.

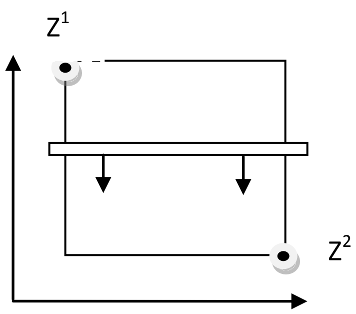

Fig. 1 SOILP is working of BBM and E-Constraint Method when (z^1, z^2) is empty

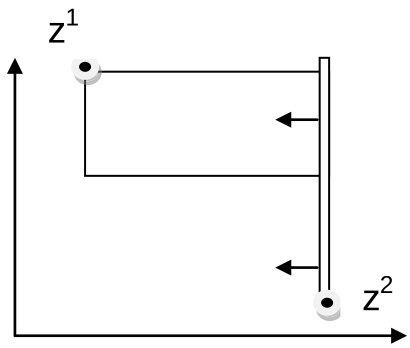

Fig. 2 (a) BOILP is working of BBM and E-Constraint Method when (z^1, z^2) is empty

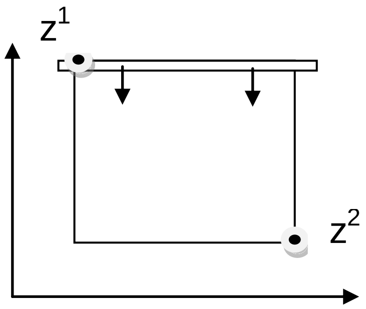

Fig. 2 (b)BOILP is working of BBM and E-Constraint Method when a rectangle is empty.

III. A TWO-STAGE APPROACH

On Our observation about Balanced Box Method in to show the main motivation of our research. From workings of BBM in figure.1, we observe that when a rectangle is empty, two BOILPs have to solved to prove that it is empty. Now suppose that whenever a rectangle is empty, we immediately switch to the E-constraint method as shown in figure.2, In this case, for each empty rectangle, only one BOILPs has to be solve. So, we conclude that if a given rectangle R (z^1, z^2) is expected to be empty, then by switching to the E-constraint method, avoid solving one redundant BOILPs. In focus of the above, our supposition and prove method solves a MOILPs in two stage.

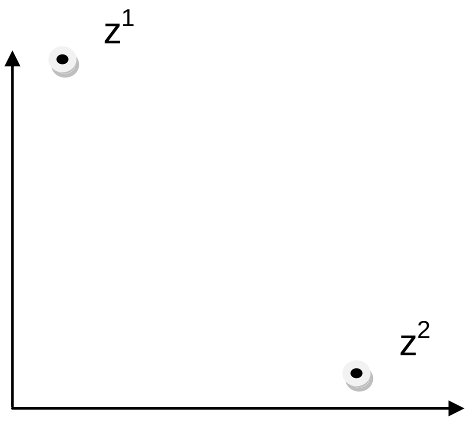

Fig. 3 MOILP is working of BBM and E-Constraint Method when (z^1, z^2) is empty

In the first stage, it employs BBM in order to generate some nondominated points from different parts of the nondominated frontier, and so speed split the search region into small rectangles. In the stage, the algorithms switches to the E-constraint method to conduct the searching in the not yet explored rectangles.

A. THE E-CONSTRAINT METHOD: The E-Constraint Method first appeared and is discussed in the details in Changkong and Haimes (1983). It is Based on a scalarization where one of the objective functions is minimized while the other objective function is bounded from above means of additional constraints,

$$(P_{E\text{-}k}) \min\{f_k(x) : f_i(x) \leq E_i , i \neq k , x \in X\},$$

Where E-k = $(E_1,\ldots, E_{k-1},\ldots E_p)^T \in \mathbb{R}^{p-1}$ and $k \in \{1,\ldots,p\}$. We denote the feasible set of the E-constraint problem $P_{E\text{-}k}$ by

$$X_k^{\epsilon} = \{x \in X: f_i(x) \leq E_i , i \neq k \}$$

Throughout this article, we assume that E-k is always chosen such that $P_{E\text{-}k}$ are feasible, i.e. $X_k^{\epsilon} \neq \emptyset$.

IV. SOLUTION DOMAIN

There are various methods for solving Multi-Objective Optimization Problems, such as weighting method, E-constraint method, evolutionary algorithms, etc. In this section, we first describe single objective optimization, Bi-objective optimization and multi-objective optimization and the principle of increase ϵ-constraint method.

A. SINGLE-OBJECTIVE OPTIMIZATION PROBLEM:

We consider the Single-Objective Optimization Problem in the form as bellow,

$$\text{Min } \{f_1(x)\}$$
$$\text{s.t. } x \in X$$

where $f_1(x)$ represent the feasible set in the decision space, x represent a decision variable vector, which belongs to the feasible solution region X. A solution x is non-dominated only if cannot be replaced by another solution which reduces one objective without increasing another. A non-dominated solution is said to be Pareto-optimal, and the image of corresponding objective value of non-dominated solutions is called the pareto front.

B. BI-OBJECTIVE OPTIMIZATION PROBLEM:

Similarly, we consider the Single-Objective Optimization Problem in the form as bellow,

$$\text{Min } \{f_1(x), f_2(x)\}$$
$$\text{s.t. } x \in X$$

where $f_1(x)$ and $f_2(x)$ represent the feasible set in the decision space, x represent a decision variable vector, which belongs to the feasible solution region X. A solution x is non-dominated only if cannot be replaced by another solution which reduces one are two objective function without increasing another. A non-dominated solution is said to be Pareto-optimal, and the image of corresponding objective value of non-dominated solutions is called the pareto front.

C. MULTI- OBJECTIVE OPTIMIZATION PROBLEM:

Similarly, we consider the Single-Objective Optimization Problem in the form as bellow,

$$\text{Min } \{f_1(x), f_2(x),\ldots\ldots\ldots f_n(x)\}$$
$$\text{s.t. } x \in X$$

where $f_1(x)$, $f_2(x),\ldots\ldots. f_n(x)$ represent the feasible set in the decision space, x represent a decision variable vector, which belongs to the feasible solution region X. A solution x is non-dominated only if cannot be replaced by another solution which reduces one, two are more than objective function without increasing another. A non-dominated solution is said to be Pareto-optimal, and the image of corresponding objective value of non-dominated solutions is called the pareto front.

D. THE INCREASE E-CONSTRAINT METHOD:

The basic idea of E-Constraint Method is to transform the Multi-Objective Problem into a series of Single-Objective and Bi-Objective Problem, which optimizes one and two objectives with restricting another by a bound E. The definition of the value of E in each iteration is one and two of critical factors for E-Constraint Method. For our problem, the Multi-Objective is considered to be a constraint and restricted by E. $[f_1^I, f_2^D]$, the range of E, is obtained by following ideal point and decline point.

- Ideal point: $f^I = (f_1^I, f_2^I, \dots\dots\dots f_n^I)$, where $f_1^I = \min\{f_1(x)\}$, $f_2^I = \min\{f_2(x)\}$ and $f_n^I = \min\{f_n(x)\}$, $x \in X$;
- Decline point: $f^D = (f_1^D, f_2^D, \dots\dots\dots f_n^D)$, where $f_1^D = \min\{f_1(x) : f_2(x) = f_2^I\}$, $f_2^D = \min$

$\{f_2(x): f_1(x) = f_1^I\}$ and $f_n^D = \min\{f_n(x): f_2(x) \text{ and } f_1(x) = f_n^I\}$

To avoid iterations that generate dominated solutions and accelerate the whole process, increase E-constraint method is proposed by mavrotas.

The value of E is also bounded by interval $[f_1^I, f_1^D]$. by varying the value of $\in$, a sequence of single and Bi-objective problems can be generated and solved.

The frame work of increase E-constraint method is shown in Algorithms.

Algorithm 1: The increase E-constraint method.

Step 1: Solution Representation

i = 1 (initialization and starting)

step 2: Compute the Ideal Point and Decline point;

step 3: $F = \{(f_1^D, f_2^I), (f_1^I, f_2^D) \text{ and } (f_n^D\, f_n^I)\}$;

step 4: **while** $i \le (f_2^D - f_2^I)$ do

step 5: solve problem and obtained an optimal solution x* and $(f_1(x^*), f_2(x^*), \dots\dots\dots f_n(x^*))$, calculate the bypass coefficient b;

step 6: $F = F \cup (f_1^*, f_2^*, \dots\dots\dots f_n^*)$;

step 7: i = i + b +1;

step 8: **end**

To obtain exact pareto front is time consuming for the increase E-constraint method.

Algorithm 2: The increase E-constraint method splitting and utilizing the SOILP and BOILP algorithms

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Data: A SOILP and BOILP problem $\mathcal{P}$ with objective function f_1 and f_2

Result: the nondominated solution to $\mathcal{P}$.

Let $t \in \{1,2\}$, and let F' be the unique value in $\{1,2\}/\{t\}$;

Let S_1 = (2,1) and S_2 = (1,2);

Let $S_1 = S_2$ = {} be empty set;

Let $f_1 = f_2 = \infty$;

For each formula t do

While $\mathcal{P}(< f_n)$ is feasible **do**

Let E = (E_1, E_2) be the solution for the range Ideal point and decline point problem $\mathcal{P}(< f_n)$**;**

add E_n to F_n;

Set $E_n = F_n$

add $F_{n'} < E_{n'}$ as a constraint to $\mathcal{P}(< f_n)$*;

end

end

return S_1 and S_2

Algorithm 3: Our Ideal point and Decline point algorithm, utilizing the MOILP algorithms

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Data: A SOILP and BOILP problem $\mathcal{P}$ with objective function f_1 and f_2 and E represent the number of formulas to use

Result: the nondominated solution to the MOILP calculate

S_1= min {$f_1(x)$ | x ∊ X} and S_2 {$f_2(x)$ | x ∊ X};

Let step = (S_1 - S_2)/2;

For each t ∊ {1,……,E} **do**

Let min = S_2 (t - 1) * step and max = S_2 (t + 1) * step;

Create $\mathcal{P}'$ as a copy of $\mathcal{P}$*;*

Add constraint $f_2(x)$ < max and $f_2(x)$ >min to $\mathcal{P}'$;

Solve $\mathcal{P}'$ in a new formula using the algorithms

End

Return: the union of all solution returned by all formulas**.**

CONCLUSION

This paper investigated and we present a simple but effective two-stage approach using algorithm for MOILP. This method combines BBM and the E-constraint method to remedy their weakness. Then increase E-constraint method are adopted to obtained the exact optimal pareto front for small size problems, the proposed method is faster, and solves less SOILP, BOILP and MOILP. Further, these basic concepts are introduced with algorithms and two stage approach MOILP convert to using for algorithms base approach in MOILP.

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