C - Compactness And Characterization Modulo An Ideal

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Abstract: *An ideal on a set X is a non-empty collection of subsets of X with heredity property which is also closed under arbitrary union and finite intersections. In this paper, we introduce C - compactness with respect to ideal and discuss some of their properties. Also, this paper gives the characterizations of C compactness with respect to ideal, some of which make use of filter.*

Keywords: F- Compact $\Im$ - compact, $\Im$C - Compact, compatible ideal $\Im$.

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I. INTRODUCTION

Given a non-empty set X, a collection $\Im$ of subsets of X is called an ideal if

(i) If $A\in \Im$ and $B \subseteq A$ implies $B \in\Im$ (heredity)

(ii) If $A\in \Im$ and $B \in\Im$ implies $A \cup B\in\Im$ (additivity)

If $X\notin \Im$, then $\Im$ is called a proper ideal.

An ideal $\Im$ is called a σ - ideal if the following holds:

If $\{A_{n:}\ n = 1,2, \ldots\}$ is a countable sub collection of $\Im$, then $\cup\{A_{n:}\ n = 1,2, \ldots\}\in\Im$

The notation $(X, \tau, \Im)$ denotes a nonempty set X, a topology τ on X and an ideal $\Im$ on X. Given point $x \in X$, $\aleph(x)$ denotes the neighbourhood system of x i.e $\aleph(x) = \{U \in \tau :\ x \in U\}$. The symbol $\wp(X)$ denotes collection of all subsets of X. Given space $(X, \tau, \Im)$ and a subset A of X, we define $A^* (\Im,\tau) = \{x \in X: U \cap A \notin \Im, \text{ for every } U\in\aleph(x)\}$

We simply write A^* for $A^*(\Im, \tau)$, when there is only one ideal $\Im$ and only one topology τ under consideration. If we define cl* on $\wp(X)$ as,

$cl^* (A) = A \cup A^*$, for all $A \in \wp(X)$,

then cl* is a Kuratowski closure operator. The topology determined by this closure operator is denoted by $\tau^* (\Im)$. $\beta (\Im, \tau) = \{U - I: U\in \tau , I \in\Im\}$ is a basis for $\tau^* (\Im)$. The first unified and extensive study on $\tau^* (\Im)$ - topology was done by Jankovic and Hamlett [2].

We shall use cl(A), int(A) will denotes closure and interior of a subset A respectively in topological space (X, τ) and cl*(A), int*(A) will denotes closure and interior of A respectively with respect to τ^*.

In topological space (X, τ), a subset U is said to be regular- open if int (cl (U)) = U. A subset U in X is regular - closed if cl (int (U)) = U. Clearly U is regular- closed (open) if and only if its complement is regular - open (closed). A subset A of (X, τ) is said to be compact if every open cover of A has a finite sub cover. If X is compact, then every closed subset of X is also compact.

Definition : 1.1 A topological space (X, τ) is said to be C - compact if and only if for every family U of open sets of X and for every $a \in I$ such that for each closed subset $A \subseteq X$, each open cover $\{U_\alpha : \alpha \wedge\}$ of A has a finite sub cover $\{ U_{\alpha i} : i = 1,2,\ldots.n\}$ such that $A \subseteq \bigcup_{i=1}^{n} cl(U_{\alpha i})$.

Newcomb introduced the concept of compactness with respect an ideal [3]. An extensive study on compactness with respect to an ideal have been made by Hamlett and Dragan Jankovic [1].

Definition: 1.2 A subset A of a space $(X, \tau, \Im)$ is said to be $\Im$ - compact if for every open cover $\{U_\alpha : \alpha \in \wedge\}$ of A, there exists finite sub collection $\{U_{\alpha i} : i = 1,2,\ldots.n\}$ such that

$$A - \bigcup_{i=1}^{n} U_{\alpha i} \in \Im.$$

A space $(X, \tau, \Im)$ is said to be $\Im$ - compact if X is $\Im$ - compact as a subset.

II. $\Im\, \mathcal{C}$- COMPACT SETS

Let us start with a definition for $\Im\ \mathcal{C}$- compact sets.

Definition : 2.1 A space $(X, \tau, \Im)$ is said to be $\Im\mathcal{C}$ -compact space if for each closed subset $A \subseteq X$ and each open cover $\{U_\alpha : \alpha \in \wedge\}$ of A has a finite sub collection $\{U_{\alpha i} : i = 1,2,\ldots.n\}$ such that $A - \cup\{ cl(U_{\alpha i}) : i = 1,2\ldots\ldots n\} \in \Im$.

The following results can be verified easily.

Results:

i) Every C-compact space is $\Im\mathcal{C}$- compact.

ii) Every closed subspace of $\Im\mathcal{C}$-compact space is $\Im\mathcal{C}$- compact.

iii) A space is C - compact if and only if it is $\Im\mathcal{C}$- compact, for the ideal $\Im = \{\phi\}$.

iv) Let $(X, \tau, \Im)$ be a $\Im\mathcal{C}$- compact. If $\Im$ is an ideal on X such that $\Im \subseteq \vartheta$, then (X, τ, ϑ) is $\vartheta\mathcal{C}$ - compact.

v) Let $\Im_f$ denotes the ideal of all finite subsets of X. Then (X, τ) is $\mathcal{C}$- compact if and only if $(X, \tau, \Im_f)$ is $\Im_f\mathcal{C}$- compact.

Example for a space which is $\Im\mathcal{C}$- compact, but not $\mathcal{C}$-compact.

Example: 2.2 Let $X = [0,1] \cup \{2, 3,\ldots.\}$. The topology τ be the induced topology obtained from the usual topology on the real line. Let $\Im = \Im c$, the ideal of countable subsets of X.

Then $(X, \tau, \Im)$ is $\Im\ \mathcal{C}$-compact, but not $\mathcal{C}$-compact.

We recall the following results [1]. Let $f: X \to Y$ be a function and let $\Im$ be an ideal on X. Then $f(\Im) = \{ f(I) : I \in \Im\}$ is a ideal on Y.

We know that the continuous image of compact set is compact. This result can be extended to $\Im\mathcal{C}$-compact.

Theorem: 2.3 Let f : (X, τ , $\Im$) $\rightarrow$ (Y, σ) be a continuous surjection and let f ($\Im$) be an ideal on Y. If X is $\Im\mathcal{C}$-compact, then Y is f ($\Im$)$\mathcal{C}$-compact.

Proof: Let f: (X, τ , $\Im$) $\rightarrow$ (Y, σ) be a continuous surjection. Let A be a closed subset in Y and let $\{U_\alpha : \alpha \in \wedge\}$ be an open cover of A in Y. Since f is continuous, $\{f^{-1}(U_\alpha): \alpha \in \wedge\}$ is an open cover of $f^{-1}(A)$ in X. Since X is $\Im\mathcal{C}$-compact and $f^{-1}(A)$ is closed in X, there is a finite subfamily $\{ U_{\alpha i} : i = 1,2,\ldots,n\}$ such

that $f^{-1}(A) - \bigcup_{i=1}^{n} cl(f^{-1}(U_{\alpha i})) \in \Im$.

Therefore $f(f^{-1}(A) - \bigcup_{i=1}^{n} cl(f^{-1}(U_{\alpha i}))) \in f(\Im)$. Since

$A - \bigcup_{i=1}^{n} f(cl(f^{-1}(U_{\alpha i}))) \subseteq f(f^{-1}(A) - \bigcup_{i=1}^{n} cl(f^{-1}(U_{\alpha i})))$, we have $A - \bigcup_{i=1}^{n} f(cl(f^{-1}(U_{\alpha i}))) \in f(\Im)$.

We note that $f^{-1}(U_{\alpha i}) \subseteq f^{-1}(cl(U_{\alpha i}))$ and $f^{-1}(cl(U_{\alpha i}))$ is closed in X.

Hence $cl(f^{-1}(U_{\alpha i})) \subseteq f^{-1}(cl(U_{\alpha i}))$. Therefore $f(cl(f^{-1}(cl(U_{\alpha i})))) \subseteq cl(U_{\alpha i})$

Hence $A - \bigcup_{i=1}^{n} cl(U_{\alpha i}) \in f(\Im)$. This shows that Y is f ($\Im$) $\mathcal{C}$-compact.

Let (X, τ) be a topological space and $\Im$ be a ideal in X. We say that the topology τ is compatible with the ideal$\Im$, denoted by $\tau \sim \Im$, if the following holds for every $A \subseteq X$: if for every $x \in A$, there exist a $U \in \aleph(x)$ such that $U \cap A \in \Im$, then $A \in \Im$.

T. R. Hamlett, D. A. Rose [4] has proved the following:

Theorem: 2.4 Let (X, τ) be a topological space with an ideal $\Im$ on X. If $\tau \sim \Im$, then

(i) $\tau^* = \{V - I: V \in \tau, I \in \Im\}$

(ii) A set $A \subseteq X$ is closed with respect to τ^* if and only if it is a union of a set which is closed with respect to τ and a set in $\Im$.

In [8], R. Vaidyanathaswamy has proved the following theorem.

Theorem: 2.5 Let (X,τ) be a topological space with $\Im$ a ideal on X. Then the following are equivalent.

(i) $\tau \sim \Im$

(ii) For every $A \subseteq X$, $A - A^* \in \Im$

We now prove that if $\tau \sim \Im$ and $\overline{A} \in \Im$, for all $A \in \Im$, then X is $\Im$C - compact with respect to τ implies X is $\Im$C - compact with respect to τ^*.

Theorem: 2.6 Let (X, τ) be a topological space and $\Im$ be a ideal on X which is compatible with τ and $\overline{I} \in \Im$, for all $I \in \Im$. If X is $\Im$ C - compact with respect to τ, then X is $\Im$ C - compact with respect to τ^*.

Proof: Let A be a closed subset with respect to τ^*. Then $A = B \cup I$, for some closed set B with respect to τ and $I \in \Im$. (This is possible as $\tau \sim \Im$).

Let $\xi = \{V_\alpha - I_\alpha : \alpha \in \wedge\}$ be an τ^* - open cover for A. Therefore $B \subseteq \cup\{V_\alpha : \alpha \in \wedge\}$

As B is closed subset of τ and (X, τ) is $\Im$ C - compact, there exist $\alpha_1, \alpha_2, \ldots \alpha_n$ such that

$B - \bigcup_{i=1}^{n} cl(V_{\alpha i}) \in \Im$. ………………..(1) Note that $cl^*(V_\alpha) = V_\alpha \cup (V_\alpha)^*$.

We claim that $cl(V_{\alpha i}) - cl^*(V_{\alpha i}) \in \Im$, for all i = 1,2,…….n.

Let $x \in cl(V_{\alpha i}) - cl^*(V_{\alpha i})$. Then there is one $U_X \in \tau$ and $I_X \in \Im$ such that $x \in U_X - I_X$ and $(U_X - I_X) \cap V_{\alpha i} = \phi$, $U_X \cap V_{\alpha i} = I_X \in \Im$ and $U_X \cap V_{\alpha i} \neq \phi$.

Let $y \in U_X \cap cl(V_{\alpha i})$. Then for every neighborhood W of y in τ, $(W \cap U_X) \cap V_{\alpha i} \neq \phi$. i.e. $W \cap (U_X \cap V_{\alpha i}) \neq \phi$. Then we get that $U_X \cap cl(V_{\alpha i}) \subseteq cl(U_X \cap V_{\alpha i}) \subseteq \overline{I} \in \Im$.

Hence to each $x \in cl(V_{\alpha i}) - cl^*(V_{\alpha i})$, there is a neighborhood $U_X \in \tau$ such that

$U_X \cap (cl(V_{\alpha i}) - cl^*(V_{\alpha i})) \subseteq U_X \cap cl(V_{\alpha i}) \in \Im$. As $\tau \sim \Im$, it follows that

$cl(V_{\alpha i}) - cl^*(V_{\alpha i}) \in \Im$. This is true for all i = 1, 2, ……….n.

From (1), $B - \bigcup_{i=1}^{n} cl^*(V_{\alpha i}) \in \Im$.

As $cl^*(V_{\alpha i}) \subseteq cl^*(V_{\alpha i} - I_{\alpha i}) \cup \overline{I_{\alpha i}}$, we have $(B \cup I) - \bigcup_{i=1}^{n} cl^*(V_{\alpha i} - I_{\alpha i}) \in \Im$.

1) i.e. $A - \bigcup_{i=1}^{n} cl^*(V_{\alpha i} - I_{\alpha i}) \in \Im$.

Hence X is $\Im$ C - compact with respect to τ^*.

It is well known that compact subset of Hausdorff space is closed. We generalize this as follows.

Theorem: 2.7 Let $(X, \tau, \Im)$ be a Hausdorff space. If $Y \subseteq X$ is $\Im$ C - compact, then Y is τ^*- closed.

Proof: Let $x \in X - Y$. Since X is Hausdorff, for each $y \in Y$, there exist τ - neighbourhood U_y, V_y of x and y respectively such that $U_y \cap V_y = \phi$. The family $\{V_y : y \in Y\}$ is an open cover of Y. Since Y is $\Im$ C - compact, then there exist a finite sub collection $\{V_{y_i} : i = 1,2,\ldots.n\}$, such that $Y - \bigcup_{i=1}^{n} cl(V_{y_i}) \in \Im$.

Let $I = Y - \bigcup_{i=1}^{n} cl(V_{y_i})$. Define $U = \cap U_{y_i}$. Then U is τ - open and $U \cap (\bigcup_{i=1}^{n} cl(V_{y_i})) = \phi$.

Therefore $U \cap Y \subseteq I$ and hence $x \notin Y$. Thus $x \notin Y \Rightarrow x \notin Y^*$. Hence Y is τ^* - closed

We now give the equivalent meaning of definition [2.1].

Theorem: 2.8 A space X is $\Im$ C - compact if and only if for each closed set A $\subseteq$ X and regular - open cover $\{U_\alpha : \alpha \in \wedge\}$ for A , there exist finite sub collection $\{U_{\alpha i} : i = 1,2\ldots n\}$ such that

$A - \bigcup_{i=1}^{n} cl(U_{\alpha i}) \in \Im$.

Proof: If X is $\Im$ C - compact, then from the definition (2.1), we get the required result.

Conversely, Let $\{U_\alpha : \alpha \in \wedge\}$be an open cover of A, then $\{int(cl(U_\alpha)) : \alpha \in \wedge\}$ is a regular open cover for A. So by assumption, there exist finite sub collection $\{int(cl(U_{\alpha i})) : i = 1,2,\ldots,n\}$

such thatA - $\bigcup_{i=1}^{n} cl(int(cl(U_{\alpha i}))) \in \Im$.

Since $U_{\alpha i} \subseteq int(cl(U_{\alpha i})) \subseteq cl(U_{\alpha i}) \Rightarrow cl(int(cl(U_{\alpha i}))) = cl(U_{\alpha i})$

Therefore $A - \bigcup_{i=1}^{n} cl(U_{\alpha i}) \in \Im$. Hence X is $\Im$ C - compact.

III. Characterization of $\Im$ C – Compact sets

We recall the following definitions.

Definition: 3.1 **H** is called a filter in X if and only if

(i) **H** $\subseteq \wp(X)$, $\wp(X)$ is the power set of X.

(ii) $\phi \notin$ H and X $\in$ **H.**

(iii) **H** is closed under finite intersection.

(iv) If F $\in$ **H** and F $\subseteq$ G, then G $\in$ **H**

It is clear that dual of a ideal in X is a filter in X.

Definition: 3.2 β is called filter base in X if and only if

(i) $\beta \subseteq \wp(X)$

(ii) $\phi \notin \beta$ and $\beta \neq \phi$

(iii) The intersection of two sets of β contains a set of β.

Definition: 3.3 **An** ultra-filter on a set X is a filter H such that there is no filter on X which is strictly finer than H.

We will say that a filter base β is a filter base on $\wp(X) - \Im$ if $\beta \subseteq \wp(X) - \Im$.

The following theorem gives several Characterizations of $\Im$ C- compact sets.

Theorem : 3.4 Let (X, τ) be a topological space and let $\Im$ be a ideal in X. The following are equivalent.

(i) X is $\Im$ C - compact.

(ii) For every closed set A of X and each regular open- cover $\{U_\alpha : \alpha \in \wedge\}$ for A, there exist a finite sub collection $\{ U_{\alpha 1}, U_{\alpha 2}, \ldots\ldots. U_{\alpha n} \}$ such that $A - \bigcup_{i=1}^{n} cl(U_{\alpha i}) \in \Im$.

(iii) For each closed set $A \subseteq X$ and each collection of regular- closed sets $\{ F_\alpha : \alpha \in \wedge\}$ such that $(\bigcap_{\alpha} F_\alpha) \cap A = \phi$, there exist a finite sub collection $\{ F_{\alpha i} : i = 1,2,\ldots..n \}$ such that $(\bigcap_{i=1}^{n} int(F_{\alpha i})) \cap A \in \Im$.

(iv) For each set $A \subset X$ and each filter base $\mathbf{H} = \{ A_\alpha : \alpha \in \wedge\}$ in $\wp(X) - \Im$, there exist an $a \in A$ such that $A_\alpha \cap cl(V) \notin \Im$, for each open set V in X containing a.

(v) For each closed set $A \subset X$ and each ultra-filter $\mathbf{M} = \{ A_\alpha : \alpha \in \wedge\}$ in $\wp(X) - \Im$, there is an $a \in A$ such that to each open set V containing a, $A_\alpha \subset cl(V)$, for some $\alpha \in \wedge$.

Proof: (i) $\Rightarrow$ (ii): If X is $\Im$ C - compact, then from the definition 1.2, we get required result.

(ii) $\Rightarrow$ (i): Let $\{U_\alpha : \alpha \in \wedge\}$ be an open cover of A, then $\{int(cl(U_\alpha)) : \alpha \in \wedge\}$ is a regular open cover for A. So by assumption, there exist finite sub collection $\{int(cl(U_{\alpha i})) : i = 1,2,\ldots.n\}$ such that

$$A - \bigcup_{i=1}^{n} cl(int(cl(U_{\alpha i}))) \in \Im.$$

Since $U_{\alpha i} \subseteq int(cl(U_{\alpha i})) \subseteq cl(U_{\alpha i}) \Rightarrow cl(int(cl(U_{\alpha i}))) = cl(U_{\alpha i})$

Therefore $A - \bigcup_{i=1}^{n} cl(U_{\alpha i}) \in \Im$. Hence X is $\Im$ C- compact.

(ii) $\Rightarrow$ (iii): Let A be a closed set in X and $\{F_\alpha : \alpha \in \wedge\}$ be a collection of regular- closed sets such that $(\cap F_\alpha) \cap A = \phi$.

Let $G_\alpha = X - F_\alpha$. Then $\{ G_\alpha : \alpha \in \wedge\}$ is a regular- open cover for A. So there exist

finite sub collection $\{ G_{\alpha 1}, G_{\alpha 2}, \ldots.. G_{\alpha n}\}$ such that $A - \bigcup_{i=1}^{n} cl(G_{\alpha i}) \in \Im$.

But $X - cl(G_{\alpha i}) = int(F_{\alpha i})$. Hence $A \cap (\bigcap_{i=1}^{n} int(F_{\alpha i})) \in \Im$.

(iii) $\Rightarrow$ (ii) : Let $\{U_\alpha : \alpha \in \wedge\}$ be a regular- open cover for A, then $A \subset \bigcup_{\alpha} U_\alpha$ implies

$(\cap (X - U_\alpha)) \cap A = \phi$. Since each $X - U_\alpha$ is regular- closed, by (iii), there exist finite

sub collection $\{ X - U_{\alpha i} : i = 1,2, \ldots\ldots..n \}$ such that $(\bigcap_{i=1}^{n} int(X - U_{\alpha i})) \cap A \in \Im$.

(ie) $A \cap (\bigcap_{i=1}^{n} (X - cl(U_{\alpha i}))) \in \Im$.

(ie) $A - (\bigcup_{i=1}^{n} cl(U_{\alpha i})) \in \Im$.

(i) $\Rightarrow$ (iv): Assume that there is a filter base $\mathbf{H} = \{ A_\alpha : \alpha \in \wedge\}$ such that (a) $A_\alpha \notin \Im$, for all $\alpha \in \wedge$ and (b) to each $x \in A$, there is an open set $V_{(x)}$ and some $A_{\alpha(x)} \in \mathbf{H}$ such that $A_{\alpha(x)} \cap cl(V_{(x)}) \in \Im$. Then the collection

$\{V_{(x)} : x \in A\}$ is an open cover for A. Hence there is a finite sub collection $\{V_{(xi)} : i = 1,2,\ldots..n\}$ such that $A - \bigcup_{i=1}^{n} cl(V_{(xi)}) \in \mathfrak{I}$.

Let $A_{\alpha 0} \in \mathbf{H}$ such that $A_{\alpha 0} \subset \bigcap_{i=1}^{n} A_{\alpha(xi)}$.

Then $A_{\alpha 0} \cap cl(V_{(xi)}) \in \mathfrak{I}$, for all i = 1,2…….n

i.e. $A_{\alpha 0} \cap (\bigcup_{i=1}^{n} cl(V_{(xi)}) \in \mathfrak{I}$ and

$A_{\alpha 0} \cap (\bigcup_{i=1}^{n} cl(V_{(xi)}) \subseteq A - (\bigcup_{i=1}^{n} cl(V_{(xi)}) \in \mathfrak{I}$, which is a contradiction as $A_{\alpha 0} \notin \mathfrak{I}$

(iv) ⇒ (v): Let A be a closed set in X and $\mathbf{M} = \{A_\alpha : \alpha \in \wedge\}$ be an ultra-filter in $\wp(X) - \mathfrak{I}$. Then $A_\alpha \notin \mathfrak{I}$, for all $\alpha \in \wedge$. By (iv), there is an $a \in A$ such that $A_\alpha \cap cl(V) \notin \mathfrak{I}$, for each open set V in X containing a. As **M** is an ultra-filter and $A_\alpha \cap (cl(V) \cap A) \notin \mathfrak{I}$, for all $\alpha \in \wedge$. Hence $cl(V) \cap A \in \mathbf{M}$.

Let $A_{\alpha 0} = cl(V) \cap A$. Thus $A_{\alpha 0} \subseteq cl(V)$.

(v) ⇒ (iii) : Let A be a closed set in X and let $\{F_\alpha : \alpha \in \wedge\}$ be a collection of regular- closed set such that $(\bigcap_{i=1}^{n} int(F_{\alpha i})) \cap A \notin \mathfrak{I}$ …….(1), for any finite subset $\{\alpha_1, \alpha_2 \ldots\ldots\ldots..\alpha_n\} \subseteq \wedge$.

In particular $A \cap int(F_\alpha) \notin \mathfrak{I}$, for all $\alpha \in \wedge$.

From (1), $\{A \cap int(F_\alpha) : \alpha \in \wedge\}$ is an ultra filter base for some ultra filter **H** in $\wp(X) - \mathfrak{I}$. By (v), there is a point $a \in A$ such that for every neighborhood V of a , cl (V) intersects every member of **H** . Therefore $cl(V) \cap (int(F_\alpha) \cap A) \neq \phi$, for all $\alpha \in \wedge$.

Let $y \in cl(V) \cap (int(F_\alpha) \cap A)$. Then $int(F_\alpha)$ is itself an open neighborhood of y. As $y \in cl(V)$, $int(F_\alpha) \cap V \neq \phi$. This is true for all open neighborhood V of a. Therefore $a \in \overline{int(F_\alpha)} \subseteq F_\alpha$

$\Rightarrow a \in (F_\alpha \cap A)$, for all $\alpha \in \wedge$.

Hence $A \cap (\bigcap_{\alpha} F_\alpha) \neq \phi$. Thus (v) ⇒ (iii) ⇒ (i).

A space is said to be quasi H-closed (denoted by QHC) if every open cover of the space contains finite sub collection whose closures cover the space. A Hausdorff space which is QHC is said to be H-closed.

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