Accurate certified domination number of graphs

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Abstract: *A dominating set D of a graph $G = (V, E)$ is an accurate dominating set, if $V - D$ has no dominating set of cardinality $|D|$. An accurate dominating set D of G is an accurate certified dominating set, if D has either zero or atleast two neighbours in $V - D$. The accurate certified domination number $\gamma_{acer}(G)$ of G is the minimum cardinality of an accurate certified dominating set of G. In this paper, we initiate a study of this new parameter and obtain some results concerning this parameter.*

Keywords: *Domination, accurate domination number, accurate certified domination number.*

I. Introduction

All graphs considered here are finite, non-trivial, undirected with loops and multiple edges. For graph theoretic terminology we refer to Harary [2]. Let $G = (V, E)$ be a graph with $|V| = p$ and $|E| = q$. Let $\Delta(G)(\delta(G))$ denote the maximum(minimum) degree and $\lceil x \rceil(\lfloor x \rfloor)$ the least (greatest) integer greater(less) than or equal to x. The neighbourhood of a vertex u is the set $N(u)$ consisting of all vertices v which are adjacent with u. The closed neighbourhood is $N[u] = N(u) \cup \{u\}$. A set of vertices in G is independent, if no two of them are adjacent. The largest number of vertices in such a set is called the vertex independence number of G and is denoted by $\beta_o(G)$. A vertex cover is vertex set S such that each edge contains atleast one vertex in S and is denoted by $\alpha_o(G)$.

A bipartite graph $G = (V, E)$ with partition $V = \{V_1, V_2\}$ is said to be a complete bipartite graph if every vertex in V_1 is connected to every vertex of V_2 and is denoted by $K_{m,n}$. A wheel graph W_p is obtained from a cycle graph C_{p-1} by adding a new vertex. That new vertex is called a Hub which is connected to all the vertices of C_{p-1}. A star graph is a complete bipartite graph if a single vertex belong to one set and all the remaining vertices belong to the other set and is denoted by $K_{1,p-1}$. The helm graph is the graph obtained from an wheel graph by adjoining a pendant edge at each node of the cycle and is denoted by H_n where $2n + 1 = p$. The diamond graph is a planar undirected graph with 4 vertices and 5 edges. A friendship graph is the graph obtained by taking m copies of the cycle graph C_3 with a vertex in common and is denoted by F_p. The corona of two graphs G_1 and G_2 is the graph $G = G_1 \circ G_2$ formed from one copy of G_1 and $|V(G_1)|$ copies of G_2 where i^{th} vertex of G_1 is adjacent to every vertex in the i^{th} copy of G_2.

A set D of vertices in a graph $G = (V, E)$ is a dominating set of G, if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ of G is the minimum cardinality of a dominating set. For a comprehensive survey of domination in graphs see [5,6].

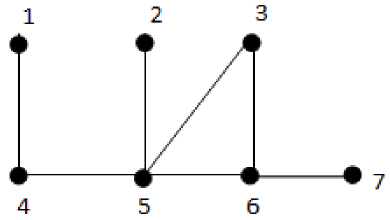
A dominating set D of $G = (V, E)$ is an **accurate dominating set**, if $V - D$ has no dominating set of cardinality $|D|$. The accurate domination number $\gamma_a(G)$ of G is the minimum cardinality of an accurate dominating set. This concept was introduced by Kulli and Kattimani [7]. A dominating set D of $G = (V, E)$ is a **certified dominating set**, if D has either zero or atleast two neighbours in $V - D$. The certified domination number $\gamma_{acer}(G)$ of G is the minimum cardinality of certified dominating set. This concept was introduced by M.Dettlaff, M. Lemanska, and J.Topp [10]. A dominating set D of a graph G is a **maximal dominating set** if $V - D$ is not a dominating set of G. The maximal domination number $\gamma_m(G)$ of G is the minimum cardinality of a maximal dominating set. [4]

II. Accurate Certified Domination Number

Definition 2.1

An accurate dominating set D of $G = (V, E)$ is an **accurate certified dominating set**, if D has either zero or atleast two neighbours in $V - D$. The accurate certified domination number $\gamma_{acer}(G)$ of G is the minimum cardinality of accurate certified dominating set.

Example 2.2 For the following graph, $V(G_1) = \{1,2,3,4,5,6,7\}$ whereas, $\{1,4,5,6\}$ satisfied accurate certified condition. Hence, $\gamma_{acer}(G_1) = 4$.



G_1
Figure 1

Accurate Certified Domination Number for some standard graphs.

Preposition 2.3

For any p vertices, $$\gamma_{acer}(P_p) = \begin{cases} \frac{p}{3} & p \equiv 0 (mod\ 3) \\ \frac{p+2}{3} & p \equiv 1 (mod\ 3) \\ \frac{p+1}{3} & p \equiv 2 (mod\ 3) \\ p & p = 2, 4 \end{cases}$$

Proof: Let p be the number of vertices in the path P_p and D and D' be the dominating set and accurate dominating set respectively.

Case 1. $p \equiv 0 (mod\ 3)$

Every dominating set is adjacent to exactly two vertices of $V - D$. That is D has atleast two neighbours in $V - D$ and $V - D$ has no dominating set of cardinality $|D|$. Therefore D satisfies accurate certified dominating set. $\therefore \gamma_{acer}(P_p) = \frac{p}{3}$.

Case 2. $p \equiv 1 (mod\ 3)$

We know, for $p \equiv 0 (mod\ 3), \gamma_{acer}(P_p) = \frac{p}{3}$.

For $p \equiv 1 (mod\ 3)$, we have one more extra vertex. So, $\gamma_{acer}(P_p)$ has $p \equiv 0 (mod\ 3)$ plus one more vertex. Therefore $\gamma_{acer}(P_p) = \frac{p-1}{3} + 1 = \frac{p+2}{3}$.

Case 3. $p \equiv 2 (mod\ 3)$

We know, for $p \equiv 0 (mod\ 3), \gamma_{acer}(P_p) = \frac{p}{3}$.

For $p \equiv 2 (mod\ 3)$, we have two more extra vertex. So, $\gamma_{acer}(P_p)$ has $p \equiv 0 (mod\ 3)$ plus one more vertex. Therefore $\gamma_{acer}(P_p) = \frac{p-2}{3} + 1 = \frac{p+1}{3}$.

Case 4. Suppose $p = 2,4$. We know, $\gamma_a(P_2) = 2$. Then clearly $\gamma_{acer}(P_2) = 2$. Also we know, $\gamma_a(P_4) = 3$. Then D' has one neighbour in $V - D'$. Which is contradiction to γ_{acer} - set. So we choose p vertices. Therefore $\gamma_{acer}(P_p) = p$.

Observation 2.4

(i) For any cycle of order $p \geq 3, \gamma_{acer}(C_p) = p$.

(ii) For any complete graph of order $p \geq 3$, $\gamma_{acer}\left(K_p\right) = \begin{cases} p & if\ p < 5 \\ \left\lfloor \frac{p}{2} \right\rfloor + 1 & if\ p \geq 5 \end{cases}$.

(iii) For any complete bipartite graph of order $p \geq 3$, $\gamma_{acer}\left(K_{m,n}\right) = \begin{cases} 4 & if\ m = n = 2 \\ m+1 & if\ m = n \\ m & if\ m < n \end{cases}$

(Where $m \geq 1, n \geq 2$ and $m + n = p$).

(iv) For any wheel of order $p \geq 4$, $\gamma_{acer}\left(W_p\right) = \begin{cases} p & if\ p = 4 \\ 1 & otherwise \end{cases}$

(v) For any helm graph of order $p \geq 7, \gamma_{acer}\left(H_n\right) = n$ (Where $2n + 1 = p$, $n \geq 3$).

(vi) For any star graph of order $p \geq 3, \gamma_{acer}\left(K_{1,p-1}\right) = 1$.

(vii) For Petersen graph, $\gamma_{acer}\left(G\right) = 6$.

(viii) For diamond graph of order $p = 4$, $\gamma_{acer}\left(G\right) = 4$.

(ix) For any friendship graph of order $p \geq 5, \gamma_{acer}\left(F_p\right) = 1$ (Where $2n + 1 = p$, $n \geq 2$).

Remark 2.5 An accurate certified dominating set of a graph G may or may not be a minimal dominating set.

Example 2.6

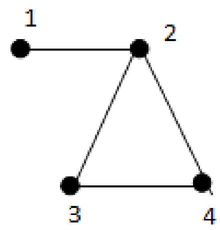


Figure 2

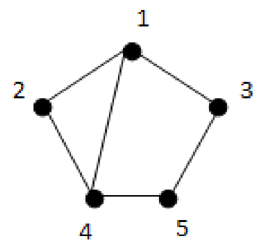


Figure 3

In Figure 2, $\gamma(G) = \{2\}$ and $\gamma_{acer}\left(G\right) = \{2\}$. Therefore $\gamma(G) = \gamma_{acer}\left(G\right)$.
In Figure 3, $\gamma(G) = \{1,5\}$ and $\gamma_{acer}\left(G\right) = \{1,2,3,4,5\}$. Therefore $\gamma(G) \neq \gamma_{acer}\left(G\right)$.

Theorem 2.7 Every support vertex of G belongs to accurate certified dominating set of G.

Proof: Let D be an accurate certified dominating set of G. Let $S = \{v_i, i = 1\ to\ n\}$ be the support vertex of $V(G) = \{V_j, j = 1\ to\ m\}$. If $\{v_i\}$ is a support vertex which is not in D, then all the pendant should be in D. If so, all the pendant has only one neighbour in $V - D$, which is contradiction to γ_{acer} set. Thus, every support vertex of G belongs to accurate certified dominating set of G.

Example 2.8

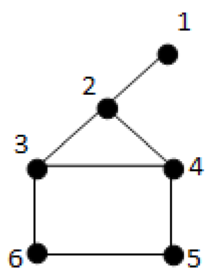


Figure 4

In Figure 4, $\gamma_{acer}\left(G\right) = \{1,2,3,4,5\}$. Thus, the support vertex $\{1\}$ belongs to accurate certified dominating set of G.

Theorem 2.9 For any graph G, $1 \leq \gamma_{acer}\left(G\right) \leq p$ and the bound is sharp.

Proof: If G is any non-trivial connected graph containing exactly one vertex of degree $\Delta(G) = p - 1$, then $\gamma_{acer}(G) = 1$, the lower bound holds. For the upper bound, Let D be an accurate dominating set of G. Then for some vertex $v \in D, N(v) = 1$. Therefore D is not a accurate certified dominating set of G. So we choose the accurate certified dominating set with $N(v) = 0$. Therefore $\gamma_{acer}(G) \leq p$. Hence $1 \leq \gamma_{acer}(G) \leq p$. For P_3, the lower bound is sharp. For C_5, the upper bound is sharp.

Theorem 2.10 For any graph G, $\gamma(G) \leq \gamma_a(G) \leq \gamma_{acer}(G)$ and $\gamma(G) \leq \gamma_{cer}(G) \leq \gamma_{acer}$(G)

Proof: Let G be a graph of order p. By theorem 2.9, $\gamma_{acer}(G) \leq p$ and by [7], $\gamma_a(G) \leq \left\lfloor \frac{p}{2} \right\rfloor + 1$. From the above, $\gamma_a(G) \leq \gamma_{acer}(G)$. By [7], every accurate dominating set of G is a dominating set of G. Hence $\gamma(G) \leq \gamma_a(G) \leq \gamma_{acer}(G)$. Also by [10], $\gamma_{cer}(G) \leq p$. Therefore $\gamma_{cer}(G) \leq \gamma_{acer}(G)$. By [10], any certified dominating is a dominating set. Hence $\gamma(G) \leq \gamma_{cer}(G) \leq \gamma_{acer}(G)$.

Theorem 2.11 For any graph G, $\left\lceil \frac{P}{1+\Delta} \right\rceil \leq \gamma_{acer}(G)$ and the bound is sharp.

Proof: Let D be a γ-set of G. Each vertex can dominate atmost itself and $\Delta(G)$ other vertices. Hence $\gamma(G) \geq \left\lceil \frac{p}{1+\Delta} \right\rceil$. By theorem 2.10, $\gamma(G) \leq \gamma_{acer}(G)$. Therefore $\left\lceil \frac{P}{1+\Delta} \right\rceil \leq \gamma_{acer}(G)$. For P_3, the bound is sharp.

Theorem 2.12 For any graph G, $\gamma_{acer}(G) \leq p - \gamma(G) + 1$.

Proof: Let D be a minimum dominating set of G and V be a vertex set of G. Then for any vertex $v \in D$,

$$\begin{aligned}\gamma_{acer}(G) &\leq (V - D) \cup \{v\} \\ &\leq p - \gamma(G) + 1.\end{aligned}$$

The sharpness is attained for C_3.

Theorem 2.13 For any graph G, $\gamma_{acer}(G) \leq \alpha_0(G) + 1$.

Proof: Let S be a vertex cover of G. We consider the following two cases.

Case 1: Suppose $|S| < \frac{p}{2}$. Then

$$\begin{aligned}\gamma_{acer}(G) &= |S| \\ &= \alpha_0(G) \\ &\leq \alpha_0(G) + 1\end{aligned}$$

Case 2: Suppose $|S| \geq \frac{p}{2}$. Then for any vertex $v \in V - S$,

$$\begin{aligned}\gamma_{acer}(G) &\leq S \cup \{v\} \\ &\leq \alpha_0 + 1.\end{aligned}$$

Corollary 2.14 For any graph G, $\gamma_{acer}(G) \leq p - \beta_0 + 1$.

Proof: By [3,5,6], $\alpha_0 + \beta_0 = p$.

By theorem 2.13, $\gamma_{acer}(G) \leq \alpha_o + 1$

$\leq p - \beta_0 + 1$.

Observation 2.15 The theorem 2.12, 2.13 and corollary 2.14 does not hold for

(i) $\gamma_{acer}(G) = p, \; p \geq 4$

(ii) $\gamma_{acer}(K_{2,2})$

Proof: Let S be a vertex cover of G and p be the number of vertices of G. We know, The number of vertices of a graph is equal to its minimum vertex cover number plus the size of maximum independent set. Of so, the vertex cover does not have total number of vertices of G. Which is contradiction to $\gamma_{acer}(G) \leq \alpha_0 + 1$.
Also $p - \gamma(G) + 1$ does not greater than or equal to $\gamma_{acer}(G) = p$. Hence proved.

Theorem 2.16 For any tree T with m cut vertices, $\gamma_{acer}(T) \leq m+1$. It is not true when $\gamma_{acer}(T_p) = p$.
Proof: Let $S = \{v_j, j = 1\ to\ m\}$ be the cut vertices of $V(G) = \{v_i, i = 1\ to\ n\}$ with $|S| = m$. Sometimes there is a vertex in S has one neighbour in $V - S$. Then for any end vertex $v \in T$,

$$\gamma_{acer}(G) \leq S \cup \{v\}$$
$$\leq m+1.$$

Corollary 2.17 For any tree T with m cut vertices and n end vertices, $\gamma_{acer}(T) \leq p - n + 1$.

Proof: By [5,6], $m + n = p$.
By Theorem 2.16, $\gamma_{acer}(T) \leq m + 1$

$$\leq p - n + 1.$$

Theorem 2.18 For any graph G, $\gamma_{acer}(G) \leq \gamma_m(G) + 1$. It is not true when $\gamma_{acer}(G) = p,\ p \geq 5$.

Proof: Let D be a γ_m-set of G. Then $V - D$ is not a dominating set of G.For some graphs D has a one neighbour in $V - D$. Then for any vertex $v \in V - D, D \cup \{v\}$ is an accurate certified dominating set of G.

$$\therefore \gamma_{acer}(G) \leq |D \cup \{v\}|$$
$$= \gamma_m(G) + 1.$$

For P_4 , the bound is sharp.

Theorem 2.19 For any graph G, $\gamma_{acer}(G) \leq \gamma(G) + p - \Delta(\bar{G})$.

Proof: Let v be a vertex of minimum degree that is $\delta(G) = \deg v$. By [4], $\gamma_m(G) \leq \gamma(G) + \delta(G)$. By [3], $\delta(G) + \Delta(\bar{G}) = p - 1$ and by theorem 2.18,

$$\gamma_{acer}(G) \leq \gamma_m(G) + 1$$
$$\leq \gamma(G) + \delta(G) + 1$$
$$\leq \gamma(G) + p - \Delta(\bar{G})$$

For C_5, the bound is sharp.

Theorem 2.20 For any connected graph G with p vertices, $\gamma_{acer}(G) + \Delta(G) \leq 2p - 1$.

Proof: Let G be a connected graph with p vertices. We know that $\Delta(G) \leq p - 1$ and by theorem 2.9 $\gamma_{acer}(G) \leq p$. Hence $\gamma_{acer}(G) + \Delta(G) \leq 2p - 1$. For K_4, the bound is sharp.

Theorem 2.21 If $G = H \circ K_1$, where H is any non-trivial connected graph then $\gamma_{acer}(G) = p$.

Proof: Let p be the number of vertices in $G = H \circ K_1$. Let l be the set of all pendant vertices in $G = H \circ K_1$ such that $|l| = \frac{p}{2}$. If $G = H \circ K_1$, then thereexist a minimal accurate certified dominating set D containing all pendant vertices and $V(H)$ of G.

Hence $\gamma_{acer}(G) = |V(H)| + |l|$

$$= \frac{p}{2} + \frac{p}{2}$$
$$= p.$$

Theorem 2.22 For the corona graph $C_m \circ P_n,\ n \geq 4,\ \gamma_{acer}(C_m \circ P_n) = m$.

Proof: Let $V(C_m) = \{v_i, i = 1\ to\ m\}$
The vertices of m^{th} copy corresponding to the path P_n is

$$V(C_m \circ P_n) = \{v_1, v_{11}, v_{12}, \ldots, v_{1n}, v_2, v_{21}, \ldots, v_{2n}, \ldots, v_m, v_{m1}, v_{m2}, \ldots, v_{mn}\}$$

Let D be a minimum accurate certified dominating set of G. We prove this result by induction on m.
Suppose $m = 3$, Then $D = \{v_i, i = 1\ to\ m\}$ dominate every vertices on the $C_3 \circ P_n$, $n \geq 4$. Also D is the accurate certified dominating set of $C_3 \circ P_n$. Thus, $\gamma_{acer}(C_3 \circ P_n) = 3,\ n \geq 4$.

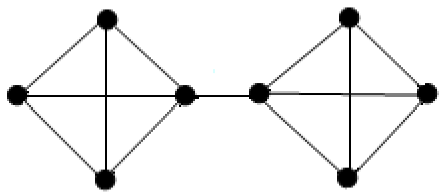
Let us assume this result is true for $m-1$. And, $\gamma_{acer}\,(C_{m-1}\circ P_n)=m-1\,,\ n\geq 4.$
Let us prove for m, Let $\{v_i, i=1\ to\ m\}$ be the vertices of C_m.
Since the result is true for $m-1$, $D=m-1$ Then for any vertex $v\in V-D, (m-1)\cup\{v\}$ is an accurate certified dominating set of G.
Thus $\gamma_{acer}\,(C_m\circ P_n)=|(m-1)\cup\{v\}|=m-1+1=m.$

Thus, $\gamma_{acer}\,(C_m\circ P_n)=m\,, n\geq 4.$

III. Accurate Certified Values for Some Graph Families

Definition 3.1

The p- barbell graph is the simple graph obtained by connecting two copies of a complete graph K_p by a bridge.



4-Barbell graph
Figure 5

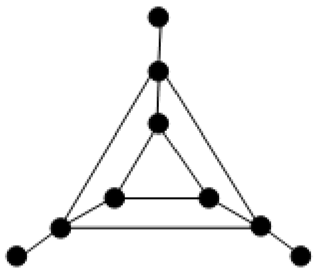
Theorem 3.1 For the barbell graph $p\geq 3$, $\gamma_{acer}\,(G)=p+1.$

Proof: The barbell graph has $2p$ vertices. Let V be the vertex set of first copy of K_p. Let U be the vertex set of second copy of K_p and $\{u_1, v_1\}$ be a bridge. Then $V\cup\{u_1\}$ is an accurate certified dominating set of G. Thus

$$\gamma_{acer}\,(G)=|V\cup\{u_1\}|$$
$$=p+1.$$

Definition 3.2

A web graph has defined as a prism graph $Y_{p+1,3}$ with the edges of the outer cycle removed and is denoted by W_p.



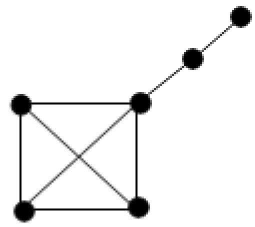
$\boldsymbol{W_3}$
Figure 6

Theorem 3.2 For a web graph W_p , $p\geq 3, \gamma(W_p)=\gamma_a(W_p)=\gamma_{cer}\,(W_p)=\gamma_{acer}\,(W_p)=p.$

Proof: Let W_p be a graph with $3p$ vertices. Let D be a dominating set of G. Then the support vertex are a minimal dominating set of W_p such that $\gamma(W_p)=p$. Since this dominating set has atleast two neighbours in $V-D$ and in $(V-D)$ there is no dominating set of cardinality p it is both certified and accurate dominating set. Also it is an accurate certified dominating set.

Definition 3.3

The lollipop graph is a special type of a graph consisting of a complete graph on m vertices and a path graph on n vertices connected with a bridge and is denoted by $K_m(P_n)$.



$\boldsymbol{K_4(P_2)}$
Figure 7

Theorem 3.3 For the lollipop graph $K_m(P_n), m \geq 3$,

$$\gamma_{acer}\left(K_m(P_n)\right) = \begin{cases} \frac{n}{3}+1 & n \equiv 0(mod\ 3) \\ \frac{n+2}{3} & n \equiv 1(mod\ 3) \\ \frac{n+4}{3} & n \equiv 2(mod\ 3) \end{cases}$$

Proof: Let u be the vertex with maximum degree in $K_m(P_n)$.
Case 1: $n \equiv 0(mod\ 3)$, By proposition 2.3, $\gamma_{acer}(P_n) = \frac{n}{3}$. Then $\gamma_{acer}(P_n) \cup \{u\}$ is an accurate certified dominating set of G. Thus $\gamma_{acer}(K_m(P_n)) = \frac{n}{3} + 1$.
Case 2: $n \equiv 1(mod\ 3)$, The vertex u dominate the vertices in K_m and one vertex in P_n. In $K_m(P_n)$, $\gamma_{acer}(P_n) = \frac{n-1}{3}$. Then $\gamma_{acer}(P_n) \cup \{u\}$ is an accurate certified dominating set of G. Thus

$$\gamma_{acer}(K_m(P_n)) = \frac{n-1}{3} + 1$$
$$= \frac{n+2}{3}.$$

Case 3: $n \equiv 2(mod\ 3)$, In $K_m(P_n)$, by preposition 2.3, $\gamma_{acer}(P_n) = \frac{n+1}{3}$. Then $\gamma_{acer}(P_n) \cup \{u\}$ is an accurate certified dominating set of G. Thus

$$\gamma_{acer}(K_m(P_n)) = \frac{n+1}{3} + 1$$
$$= \frac{n+4}{3}.$$

Theorem 3.4 For the lollipop graph $K_m(P_2), m \geq 3, \gamma_{acer}(K_m(P_2)) = \left\lfloor \frac{m}{2} \right\rfloor + 2$.

Proof: Let V be the vertex set of K_m. Let U be the vertex set of P_2 and D be an accurate certified dominating set of $K_m(P_2)$. Thus

$$|D| = \left\lfloor \frac{m}{2} \right\rfloor + |U(P_2)|$$
$$= \left\lfloor \frac{m}{2} \right\rfloor + 2.$$

IV. Nordhaus-Gaddum Type Results

In 1956 the original paper [1] by Nordhaus and Gaddum appeared. In it they gave sharp bounds on the sum and product of the chromatic numbers of a graph and its complement. Since then such results have been given for several parameters.

Theorem 4.1 If graphs G and $\bar{G}$ have no isolated vertices, then

$$\gamma_{acer}(G) + \gamma_{acer}(\bar{G}) \leq 2p$$
$$\gamma_{acer}(G)\gamma_{acer}(\bar{G}) \leq p^2$$

Furthermore the bounds are attained if $G = C_4$.

Proof: Let G and $\bar{G}$ have no isolated vertices and p be the number of vertices of G and $\bar{G}$. By theorem 2.9, $\gamma_{acer}(G) \leq p$. Since $\bar{G}$ has no isolated vertices, $\gamma_{acer}(\bar{G}) \leq p$. Thus both upper bound holds. Clearly, if $G = C_4$, then $\gamma_{acer}(G) = 4$ and $\gamma_{acer}(\bar{G}) = 4$. Therefore both bounds are attained.

Theorem 4.2 Nordhaus-Gaddum result for p- barbell graph

$$\gamma_{acer}(G) + \gamma_{acer}(\bar{G}) = 2(p+1)$$
$$\gamma_{acer}(G)\gamma_{acer}(\bar{G}) = (p+1)^2$$

Proof: Let G and $\bar{G}$ be the p-barbell graph and its complement respectively. The p-barbell graph has $2p$ vertices. By theorem 3.1, $\gamma_{acer}(G) = p + 1$. Let D be an accurate dominating set of $\bar{G}$. We know by [7], $|D| = \left\lfloor \frac{p}{2} \right\rfloor + 1$. Also D has atleast two neighbours in $V - D$. Therefore D is also an accurate certified dominating set of $\bar{G}$. But here we have $2p$ vertices. Therefore $\gamma_{acer}(\bar{G}) = p + 1$. Hence $\gamma_{acer}(G) + \gamma_{acer}(\bar{G}) = 2(p+1)$ and $\gamma_{acer}(G)\gamma_{acer}(\bar{G}) = (p+1)^2$.

Theorem 4.3 Nordhaus-Gaddum result for web graph

$$\gamma_{acer}(G) + \gamma_{acer}(\bar{G}) \leq \frac{5p+2}{2}$$
$$\gamma_{acer}(G)\gamma_{acer}(\bar{G}) \leq \frac{3p^2+2p}{2}$$

Proof: Let G and $\bar{G}$ be the web graph and its complement respectively. The web graph has $3p$ vertices. By theorem 3.2, $\gamma_{acer}(G) = p$. Let D be an accurate dominating set of $\bar{G}$. We know by [7], $|D| = \left\lfloor \frac{p}{2} \right\rfloor + 1$. Also D has atleast two neighbours in $V - D$. Therefore D is also an accurate certified dominating set of $\bar{G}$. But here we have $3p$ vertices. Therefore $\gamma_{acer}(\bar{G}) = \left\lfloor \frac{3p}{2} \right\rfloor + 1 \leq \frac{3p+2}{2}$. Hence $\gamma_{acer}(G) + \gamma_{acer}(\bar{G}) \leq \frac{5p+2}{2}$ and $\gamma_{acer}(G)\gamma_{acer}(\bar{G}) \leq \frac{3p^2+2p}{2}$.

Theorem 4.4 Nordhaus-Gaddum result for lollipop graph $K_m(P_2), m \geq 3$

$$\gamma_{acer}(G) + \gamma_{acer}(\bar{G}) \leq \frac{m+8}{2}$$
$$\gamma_{acer}(G)\gamma_{acer}(\bar{G}) \leq m + 4$$

Proof: Let G and $\bar{G}$ be the lollipop graph and its complement respectively. The lollipop graph has $m + n = p$ vertices. By theorem 3.4, $\gamma_{acer}(G) = \left\lfloor \frac{m}{2} \right\rfloor + 2 \leq \frac{m}{2} + 2$. Let D be a dominating set of $\bar{G}$. In $\bar{G}$, we have exactly one vertex of degree $\Delta(G) = p - 2$. Therefore $|D| = 2$ and $V - D$ has no dominating set of cardinality $|D|$. Also D has atleast two neighbours in $V - D$. Therefore $\gamma_{acer}(\bar{G}) = 2$. Hence $\gamma_{acer}(G) + \gamma_{acer}(\bar{G}) \leq \frac{m+8}{2}$ and $\gamma_{acer}(G)\gamma_{acer}(\bar{G}) \leq m + 4$.

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