Accurate certified domination number of graphs

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Abstract: A dominating set \( D \) of a graph \( G = (V, E) \) is an accurate dominating set, if \( V - D \) has no dominating set of cardinality \( |D| \). An accurate dominating set \( D \) of \( G \) is an accurate certified dominating set, if \( D \) has either zero or atleast two neighbours in \( V - D \). The accurate certified domination number \( \gamma_{accd}(G) \) of \( G \) is the minimum cardinality of an accurate certified dominating set of \( G \). In this paper, we initiate a study of this new parameter and obtain some results concerning this parameter.

Keywords: Domination, accurate domination number, accurate certified domination number.

I. Introduction

All graphs considered here are finite, non-trivial, undirected with loops and multiple edges. For graph theoretic terminology we refer to Harary [2]. Let \( G = (V, E) \) be a graph with \( |V| = p \) and \( |E| = q \). Let \( \Delta(G)(\delta(G)) \) denote the maximum(minimum) degree and \( \lfloor x \rfloor(\lceil x \rceil) \) the least (greatest) integer greater(less) than or equal to \( x \). The neighbourhood of a vertex \( u \) is the set \( N(u) \) consisting of all vertices \( v \) which are adjacent with \( u \). The closed neighbourhood is \( N[u] = N(u) \cup \{u\} \). A set of vertices in \( G \) is independent, if no two of them are adjacent. The largest number of vertices in such a set is called the vertex independence number of \( G \) and is denoted by \( \beta(G) \). A vertex cover is vertex set \( S \) such that each edge contains at least one vertex in \( S \) and is denoted by \( \alpha_c(G) \).

A bipartite graph \( G = (V, E) \) with partition \( V = \{V_1, V_2\} \) is said to be a complete bipartite graph if every vertex in \( V_1 \) is connected to every vertex of \( V_2 \) and is denoted by \( K_{m,n} \). A wheel graph \( W_p \) is obtained from a cycle graph \( C_{p-1} \) by adding a new vertex. That new vertex is called a Hub which is connected to all the vertices of \( C_{p-1} \). A star graph is a complete bipartite graph if a single vertex belong to one set and all the remaining vertices belong to the other set and is denoted by \( K_{1,p-1} \). The helm graph is the graph obtained from an wheel graph by adjoining a pendant edge at each node of the cycle and is denoted by \( H_n \) where \( 2n + 1 = p \). The diamond graph is a planar undirected graph with 4 vertices and 5 edges. A friendship graph is the graph obtained by taking \( m \) copies of the cycle graph \( C_3 \) with a vertex in common and is denoted by \( F_m \). The corona of two graphs \( G_1 \) and \( G_2 \) is the graph \( G = G_1 \odot G_2 \) formed from one copy of \( G_1 \) and \( |V(G_1)| \) copies of \( G_2 \) where \( i^{th} \) vertex of \( G_1 \) is adjacent to every vertex in the \( i^{th} \) copy of \( G_2 \).

A set \( D \) of vertices in a graph \( G = (V, E) \) is a dominating set of \( G \), if every vertex in \( V - D \) is adjacent to some vertex in \( D \). The domination number \( \gamma(G) \) of \( G \) is the minimum cardinality of a dominating set. For a comprehensive survey of domination in graphs see [5,6].

A dominating set \( D \) of \( G = (V, E) \) is an **accurate dominating set**, if \( V - D \) has no dominating set of cardinality \( |D| \). The accurate domination number \( \gamma_a(G) \) of \( G \) is the minimum cardinality of an accurate dominating set. This concept was introduced by Kulli and Kattimani [7]. A dominating set \( D \) of \( G = (V, E) \) is a **certified dominating set**, if \( D \) has either zero or atleast two neighbours in \( V - D \). The certified domination number \( \gamma_c(G) \) of \( G \) is the minimum cardinality of certified dominating set. This concept was introduced by M.Dettlaff, M. Lemanska, and J.Topp [10]. A dominating set \( D \) of a graph \( G \) is a **maximal dominating set** if \( V - D \) is not a dominating set of \( G \). The maximal domination number \( \gamma_m(G) \) of \( G \) is the minimum cardinality of a maximal dominating set. [4]
II. Accurate Certified Domination Number

Definition 2.1
An accurate dominating set \( D \) of \( G = (V, E) \) is an accurate certified dominating set, if \( D \) has either zero or at least two neighbours in \( V - D \). The accurate certified domination number \( \gamma_{acer}(G) \) of \( G \) is the minimum cardinality of accurate certified dominating set.

Example 2.2 For the following graph, \( V(G_1) = \{1,2,3,4,5,6,7\} \) whereas, \( \{1,4,5,6\} \) satisfied accurate certified condition. Hence, \( \gamma_{acer}(G_1) = 4 \).

Accurate Certified Domination Number for some standard graphs.

Preposition 2.3
For any \( p \) vertices, \( \gamma_{acer}(P_p) = \begin{cases} \frac{p}{3} & p \equiv 0 \text{ (mod 3)} \\ \frac{p+2}{3} & p \equiv 1 \text{ (mod 3)} \\ \frac{p+1}{3} & p \equiv 2 \text{ (mod 3)} \end{cases} \)

\( \frac{p}{3} \quad p = 2, 4 \)

Proof: Let \( p \) be the number of vertices in the path \( P_p \) and \( D \) and \( D' \) be the dominating set and accurate dominating set respectively.

Case 1. \( p \equiv 0 \text{ (mod 3)} \)
Every dominating set is adjacent to exactly two vertices of \( V - D \). That is \( D \) has at least two neighbours in \( V - D \) and \( V - D \) has no dominating set of cardinality \( |D| \). Therefore \( D \) satisfies accurate certified dominating set.
\( \therefore \gamma_{acer}(P_p) = \frac{p}{3} \).

Case 2. \( p \equiv 1 \text{ (mod 3)} \)
We know, for \( p \equiv 0 \text{ (mod 3)} \), \( \gamma_{acer}(P_p) = \frac{p}{3} \).
For \( p \equiv 1 \text{ (mod 3)} \), we have one more extra vertex. So, \( \gamma_{acer}(P_p) \) has \( p \equiv 0 \text{ (mod 3)} \) plus one more vertex. Therefore \( \gamma_{acer}(P_p) = \frac{p-1}{3} + 1 = \frac{p+2}{3} \).

Case 3. \( p \equiv 2 \text{ (mod 3)} \)
We know, for \( p \equiv 0 \text{ (mod 3)} \), \( \gamma_{acer}(P_p) = \frac{p}{3} \).
For \( p \equiv 2 \text{ (mod 3)} \), we have two more extra vertex. So, \( \gamma_{acer}(P_p) \) has \( p \equiv 0 \text{ (mod 3)} \) plus one more vertex. Therefore \( \gamma_{acer}(P_p) = \frac{p-2}{3} + 1 = \frac{p+1}{3} \).

Case 4. Suppose \( p = 2, 4 \). We know, \( \gamma_{acer}(P_2) = 2 \). Then clearly \( \gamma_{acer}(P_2) = 2 \). Also we know, \( \gamma_{acer}(P_4) = 3 \). Then \( D' \) has one neighbour in \( V - D \). Which is contradiction to \( \gamma_{acer} \) - set. So we choose \( p \) vertices. Therefore \( \gamma_{acer}(P_p) = p \).

Observation 2.4
(i) For any cycle of order \( p \geq 3 \), \( \gamma_{acer}(C_p) = p \).
For any complete graph of order \( p \geq 3 \), \( \gamma_{\text{acer}}(K_p) = \begin{cases} p & \text{if } p < 5 \\ \left\lfloor \frac{p}{2} \right\rfloor + 1 & \text{if } p \geq 5 \end{cases} \).

For any complete bipartite graph of order \( p \geq 3 \), \( \gamma_{\text{acer}}(K_{m,n}) = \begin{cases} 4 & \text{if } m = n = 2 \\ m + 1 & \text{if } m = n \\ m & \text{if } m < n \end{cases} \) (Where \( m \geq 1, n \geq 2 \) and \( m + n = p \)).

For any wheel of order \( p \geq 4 \), \( \gamma_{\text{acer}}(W_p) = \begin{cases} p & \text{if } p = 4 \\ 1 & \text{otherwise} \end{cases} \).

For any helm graph of order \( p \geq 7 \), \( \gamma_{\text{acer}}(H_n) = n \) (Where \( 2n + 1 = p, n \geq 3 \)).

For any star graph of order \( p \geq 4 \), \( \gamma_{\text{acer}}(W_p) = \begin{cases} p & \text{if } p = 4 \\ 1 & \text{otherwise} \end{cases} \).

For any友谊 graph of order \( p \geq 5 \), \( \gamma_{\text{acer}}(F_p) = 1 \) (Where \( 2n + 1 = p, n \geq 2 \)).

Remark 2.5 An accurate certified dominating set of a graph \( G \) may or may not be a minimal dominating set.

Example 2.6

In Figure 2, \( \gamma(G) = \{2\} \) and \( \gamma_{\text{acer}}(G) = \{2\} \). Therefore \( \gamma(G) = \gamma_{\text{acer}}(G) \).

In Figure 3, \( \gamma(G) = \{1, 5\} \) and \( \gamma_{\text{acer}}(G) = \{1, 2, 3, 4, 5\} \). Therefore \( \gamma(G) \neq \gamma_{\text{acer}}(G) \).

Theorem 2.7 Every support vertex of \( G \) belongs to accurate certified dominating set of \( G \).

Proof: Let \( D \) be an accurate certified dominating set of \( G \). Let \( S = \{v_i, i = 1 \text{ to } n\} \) be the support vertex of \( V(G) = \{V_j, j = 1 \text{ to } m\} \). If \( \{v_i\} \) is a support vertex which is not in \( D \), then all the pendant should be in \( D \). If so, all the pendant has only one neighbour in \( V - D \), which is contradiction to \( \gamma_{\text{acer}} \) set. Thus, every support vertex of \( G \) belongs to accurate certified dominating set of \( G \).

Example 2.8

In Figure 4, \( \gamma_{\text{acer}}(G) = \{1, 2, 3, 4, 5\} \). Thus, the support vertex \( \{1\} \) belongs to accurate certified dominating set of \( G \).

Theorem 2.9 For any graph \( G \), \( 1 \leq \gamma_{\text{acer}}(G) \leq p \) and the bound is sharp.
Proof: If $G$ is any non-trivial connected graph containing exactly one vertex of degree $\Delta(G) = p - 1$, then $\gamma_{acer}(G) = 1$, the lower bound holds. For the upper bound, Let $D$ be an accurate dominating set of $G$. Then for some vertex $v \in D, N(v) = 1$. Therefore $D$ is not a accurate certified dominating set of $G$. So we choose the accurate certified dominating set with $N(v) = 0$. Therefore $\gamma_{acer}(G) \leq p$. Hence $1 \leq \gamma_{acer}(G) \leq p$. For $P_3$, the lower bound is sharp. For $C_3$, the upper bound is sharp.

Theorem 2.10 For any graph $G$, $\gamma(G) \leq \gamma_a(G) \leq \gamma_{acer}(G)$ and $\gamma(G) \leq \gamma_{cer}(G) \leq \gamma_{acer}(G)$

Proof: Let $G$ be a graph of order $p$. By theorem 2.9, $\gamma_{acer}(G) \leq p$ and by [7], $\gamma_a(G) \leq \left\lceil \frac{p}{2} \right\rceil + 1$. From the above, $\gamma_a(G) \leq \gamma_{acer}(G)$. By [7], every accurate dominating set of $G$ is a dominating set of $G$. Hence $\gamma(G) \leq \gamma_a(G) \leq \gamma_{acer}(G)$. Also by [10], $\gamma_{cer}(G) \leq p$. Therefore $\gamma_{cer}(G) \leq \gamma_{acer}(G)$. By [10], any certified dominating is a dominating set. Hence $\gamma(G) \leq \gamma_{cer}(G) \leq \gamma_{acer}(G)$.

Theorem 2.11 For any graph $G$, $\left\lceil \frac{p}{1+\alpha} \right\rceil \leq \gamma_{acer}(G)$ and the bound is sharp.

Proof: Let $D$ be a $\gamma$-set of $G$. Each vertex can dominate atmost itself and $\Delta(G)$ other vertices. Hence $\gamma(G) \geq \left\lceil \frac{p}{1+\alpha} \right\rceil$. By theorem 2.10, $\gamma(G) \leq \gamma_{acer}(G)$. Therefore $\left\lceil \frac{p}{1+\alpha} \right\rceil \leq \gamma_{acer}(G)$. For $P_3$, the bound is sharp.

Theorem 2.12 For any graph $G$, $\gamma_{acer}(G) \leq p - \gamma(G) + 1$.

Proof: Let $D$ be a minimum dominating set of $G$ and $V$ be a vertex set of $G$. Then for any vertex $v \in D,$ $\gamma_{acer}(G) \leq (V - D) \cup \{v\}$ $\leq p - \gamma(G) + 1.$

The sharpness is attained for $C_3$.

Theorem 2.13 For any graph $G$, $\gamma_{acer}(G) \leq \alpha_0(G) + 1$.

Proof: Let $S$ be a vertex cover of $G$. We consider the following two cases.

Case 1: Suppose $|S| < \frac{p}{2}$. Then $\gamma_{acer}(G) = |S|$ $= \alpha_0(G)$ $\leq \alpha_0(G) + 1$

Case 2: Suppose $|S| \geq \frac{p}{2}$. Then for any vertex $v \in V - S,$ $\gamma_{acer}(G) \leq S \cup \{v\}$ $\leq \alpha_0 + 1.$

Corollary 2.14 For any graph $G$, $\gamma_{acer}(G) \leq p - \beta_0 + 1$.

Proof: By [3.5.6], $\alpha_0 + \beta_0 = p$.

By theorem 2.13, $\gamma_{acer}(G) \leq \alpha_0 + 1$ $\leq p - \beta_0 + 1.$

Observation 2.15 The theorem 2.12, 2.13 and corollary 2.14 does not hold for

(i) $\gamma_{acer}(G) = p, p \geq 4$

(ii) $\gamma_{acer}(K_{2,2})$

Proof: Let $S$ be a vertex cover of $G$ and $p$ be the number of vertices of $G$. We know, The number of vertices of a graph is equal to its minimum vertex cover number plus the size of maximum independent set. Of so, the vertex cover does not have total number of vertices of $G$. Which is contradiction to $\gamma_{acer}(G) \leq \alpha_0 + 1$.

Also $p - \gamma(G) + 1$ does not greater than or equal to $\gamma_{acer}(G) = p$. Hence proved.
Theorem 2.16 For any tree $T$ with $m$ cut vertices, $\gamma_{acer}(T) \leq m + 1$. It is not true when $\gamma_{acer}(T_p) = p$.

Proof: Let $S = \{v_j, j = 1 \text{ to } m\}$ be the cut vertices of $V(G) = \{v_i, i = 1 \text{ to } n\}$ with $\vert S \vert = m$. Sometimes there is a vertex in $S$ has one neighbour in $V - S$. Then for any end vertex $v \in T$,
$$\gamma_{acer}(G) \leq S \cup \{v\} \leq m + 1.$$ 

Corollary 2.17 For any tree $T$ with $m$ cut vertices and $n$ end vertices, $\gamma_{acer}(T) \leq p - n + 1$.

Proof: By [5,6], $m + n = p$. By Theorem 2.16, $\gamma_{acer}(T) \leq m + 1 \leq p - n + 1$.

Theorem 2.18 For any graph $G$, $\gamma_{acer}(G) \leq \gamma_m(G) + 1$. It is not true when $\gamma_{acer}(G) = p$, $p \geq 5$.

Proof: Let $D$ be a $\gamma_m$-set of $G$. Then $V - D$ is not a dominating set of $G$. For some graphs $D$ has a one neighbour in $V - D$. Then for any vertex $v \in V - D, D \cup \{v\}$ is an accurate certified dominating set of $G$.

$$\gamma_{acer}(G) \leq \vert D \cup \{v\}\vert = \gamma_m(G) + 1.$$ 

For $P_4$, the bound is sharp.

Theorem 2.19 For any graph $G$, $\gamma_{acer}(G) \leq \gamma(G) + p - \Delta(G)$.

Proof: Let $v$ be a vertex of minimum degree that is $\delta(G) = \deg v$. By [4], $\gamma_m(G) \leq \gamma(G) + \delta(G)$. By [3], $\delta(G) + \Delta(G) = p - 1$ and by theorem 2.18,
$$\gamma_{acer}(G) \leq \gamma_m(G) + 1 \leq \gamma(G) + \delta(G) + 1 \leq \gamma(G) + p - \Delta(G)$$

For $C_5$, the bound is sharp.

Theorem 2.20 For any connected graph $G$ with $p$ vertices, $\gamma_{acer}(G) + \Delta(G) \leq 2p - 1$.

Proof: Let $G$ be a connected graph with $p$ vertices. We know that $\Delta(G) \leq p - 1$ and by theorem 2.9 $\gamma_{acer}(G) \leq p$. Hence $\gamma_{acer}(G) + \Delta(G) \leq 2p - 1$. For $K_4$, the bound is sharp.

Theorem 2.21 If $G = H \circ K_1$, where $H$ is any non-trivial connected graph then $\gamma_{acer}(G) = p$.

Proof: Let $p$ be the number of vertices in $G = H \circ K_1$. Let $l$ be the set of all pendant vertices in $G = H \circ K_1$ such that $\vert l \vert = \frac{p}{2}$. If $G = H \circ K_1$, then there exist a minimal accurate certified dominating set $D$ containing all pendant vertices and $V(H)$ of $G$.

Hence $\gamma_{acer}(G) = \vert V(H)\vert + \vert l\vert = \frac{p}{2} + \frac{p}{2} = p$.

Theorem 2.22 For the corona graph $C_m \circ P_n, n \geq 4$, $\gamma_{acer}(C_m \circ P_n) = m$.

Proof: Let $V(C_m) = \{v_i, i = 1 \text{ to } m\}$

The vertices of $m^{th}$ copy corresponding to the path $P_n$ is

$$V(C_m \circ P_n) = \{v_1, v_1, v_12, ..., v_{1n}, v_2, v_22, ..., v_{2n}, ..., v_m, v_m1, v_m2, ..., v_{mn}\}$$

Let $D$ be a minimum accurate certified dominating set of $G$. We prove this result by induction on $m$.

Suppose $m = 3$, then $D = \{v_j, i = 1 \text{ to } m\}$ dominate every vertices on the $C_3 \circ P_n, n \geq 4$. Also $D$ is the accurate certified dominating set of $C_3 \circ P_n$. Thus, $\gamma_{acer}(C_3 \circ P_n) = 3, n \geq 4$. 

Let us assume this result is true for $m - 1$. And, $\gamma_{acer}(C_{m-1} \ast P_n) = m - 1$, $n \geq 4$.

Let us prove for $m$. Let $\{v_i, i = 1 \text{ to } m\}$ be the vertices of $C_m$.

Since the result is true for $m - 1$, $D = m - 1$ Then for any vertex $v \in V - D, (m - 1) \cup \{v\}$ is an accurate certified dominating set of $G$.

Thus $\gamma_{acer}(C_m \ast P_n) = |(m - 1) \cup \{v\}| = m - 1 + 1 = m$.

Thus, $\gamma_{acer}(C_m \ast P_n) = m, n \geq 4$.

III. Accurate Certified Values for Some Graph Families

Definition 3.1

The $p$-barbell graph is the simple graph obtained by connecting two copies of a complete graph $K_p$ by a bridge.

![4-Barbell graph](image)

4-Barbell graph

Figure 5

Theorem 3.1 For the barbell graph $p \geq 3, \gamma_{acer}(G) = p + 1$.

Proof: The barbell graph has $2p$ vertices. Let $V$ be the vertex set of first copy of $K_p$. Let $U$ be the vertex set of second copy of $K_p$ and $\{u_1, v_1\}$ be a bridge. Then $V \cup \{u_1\}$ is an accurate certified dominating set of $G$. Thus

$\gamma_{acer}(G) = |V \cup \{u_1\}|$

$= p + 1$.

Definition 3.2

A web graph has defined as a prism graph $Y_{p+1,3}$ with the edges of the outer cycle removed and is denoted by $W_p$.

![Web graph](image)

$W_3$

Figure 6

Theorem 3.2 For a web graph $W_p$, $p \geq 3, \gamma(W_p) = \gamma_{a}(W_p) = \gamma_{cer}(W_p) = \gamma_{acer}(W_p) = p$.

Proof: Let $W_p$ be a graph with $3p$ vertices. Let $D$ be a dominating set of $G$. Then the support vertex are a minimal dominating set of $W_p$ such that $\gamma(W_p) = p$. Since this dominating set has atleast two neighbours in $V - D$ and in $(V - D)$ there is no dominating set of cardinality $p$ it is both certified and accurate dominating set. Also it is an accurate certified dominating set.
Definition 3.3

The lollipop graph is a special type of a graph consisting of a complete graph on \( m \) vertices and a path graph on \( n \) vertices connected with a bridge and is denoted by \( K_m(P_n) \).

\[
K_4(P_2)
\]

Figure 7

\[\text{Theorem 3.3} \]

For the lollipop graph \( K_m(P_n) \), \( m \geq 3 \),

\[
\gamma_{acer}(K_m(P_n)) = \begin{cases} 
\frac{n+1}{3} & n \equiv 0 \pmod{3} \\
\frac{n+2}{3} & n \equiv 1 \pmod{3} \\
\frac{n+4}{3} & n \equiv 2 \pmod{3} 
\end{cases}
\]

**Proof:** Let \( u \) be the vertex with maximum degree in \( K_m(P_n) \).

**Case 1:** \( n \equiv 0 \pmod{3} \), by proposition 2.3, \( \gamma_{acer}(P_n) = \frac{n}{3} \). Then \( \gamma_{acer}(P_n) \cup \{u\} \) is an accurate certified dominating set of \( G \). Thus \( \gamma_{acer}(K_m(P_n)) = \frac{n}{3} + 1 \).

**Case 2:** \( n \equiv 1 \pmod{3} \), the vertex \( u \) dominate the vertices in \( K_m \) and one vertex in \( P_n \). In \( K_m(P_n) \), \( \gamma_{acer}(P_n) = \frac{n-1}{3} \). Then \( \gamma_{acer}(P_n) \cup \{u\} \) is an accurate certified dominating set of \( G \). Thus

\[
\gamma_{acer}(K_m(P_n)) = \frac{n-1}{3} + 1 = \frac{n+2}{3}.
\]

**Case 3:** \( n \equiv 2 \pmod{3} \), in \( K_m(P_n) \), by preposition 2.3, \( \gamma_{acer}(P_n) = \frac{n+1}{3} \). Then \( \gamma_{acer}(P_n) \cup \{u\} \) is an accurate certified dominating set of \( G \). Thus

\[
\gamma_{acer}(K_m(P_n)) = \frac{n+1}{3} + 1 = \frac{n+4}{3}.
\]

**Theorem 3.4** For the lollipop graph \( K_m(P_2) \), \( m \geq 3 \), \( \gamma_{acer}(K_m(P_2)) = \left\lceil \frac{m}{2} \right\rceil + 2 \).

**Proof:** Let \( V \) be the vertex set of \( K_m \). Let \( U \) be the vertex set of \( P_2 \) and \( D \) be an accurate certified dominating set of \( K_m(P_2) \). Thus

\[
|D| = \left\lceil \frac{m}{2} \right\rceil + |U(P_2)| = \left\lceil \frac{m}{2} \right\rceil + 2.
\]

**IV. Nordhaus-Gaddum Type Results**

In 1956 the original paper [1] by Nordhaus and Gaddum appeared. In it they gave sharp bounds on the sum and product of the chromatic numbers of a graph and its complement. Since then such results have been given for several parameters.

**Theorem 4.1** If graphs \( G \) and \( \bar{G} \) have no isolated vertices, then
\[ \gamma_{acer}(G) + \gamma_{acer}(\bar{G}) \leq 2p \]
\[ \gamma_{acer}(G)\gamma_{acer}(\bar{G}) \leq p^2 \]
Furthermore the bounds are attained if \( G = C_4 \).

**Proof:** Let \( G \) and \( \bar{G} \) have no isolated vertices and \( p \) be the number of vertices of \( G \) and \( \bar{G} \). By theorem 2.9, \( \gamma_{acer}(G) \leq p \). Since \( \bar{G} \) has no isolated vertices, \( \gamma_{acer}(\bar{G}) \leq p \). Thus both upper bound holds. Clearly, if \( G = C_4 \), then \( \gamma_{acer}(G) = 4 \) and \( \gamma_{acer}(\bar{G}) = 4 \). Therefore both bounds are attained.

**Theorem 4.2** Nordhaus-Gaddum result for \( p \) - barbell graph
\[ \gamma_{acer}(G) + \gamma_{acer}(\bar{G}) = 2(p + 1) \]
\[ \gamma_{acer}(G)\gamma_{acer}(\bar{G}) = (p + 1)^2 \]

**Proof:** Let \( G \) and \( \bar{G} \) be the \( p \)-barbell graph and its complement respectively. The \( p \)-barbell graph has \( 2p \) vertices. By theorem 3.1, \( \gamma_{acer}(\bar{G}) = p + 1 \). Let \( D \) be an accurate dominating set of \( \bar{G} \). We know by [7], \( |D| = \left[ \frac{p}{2} \right] + 1 \). Also \( D \) has at least two neighbours in \( V - D \). Therefore \( D \) is also an accurate certified dominating set of \( \bar{G} \). But here we have \( 2p \) vertices. Therefore \( \gamma_{acer}(\bar{G}) = p + 1 \). Hence \( \gamma_{acer}(G) + \gamma_{acer}(\bar{G}) = 2(p + 1) \) and \( \gamma_{acer}(G)\gamma_{acer}(\bar{G}) = (p + 1)^2 \).

**Theorem 4.3** Nordhaus-Gaddum result for web graph
\[ \gamma_{acer}(G) + \gamma_{acer}(\bar{G}) \leq \frac{5p+2}{2} \]
\[ \gamma_{acer}(G)\gamma_{acer}(\bar{G}) \leq \frac{3p^2+2p}{2} \]

**Proof:** Let \( G \) and \( \bar{G} \) be the web graph and its complement respectively. The web graph has \( 3p \) vertices. By theorem 3.2, \( \gamma_{acer}(G) = p \). Let \( D \) be an accurate dominating set of \( \bar{G} \). We know by [7], \( |D| = \left[ \frac{p}{2} \right] + 1 \). Also \( D \) has at least two neighbours in \( V - D \). Therefore \( D \) is also an accurate certified dominating set of \( \bar{G} \). But here we have \( 3p \) vertices. Therefore \( \gamma_{acer}(\bar{G}) = \left[ \frac{3p}{2} \right] + 1 \leq \frac{3p+2}{2} \). Hence \( \gamma_{acer}(G) + \gamma_{acer}(\bar{G}) \leq \frac{5p+2}{2} \) and \( \gamma_{acer}(G)\gamma_{acer}(\bar{G}) \leq \frac{3p^2+2p}{2} \).

**Theorem 4.4** Nordhaus-Gaddum result for lollipop graph \( K_m(P_2), m \geq 3 \)
\[ \gamma_{acer}(G) + \gamma_{acer}(\bar{G}) \leq \frac{m+n}{2} \]
\[ \gamma_{acer}(G)\gamma_{acer}(\bar{G}) \leq m + 4 \]

**Proof:** Let \( G \) and \( \bar{G} \) be the lollipop graph and its complement respectively. The lollipop graph has \( m + n = p \) vertices. By theorem 3.4, \( \gamma_{acer}(G) = \left[ \frac{m}{2} \right] + 2 \leq \frac{m}{2} + 2 \). Let \( D \) be a dominating set of \( \bar{G} \). In \( \bar{G} \), we have exactly one vertex of degree \( \Delta(\bar{G}) = p - 2 \). Therefore \( |D| = 2 \) and \( V - D \) has no dominating set of cardinality \( |D| \).
Also \( D \) has at least two neighbours in \( V - D \). Therefore \( \gamma_{acer}(\bar{G}) = 2 \). Hence \( \gamma_{acer}(G) + \gamma_{acer}(\bar{G}) \leq \frac{m+n}{2} \) and \( \gamma_{acer}(G)\gamma_{acer}(\bar{G}) \leq m + 4 \).

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