A study on multi-rational numbers forming a multi-field

Debjyoti Chatterjee[#1], S. K. Samanta[*2]

[#]*Assistant Professor, Department of Mathematics, Jalpaiguri Government Engineering College, Jalpaiguri-735101, W.B., India.*
[*]*Professor, Department of Mathematics, Visva-Bharati, Santiniketan – 731235, W.B., India*

Abstract: *In an attempt to develop multi number system, in this paper, we introduce a concept of multi-rational number system which forms a multi-field. It is also shown that the multi-rational number system is an extension of multi-integer system.*

Keywords: *Multiset, Multi-natural number, Multi-integer, Multi-rational number, Multi-ring, Multi-integral domain, Multi-field.*

I. INTRODUCTION

In classical set theory, a set is a well-defined collection of distinct objects. If the repeated occurrences of any object are allowed in a collection, then that mathematical structure is called a multiset (mset in short). In many situations, it is more convenient to consider a collection like multiset. e.g., repeated roots of an equation, repeated eigen values of a matrix, prime factors of a positive integer, repeated observations in a statistical sample, data structure, information retrieval on the web, multi-criteria decision making, knowledge presentation in data based system, biological systems and membrane computing [20,22,24,25]. More studies on multisets can be found in [2,4,5,9,12,14,16,17]. The term multiset as Knuth notes [17], was first suggested by N. G. de Bruijn [10] in a private correspondence to him. N. G. de Bruijn's interests in multisets grew out of his investigations into the combinatorial properties of the set of divisors of a number. A number or any of its divisors is expressible as a multiset of prime factors [2,17]. But it have now become an area of special interest in various subjects like mathematics, statistics, computer science, physics and philosophy [2,8,10,12,22,24,25]. Many authors like Yagar [25], Miyamoto [20], Hickman [15], Blizard [4], Girish and John [12,13,23], D. Singh [24], A. M. Ibrahim [24] etc. have studied the properties of multisets. Some authors have also generalized the notion of multisets to form fuzzy multisets [18], Intuitionistic fuzzy multisets [3,23], soft multisets [1,13,19] etc. Various research work on the multiset ordering [4,11,24], relations and functions in multiset context [20], multiset topology [12,13], multi group theory [21] etc. have been done recently by some researchers.

In order to develop various structures on multisets we have started from the beginning. Our motif is to develop a multi number system which a generalization of the ordinary number system and also compatible with the multiset setting as number system plays an important role in mathematics. In a previous papers [6], we have introduced a concept of multi-natural number system from the axiomatic point of view and study its properties related to compositions and order relations. After that in another paper [7],we introduce concept of multi-integer system. In this paper, we extend it to develop multi-rational number system and to study their properties. The organization of the paper is as follows:

Section 2 is the preliminary part where some definitions and results regarding multisets, multi-natural numbers and multi-integers have been introduced. In section 3, the notion of multi-fractional system together with binary operations and order relation defined on it has been introduced. Several properties regarding multi-fractional system have been studied and notions like multi-distributive property, multi-rational number, multi-field etc. have been also defined in this section. Finally, Multi-rational number system has been introduced; its isomorphism with multi-fractional system and its existence and uniqueness have been established. The straightforward proofs of the propositions have been omitted.

II. PRELIMINARIES

Definition 2.1 [12] A multiset (or mset, in short) M drawn from a set X is represented by a function $Count_M$ or C_M defined as $C_M: X \to N \cup \{0\}$ where N represents the set of all natural numbers. Let M be an mset drawn from the set $X = \{x_1, x_2, x_3, \dots, x_n\}$ with x_i appearing k_i times in M. It is denoted by $x_i \in^{k_i} M$. The mset M drawn from the set X is then denoted by $\{k_1/x_1, k_2/x_2, \ldots, k_n/x_n\}$. Also $C_M(x)$ is the number of occurrences of the element x in the mset M. However, those elements which are not included in the mset M have zero count.

Example 2.2 Let $X = \{a, b, c, d, e\}$ be any set. Then $M = \{3/a, 2/b, 1/e\}$ is an mset drawn from X.
Definition 2.3 [12] Let A, B and M be three msets drawn from a set X. Then the followings are defined:

(i) $A = B$ if $C_A(x) = C_B(x)$ for all $x \in X$.
(ii) $A \subseteq B$ if $C_A(x) \leq C_B(x)$ for all $x \in X$, thn we call A to be a submset of B.
(iii) $M = A \cup B$ if $C_M(x) = \max\{C_A(x), C_B(x)\}$ for all $x \in X$.
(iv) $M = A \cap B$ if $C_M(x) = \min\{C_A(x), C_B(x)\}$ for all $x \in X$.
(v) $M = A \oplus B$ if $C_M(x) = C_A(x) + C_B(x)$ for all $x \in X$.
(vi) $M = A \ominus B$ if $C_M(x) = \max\{C_A(x) - C_B(x), 0\}$ for all $x \in X$.

Where $\oplus$ and $\ominus$ represents mset addition and mset subtraction resoectively. Let M be an mset drawn from a set X, then the support set of M denoted by M^* is a subset of X and $M^* = \{x \in X : C_M(x) > 0\}$. i.e., M^* is an ordinary set and it is also called root set. The cardinality of an mset M drawn from a set X is denoted by $card(M)$ or $|M|$ and is given by $|M| = \sum_{x \in X} C_M(x)$.

Remark 2.4 [12, 16] A domain X is defined as a set of elements from which msets are constructed. The mset space $[X]^m$ is the set of all msets whose elements are in X such that no element in the mset occurs more than m times. The mset space $[X]^\infty$ is the set of all msets over a domain X such that there is no limit on the number of occurrences of an element in an mset. If $X = \{x_1, x_2, x_3, \ldots, x_k\}$, then $[X]^m = \{\{m_1/x_1, m_2/x_2, \ldots, m_k/x_k\}$: for $i = 1,2, \ldots, k;\ m_i \in \{0,1,2, \ldots, m\}\}$.

Definition 2.5 [12, 16] Let X be a crisp set and $[X]^m$ be the mset space defined over X, then the complement M^c of M in $[X]^m$ is an element of $[X]^m$ such that $C_{M^c}(x) = m - C_M(x)$ for all $x \in X$.

Definition 2.6 (Different types of msets)

(i) [14] Whole submset: A submset P of an mset M (i.e., $P \subseteq M$) is a whole submset of M if each element in P has full multiplicity as in M. i.e., $C_P(x) = C_M(x)$ for all $x \in P^*$.
(ii) [14] Partial whole submset: A submset P of an mset M (i.e., $P \subseteq M$) is a partial whole submset of M if at least one element in P has the full multiplicity as in M. i.e., $C_P(x) = C_M(x)$ for some $x \in P^*$.
(iii) [14] Full submset: A submset P of an mset M (i.e., $P \subseteq M$) is a full submset of M if $P^* = M^*$ and $C_P(x) \leq C_M(x)$ for all $x \in P^*$.
(iv) [6] Single whole submset and single submset: A submset P of an mset M drawn from a set X is a single whole submset if $C_P(x)$ is either C_M or 0 for all $x \in P^*$ and $\{x \in P^* : C_P(x) = C_M(x)\}$ is a singleton set, say $\{a\}$, then let us denote it as $M_{\{a\}}\ (= P)$, i.e., a single whole submset is such a submset of an mset for which exactly one element of the support set belongs to it with the same count as in the mset.
An mset is called a single mset if it has a singleton support set and a submset P of an mset M drawn from a set X is a single submset if P is a single mset.
So, immediately, each mset can be expressed as a union of all its single whole submsets. Therefore, $M = \bigcup_{a \in M^*} M_{\{a\}}$.
In this connection, we note that single whole submsets are pair wise disjoint.

Definition 2.7 [6] (Axiomatic definition of multi-natural numbers)
Let $(N, 1, \sigma)$ be the unique ordinary natural number system defined by Peano. Then,
Axiom 1: For all $p, q \in N$, there exist a multi-natural number denoted by $\mathrm{N}_\mathrm{p}^\mathrm{q}$.
Axiom 2: Two multi-natural numbers N_p^q and N_r^s are equal if and only if $p = r$ and $q = s$.
Axiom 3: For any multi-natural number $N_p^q, p, q \in N$, there exist a multi-natural number $N_{\sigma(p)}^q$ (defined to be the support successor of N_p^q) and another there exist a multi-natural number $\mathrm{N}_\mathrm{p}^{\sigma(\mathrm{q})}$ (defined to be multiplicity successor of N_p^q).
Axiom 4: N_1^q for all $q \in N$ is not a support successor of any multi-natural number. Also, N_p^1 for all $p \in N$ is not a multiplicity successor of any multi-natural number.

Axiom 5: Let $P(N_p^q)$ be any proposition involving N_p^q. Suppose that $P(N_1^1)$ is true. Also suppose that whenever $P(N_p^q)$ is true, then $P(N_{\sigma(p)}^q)$ and $P(N_p^{\sigma(q)})$ both are also true. Then $P(N_p^q)$ is true for every multi-natural number N_p^q.
The set of all multi-natural numbers is denoted by $m(N)$. $p \in N$ and $q \in N$ are respectively the support and the multiplicity of a multi-natural number N_p^q.

Definition 2.8 [6] (Successor Functions) $S: m(N) \to m(N)$ defined by $S(N_p^q) = N_{\sigma(p)}^q$ is called the support successor function. $M: m(N) \to m(N)$ defined by $M(N_p^q) = N_p^{\sigma(q)}$ is called the multiplicity successor function. S and M both are one to one since σ is one to one.

Definition 2.9 [6] (Definition of addition) There exists a unique function $A: m(N) \times m(N) \to m(N)$ with the following properties:
Axiom 1: $A(N_p^q, N_1^1) = S(N_p^q)$, $N_p^q \in m(N)$,
Axiom 2: $A\left(N_p^q, S(N_n^m)\right) = S(A(N_p^q, N_n^m))$, $N_p^q, N_n^m \in m(N)$,
Axiom 3: $A(N_p^q, M(N_n^m)) = M^{(q)}(A(N_p^q, N_n^m))$, $N_p^q, N_n^m \in m(N)$ which is called addition of two multi-natural numbers and it is given by $A(N_p^q, N_n^m) = N_{p+n}^{qm}, N_p^q, N_n^m \in m(N)$. $A(N_p^q, N_n^m)$ is also denoted by $N_p^q + N_n^m$.
Proposition 2.10 [6] (Properties of addition)

- (i) $S(N_p^q) = N_p^q + N_1^1$, for all $N_p^q \in m(N)$,
- (ii) $N_p^q + (N_k^t + N_1^1) = (N_p^q + N_k^t) + N_1^1$ for all $N_p^q, N_k^t \in m(N)$,
- (iii) $N_1^1 + N_p^q = N_p^q + N_1^1$ for all $N_p^q \in m(N)$,
- (iv) $(N_p^q + N_1^1) + N_k^t = (N_p^q + N_k^t) + N_1^1)$ for all $N_p^q, N_k^t \in m(N)$,
- (v) The commutative law of addition: $N_p^q + N_k^t = N_k^t + N_p^q$ for all $N_p^q, N_k^t \in m(N)$,
- (vi) The associative law of addition: $(N_p^q + N_k^t) + N_n^m = N_k^t + (N_p^q + N_n^m)$ for all $N_p^q, N_k^t, N_n^m \in m(N)$,
- (vii) The cancellation law for addition: $N_p^q + N_k^t = N_p^q + N_n^m \Rightarrow N_k^t = N_n^m$ for all $N_p^q, N_k^t, N_n^m \in m(N)$.

Example 2.11 For two multi-natural numbers N_5^6 and N_3^4, $N_5^6 + N_3^4 = N_{5+3}^{6.4} = N_8^{24}$.

Definition 2.12 [6] (Definition of multiplication)
There exists a unique function $P: m(N) \times m(N) \to m(N)$ with the following properties:

- (i) $P(N_p^q, N_1^1) = N_p^q$, $N_p^q \in m(N)$,
- (ii) $P\left(N_p^q, S(N_n^m)\right) = S^{(p)}(P(N_p^q, N_n^m))$, $N_p^q, N_n^m \in m(N)$,
- (iii) $P\left(N_p^q, M(N_n^m)\right) = M^{(p)}(P(N_p^q, N_n^m))$, $N_p^q, N_n^m \in m(N)$ which is called multiplication of two multi-natural numbers and it is given by $P(N_p^q, N_n^m) = N_p^q . N_n^m$, $N_p^q, N_n^m \in m(N)$. $P(N_p^q, N_n^m)$ is also denoted by $N_p^q . N_n^m$.

Properties 2.13. [6] (Properties of multiplication)

- (i) $N_p^q . N_1^1 = N_p^q = N_1^1 . N_p^q$, for all $N_p^q \in m(N)$,
- (ii) The commutative law of multiplication: $N_p^q . N_k^t = N_k^t . N_p^q$ for all $N_p^q, N_k^t \in m(N)$,
- (iii) The associative law of multiplication:: $(N_p^q . N_k^t) . N_n^m = N_k^t . (N_p^q . N_n^m)$ for all $N_p^q, N_k^t, N_n^m \in m(N)$,
- (iv) P does not obey distributive property over A. i.e., $N_p^q . (N_k^t + N_n^m) \neq N_p^q . N_k^t + N_p^q . N_n^m$, $N_p^q, N_k^t, N_n^m \in m(N)$.

Example 2.14. For two multi-natural numbers N_5^6 and N_3^4, $N_5^6 . N_3^4 = N_{5.3}^{6.4} = N_{15}^{24}$.

Definition 2.15. [6] (Order on $m(N)$) For $N_p^q, N_n^m \in m(N)$, $N_p^q = N_n^m$ if and only if ($p = m$ as well as $q = n$). Also, for $N_p^q, N_n^m \in m(N)$, N_p^q is greater than N_n^m, i.e., $N_p^q > N_n^m$ if there exists $N_r^s \in m(N)$ such that $N_p^q = N_n^m + N_r^s$ ($= N_{n+r}^{ms}$), i.e., if ($p > n$ as well as $m|q$). Again, N_p^q is greater than or equal to N_n^m and we write $N_p^q \geq N_n^m$ if $N_p^q > N_n^m$ or $N_p^q = N_n^m$, i.e., if ($p > n$ as well as $m|q$) or ($p = n$ as well as $q = m$). The relation $\geq$ defined on $m(N)$ is a partial order relation which is not total.

Definition 2.16. [6] (Multi number of elements in a multiset) Let N be a single mset. Also, let x is the only element of N with $C_N(x) = n$. Then, we define N_1^n as the multi number of elements in N.Next, we consider an mset M whose support $N^* = \{x_1, x_2, \dots, x_n\}$ is a finite set and multiplicity of each of its elements is finite and is given by the count function as $C_N(x_i) = t_i, i = 1,2, \dots, n$.Then we define the multi number of elements in M as the sum of the multi numbers of the elements in all its single whole submsets, i.e., $N_1^{t_1} + N_1^{t_2} + \cdots N_1^{t_n} = N_n^{t_1 t_2 \dots t_n}$.

Example 2.17.

(i) The multi number of elements in the multiset $\{a, a, a\}$ is N_1^3.
(ii) The multi number of elements in the multiset $\{b, b\}$ is N_1^2.
(iii) The multi number of elements in the multiset $\{a, a, a, b, b, c, c\}$ is $N_1^3 + N_1^2 + N_1^2 = N_2^6 + N_1^2 = N_3^{12}$.
(iv) The multi number of elements in the multiset $\{a, a, a, a, a, a, a, a, a, a, a, a, b, c\}$ is $N_1^{12} + N_1^1 + N_1^1 = N_2^{12} + N_1^1 = N_3^{12}$.
(v) The multi number of roots of the equation $(x-1)^2(x-2)^3 = 0$ is $N_1^2 + N_1^3 = N_2^6$.

Remark 2.18. Now we shall represent multi-integer system in terms of multi-natural numbers that we have already constructed in a previous paper [8]. First of all, we shall introduce the concept of multi-difference system together with some binary operations and order relation. Let us now introduce the following binary relation on $m(N) \times m(N)$:

Definition 2.19. [7] For $(N_a^b, N_c^d), (N_p^q, N_r^s) \in m(N) \times m(N)$, we say (N_a^b, N_c^d) is equivalent to (N_p^q, N_r^s) and we write $(N_a^b, N_c^d) \sim (N_p^q, N_r^s)$ if and only if $N_a^b + N_r^s = N_c^d + N_p^q$.

Theorem 2.20. [7] The relation $\sim$ is an equivalence relation defined on $m(N) \times m(N)$.

Remark 2.21. [7] The set of all equivalence classes of $m(N) \times m(N)$ is denoted by $m_d(Z)$ and is called multi-difference system. An element $[(N_a^b, N_c^d)]$ of $m_d(Z)$ is simply denoted by $[N_a^b, N_c^d]$ and $[N_a^b, N_c^d] = [N_p^q, N_r^s]$ if and only if $N_a^b + N_r^s = N_c^d + N_p^q$.

Remark 2.22. [7] For $[N_a^b, N_c^d], [N_p^q, N_r^s] \in m_d(Z)$, $[N_a^b, N_c^d] = [N_p^q, N_r^s] \Leftrightarrow a - c = p - r$ and $\frac{b}{d} = \frac{q}{s}$.

Lemma 2.23. [7] $[N_a^b, N_c^d] = [N_a^b + N_k^t, N_c^d + N_k^t] = [N_k^t + N_a^b, N_k^t + N_c^d]$ for all $[N_a^b, N_c^d], [N_p^q, N_r^s] \in m_d(Z)$.

Definition 2.24. [7] (Addition on $m_d(Z)$) There exist a well-defined binary operation $\oplus$ on $m_d(Z)$ defined by $[N_a^b, N_c^d] \oplus [N_p^q, N_r^s] = [N_a^b + N_p^q, N_c^d + N_r^s]$, $[N_a^b, N_c^d], [N_p^q, N_r^s] \in m_d(Z)$.

Proposition 2.25. [7] (Properties of $m_d(Z)$)

(i) $\oplus$ is commutative on $m_d(Z)$.
(ii) $\oplus$ is associative on $m_d(Z)$.
(iii) $[N_1^1, N_1^1]$ is the identity element in $m_d(Z)$ for $\oplus$.
(iv) For each $[N_a^b, N_c^d] \in m_d(Z)$, its $\oplus -$ inverse exists and is given by $[N_c^d, N_a^b] \in m_d(Z)$ such that $[N_a^b, N_c^d] \oplus [N_c^d, N_a^b] = [N_1^1, N_1^1]$.

Remark 2.26. [7] $(m_d(Z), \oplus)$ is a commutative group.

Remark 2.27. [7] $[N_a^b, N_c^d] \oplus [N_p^q, N_r^s] = [N_{a+p}^{bq}, N_{c+r}^{ds}]$, $[N_a^b, N_c^d], [N_p^q, N_r^s] \in m_d(Z)$.

Definition 2.28. [9] (Multiplication on $m_d(Z)$) There exists a well-defined binary operation $\odot$ on $m_d(Z)$ defined by $[N_a^b, N_c^d] \odot [N_p^q, N_r^s] = [N_{ap+cr}^{bq}, N_{ar+cp}^{ds}]$, $[N_a^b, N_c^d], [N_p^q, N_r^s] \in m_d(Z)$.

Proposition 2.29. [7] (Properties of multiplication on $m_d(Z)$)

(i) $\odot$ is commutative on $m_d(Z)$.
(ii) $\odot$ is associative on $m_d(Z)$.
(iii) The identity element exists for $\odot$ in $m_d(Z)$ and is $[N_2^1, N_1^1]$.
(iv) $[N_a^1, N_b^1] \odot ([N_p^q, N_r^s] \oplus [N_x^y, N_z^t]) = ([N_a^1, N_b^1] \odot [N_p^q, N_r^s]) \oplus ([N_a^1, N_b^1] \odot [N_x^y, N_z^t])$.

(v) (Remark on distributive property) $[N_a^b, N_c^d] \odot ([N_p^q, N_r^s] \oplus [N_x^y, N_z^t]) \neq ([N_a^b, N_c^d] \odot [N_p^q, N_r^s]) \oplus ([N_a^b, N_c^d] \odot [N_x^y, N_z^t])$ in general.
Actually, $[N_a^b, N_c^d] \odot ([N_p^q, N_r^s] \oplus [N_x^y, N_z^t]) = [N_a^b, N_c^d] \odot [N_{p+x}^{qy}, N_{r+z}^{st}] = [N_{ap+ax+cr+cz}^{bqy}, N_{ar+az+cp+cx}^{dst}]$.
But, $([N_a^b, N_c^d] \odot [N_p^q, N_r^s]) \oplus ([N_a^b, N_c^d] \odot [N_x^y, N_z^t]) = [N_{ap+cr}^{bq}, N_{ar+cp}^{ds}] \oplus [N_{ax+cz}^{by}, N_{az+cx}^{dt}] = [N_{ap+cr+ax+cz}^{b^2qy}, N_{ar+cp+az+cx}^{d^2st}] = [N_2^b, N_1^d] \odot [N_{ap+cr+ax+cz}^{bqy}, N_{ar+cp+az+cx}^{dst}]$.

(vi) (Multi-distributive property) For all $[N_a^b, N_c^d], [N_p^q, N_r^s], [N_x^y, N_z^t] \in m_d(Z)$, $[N_2^b, N_1^d] \odot \left([N_a^b, N_c^d] \odot \left([N_p^q, N_r^s] \oplus [N_x^y, N_z^t]\right)\right) = ([N_a^b, N_c^d] \odot [N_p^q, N_r^s]) \oplus ([N_a^b, N_c^d] \odot [N_x^y, N_z^t])$. Let us define the above property to be the multi-distributive property of $\odot$ over $\oplus$ on $m_d(Z)$.

Definition 2.30. [7] The subset $m_d(N_Z)$ of $m_d(Z)$ is defined by $m_d(N_Z) = \{[N_n^m + N_1^1, N_1^1] \in m_d(Z): N_n^m \in m(N)\}$.

Proposition 2.31. [7] $[N_u^v, N_w^x] \in m_d(N_Z) \Leftrightarrow u - w \in N$ and $x|v$.
Theorem 2.32. [7] For the set $m_d(N_Z)$ the following hold:

(i) $(m_d(N_Z),\oplus)$ is a sub semi group of $(m_d(Z),\oplus)$.
(ii) $(m_d(N_Z),\odot)$ is a sub semi group of $(m_d(Z),\odot)$.
(iii) $(m_d(N_Z),\oplus)$ is isomorphic to $(m(N),+)$ and $(m_d(N_Z),\odot)$ is isomorphic to $(m(N),\cdot)$ as semigroup under the same isomorphism.
(iv) For every $x \in m_d(Z)$, there exist $y, z \in m_d(N_Z)$ such that $x = y \oplus (-z)$.

Definition 2.33. [7] Each member of $m_d(Z)$ is called a multi-integer. Each member of $m_d(N_Z)$ is called a positive multi-integer.

Remark 2.34. [7] $(m_d(N_Z),\oplus)$ is isomorphic to $(m(N),+)$ and $(m_d(N_Z),\odot)$ is isomorphic to $(m(N),\cdot)$ as semigroup under the same isomorphism. So, each member of $m(N)$ is also called a positive multi-integer.

Definition 2.35. [7] The subset $(-m_d(N_Z))$ of $m_d(Z)$ is defined by $(-m_d(N_Z)) = \{[N_c^d, N_a^b]: [N_a^b, N_c^d] \in m_d(N_Z)\}$. Every member of $(-m_d(N_Z))$ is called a negative multi-integer.

Definition 2.36. [7] (Positive multi-integer, negative multi-integer, zero, special multi-integer, multi-zero) $m_d(Z_S) = m_d(Z) - (m_d(N_Z) \cup (-(m_d(N_Z)) \cup \{[N_1^1, N_1^1]\})$. $[N_1^1, N_1^1]$ is called zero and every member of $m_d(Z_S)$ is called a special multi-integer. Any multi-integer of the form $[N_a^p, N_a^q]$ is called a multi-zero which is obviously a special multi-integer or zero.

Theorem 2.37. [7] If product of two multi-integer be zero, then at least one of them must be a multi-zero.
Definition 2.38. [7] (Order on $m_d(Z)$) Let $[N_a^b, N_c^d], [N_p^q, N_r^s] \in m_d(Z)$. Then $[N_a^b, N_c^d] > [N_p^q, N_r^s]$ if and only if $[N_a^b, N_c^d] \oplus (-[N_p^q, N_r^s]) \in m_d(N_Z)$. i.e., there exist $[N_n^m + N_1^1, N_1^1] \in m_d(N_Z)$ such that $[N_a^b, N_c^d] \oplus (-[N_p^q, N_r^s]) = [N_n^m + N_1^1, N_1^1]$ or, $[N_a^b, N_c^d] = [N_p^q, N_r^s] \oplus [N_n^m + N_1^1, N_1^1]$. Also, $[N_a^b, N_c^d] \geq [N_p^q, N_r^s]$ if and only if $[N_a^b, N_c^d] > [N_p^q, N_r^s]$ or $[N_a^b, N_c^d] = [N_p^q, N_r^s]$.

Remark 2.39. [7] $\geq$ defined on $m_d(Z)$ is a partial order relation. So, $(m_d(Z),\geq)$ is a poset but not a chain. Immediately, $(m_d(Z),\geq)$ do not obey law of trichotomy. e.g., $[N_2^3 + N_1^1, N_1^1]$ and $[N_2^2 + N_1^1, N_1^1]$ are two incomparable elements of $(m_d(Z),\geq)$.

Proposition 2.40. [7] (Properties of order)

(i) For $[N_a^b, N_c^d], [N_p^q, N_r^s] \in m_d(Z)$, $[N_a^b, N_c^d] > [N_p^q, N_r^s] \Rightarrow a - c > p - r$ and $dq|bs$ and conversely.
(ii) For all $[N_a^b, N_c^d] \in m_d(Z)$, $[N_a^b, N_c^d] \ngtr [N_a^b, N_c^d]$.
(iii) For $[N_a^b, N_c^d], [N_e^f, N_g^h] \in m_d(Z)$, $[N_a^b, N_c^d] > [N_e^f, N_g^h] \Leftrightarrow [N_a^b, N_c^d] \oplus [N_u^v, N_w^x] > [N_e^f, N_g^h]$ for all $[N_u^v, N_w^x] \in m_d(N_Z)$.

(iv) For $[N_a^b, N_c^d], [N_e^f, N_g^h], [N_u^v, N_w^x], [N_p^q, N_r^s] \in m_d(Z)$, $[N_a^b, N_c^d] > [N_e^f, N_g^h]$ and $[N_u^v, N_w^x] > [N_p^q, N_r^s] \Rightarrow [N_a^b, N_c^d] \oplus [N_u^v, N_w^x] > [N_e^f, N_g^h] \oplus [N_p^q, N_r^s]$.

(v) For $[N_a^b, N_c^d], [N_e^f, N_g^h] \in m_d(Z)$, $[N_a^b, N_c^d] \geq [N_e^f, N_g^h] \Rightarrow [N_a^b, N_c^d] \oplus [N_2^1, N_1^1] > [N_e^f, N_g^h]$.

(vi) For all $[N_a^b, N_c^d] \in m_d(Z)$, $[N_a^b, N_c^d] \oplus [N_e^f, N_g^h] > [N_a^b, N_c^d]$ for all $[N_e^f, N_g^h] \in m_d(N_Z)$.

(vii) For $[N_a^b, N_c^d], [N_e^f, N_g^h] \in m_d(Z)$, $[N_a^b, N_c^d] > [N_e^f, N_g^h] \Leftrightarrow [N_a^b, N_c^d] \odot [N_p^q, N_r^s] > [N_e^f, N_g^h] \odot [N_p^q, N_r^s]$ for all $[N_p^q, N_r^s] \in m_d(N_Z)$.

Definition 2.41. [7] (General multiset, Real multiset and Natural multiset)

(i) Let X be a non-empty set. A general multiset (General mset in short) M drawn from X is characterized by a relation ρ_M between X and R (R being the set of all real numbers).
If $(x, r) \in \rho_M$ for some $x \in X$ and $r \in R - \{0\}$, then we represent it by writing $X_x^r \in M$.

(ii) Let X be a non-empty set. A real multiset (Real mset in short) M drawn from X is characterized by a function $Count_M$ or $C_M: X \to R$.
If $C_M(x) = r$ for some $x \in X$ and $r \in R - \{0\}$, then we represent it by writing $X_x^r \in M$. Also, we shall denote a real mset M drawn from X as $\{X_{x_1}^{k_1}, X_{x_2}^{k_2}, \ldots, X_{x_n}^{k_n}, \ldots\}$ where $C_M(x_i) = k_i$, $x_i \in X$ and $r \in R - \{0\}$.

(iii) Let X be a non-empty set. A natural multiset (Natural mset in short) M drawn from X is characterized by a function $Count_M$ or $C_M: X \to N \cup \{0\}$.
If $C_M(x) = r$ for some $x \in X$ and $r \in N - \{0\}$, then we represent it by writing $X_x^r \in M$. Also, we shall denote a natural mset M drawn from X as $\{X_{x_1}^{k_1}, X_{x_2}^{k_2}, \ldots, X_{x_n}^{k_n}, \ldots\}$ where $C_M(x_i) = k_i$, $x_i \in X$ and $r \in N - \{0\}$. $k_i \in N - \{0\}$ is called the multiplicity of the element $x_i \in X$ in M.

Example 2.42. Consider the set $X = \{a, b, c\}$. Consider the relation ρ_M between X and R where $\rho_M = \{\left(a, \frac{1}{4}\right), (b, 3), (b, \sqrt{2})\}$. Then ρ_M represents a general mset M drawn from X which is given by $\{X_a^{\frac{1}{4}}, X_b^3, X_b^{\sqrt{2}}\}$. Next consider the function $C_M: X \to R$ defined by $C_M(a) = \frac{1}{4}$, $C_M(b) = 3$, $C_M(c) = 0$. Then C_M represents a real mset M drawn from X which is given by $M = \{X_a^{\frac{1}{4}}, X_b^3\}$. Finally, consider the function $C_M: X \to N \cup \{0\}$ defined by $C_M(a) = 1$, $C_M(b) = 3$, $C_M(c) = 0$. Then C_M represents a natural mset M drawn from X which is given by $M = \{X_a^1, X_b^3\}$. It is worth noting that $m(N)$ is a general mset drawn from N.

Remark 2.43. [7]

(i) Clearly, general mset is a generalization of real mset. Also, real mset is a generalization of natural mset.

(ii) Let A' and B' be two general msets drawn from the sets A and B respectively. If for $a \in A \cap B$ and $r \in R - \{0\}$, $A_a^r \in A'$ and $B_a^r \in B'$, then we shall consider $A_a^r = B_a^r$.

(iii) We note that for all $i, j \in N$, Z_j^i and N_j^i both are immediately identical. i.e., $Z_j^i = N_j^i$ for all $i, j \in N$.

(iv) Let X be a non-empty set. Let us denote the general mset drawn from X and characterized by the universal relation between X and R as $\pi(X)$ and accordingly denote the relation $X \times R$ between X and R as $\rho_\pi(X)$ as the most general mset drawn from X.

Theorem 2.44. [7] (Isomorphism theorem) Let us consider the general mset $m(\hat{Z})$ drawn from Z characterized by universal relation $\rho_{m(\hat{Z})} = Z \times Q^+$ (Q^+ is the set of all positive rational numbers). i.e., $Z_p^q \in m(\hat{Z})$ if and only if $p \in Z$ and $q \in Q^+$. Consider two binary operations $\widehat{\oplus}$ and $\widehat{\odot}$ on $m(\hat{Z})$ as follows: for $Z_p^q, Z_r^s \in m(\hat{Z})$, $Z_p^q \widehat{\oplus} Z_r^s = Z_{p+r}^{qs}$ and $Z_p^q \widehat{\odot} Z_r^s = Z_{pr}^{qs}$. Also define $\widehat{\geq}$ on $m(\hat{Z})$ as follows: for $Z_p^q, Z_r^s \in m(\hat{Z})$, $Z_p^q \widehat{>} Z_r^s$ if and only if there exists $Z_a^b \in m(\hat{Z})$ with $a, b \in N$ such that $Z_p^q = Z_r^s \widehat{\oplus} Z_a^b$. For $Z_p^q, Z_r^s \in m(\hat{Z})$ define $Z_p^q = Z_r^s$ if and only if $p = r$ and $q = s$. Also, for $Z_p^q, Z_r^s \in m(\hat{Z})$ define $Z_p^q \widehat{\geq} Z_r^s$ if and only if $Z_p^q \widehat{>} Z_r^s$ or $Z_p^q = Z_r^s$. Then $(m_d(Z), \oplus, \odot, \geq)$ $(m(\hat{Z}), \widehat{\oplus}, \widehat{\odot}, \widehat{\geq})$ are isomorphic as the mapping $\tau: m_d(Z) \to m(\hat{Z})$ defined by $\tau([N_a^b, N_c^d]) = Z_{a-c}^{\frac{b}{d}}$, $[N_a^b, N_c^d] \in m_d(Z)$ is an isomorphism.

Remark 2.45. [7] (Properties of $(m(\hat{Z}), \widehat{\oplus}, \widehat{\odot}, \widehat{\geq})$)

Since $(m_d(Z), \oplus, \odot, \geq)$ and $(m(\hat{Z}), \widehat{\oplus}, \widehat{\odot}, \widehat{\geq})$ are isomorphic, so $(m(\hat{Z}), \widehat{\oplus})$ is a commutative group, $(m(\hat{Z}), \widehat{\odot})$ is a commutative monoid and $\widehat{\odot}$ obeys multi-distributive property over $\widehat{\oplus}$. Also, $(m(\hat{Z}), \widehat{\geq})$ is a poset. Moreover, $\widehat{\geq}$ defined on $m(\hat{Z})$ is an extension of $\geq$ defined on $m(N)$.

Remark 2.46. [7] $(m(\hat{Z}), \widehat{\oplus})$ is a commutative group and $(m(\hat{Z}), \widehat{\odot})$ is a commutative monoid but $(m(\hat{Z}), \widehat{\oplus}, \widehat{\odot})$ is not a ring, since $\widehat{\odot}$ can not be distributed over $\widehat{\oplus}$. But $\widehat{\odot}$ obeys multi-distributive property over $\widehat{\oplus}$.

Definition 2.47. [7] (General mset drawn from a ring) Let $(X, +, \cdot)$ be a ring. Let M be a general mset drawn from X. Consider two functions $\oplus: M \times M \to \pi(X)$ and $\odot: M \times M \to \pi(X)$ defined as follows: for $X_a^r, X_b^s \in M$, $X_a^r \oplus X_b^s = X_{a+b}^{rs}$ and $X_a^r \odot X_b^s = X_{ab}^{rs}$. $\oplus$ and $\odot$ respectively called m-addition and m-multiplication defined on M induced by the ring $(X, +, \cdot)$. Also, let M be closed under $\oplus$ and $\odot$. Then immediately, $\oplus$ obeys commutative and associative property on M. So, $(M, \oplus)$ is a commutative semi group. Also, immediately, $\odot$ obeys associative property on M. So, $(M, \odot)$ is a semi group. M is defined to be a general mset drawn from the ring $(X, +, \cdot)$.

Theorem 2.48. [7] Let M be a general mset drawn from a ring $(X, +, \cdot)$. Then, $\odot$ obey multi-distributive property over $\oplus$.

Definition 2.49. [7] (Multi-ring) Let M be a general mset drawn from a ring $(X, +, \cdot)$. Let $\oplus$ and $\odot$ are called m-addition and m-multiplication respectively defined on M induced by the ring $(X, +, \cdot)$. If the structure $(M, \oplus, \odot)$ satisfies the followings:

(1) $(M, \oplus)$ is an abelian group
(2) $(M, \odot)$ is a semigroup and
(3) $\odot$ is distributive over $\oplus$, then $(M, \oplus, \odot)$ is called a multi-ring induced by the ring $(X, +, \cdot)$.

Theorem 2.50. [7]) Let M be a general mset drawn from a ring $(X, +, \cdot)$. Let $\oplus$ and $\odot$ are called m-addition and m-multiplication respectively defined on M induced by the ring $(X, +, \cdot)$. Then $(M, \oplus, \odot)$ will be a multi-ring induced by the ring $(X, +, \cdot)$ if and only if the following conditions are satisfied:

(1) There exists $X_\theta^1 \in M$ (θ being the zero element in the ring $(X, +, \cdot)$).
(2) For $a \in X$ and $r \in R - \{0\}$, $X_a^r \in M \Rightarrow X_{(-a)}^{\frac{1}{r}} \in M$.

Example 2.51. [7]

(i) Let us consider the ring $(X, +, \cdot)$ where $X = Z_4$, the set of all residue classes modulo 4, also, $+$ and $\cdot$ are respectively addition and multiplication modulo 4. Consider the general mset M characterized by the relation $\rho_M = X \times G$ where $G = \{2^n : n \in Z\}$ between X and G. Then, for all $a \in X$ and for all $r \in G$, $X_a^r \in M$. Let $\oplus$ and $\odot$ are m-addition and m-multiplication respectively defined on M induced by the ring $(X, +, \cdot)$. Then $(M, \oplus, \odot)$ forms a multi-ring induced by the ring $(X, +, \cdot)$.
(ii) $(m(\hat{Z}), \widehat{\oplus}, \widehat{\odot})$ is a a multi-ring induced by the ring $(Z, +, \cdot)$.

Remark 2.52. [7] Let $(M, \oplus, \odot)$ be a multi-ring induced by the ring $(X, +, \cdot)$ where M is a general mset drawn from the ring $(X, +, \cdot)$. θ be the zero element in the ring $(X, +, \cdot)$. Then X_θ^1 must be the zero element in $(M, \oplus, \odot)$. Let us also define any element in M of the form X_θ^r for some $r \in R - \{0\}$ to be the multi-zero elements of M such that the product of any element of the multi-ring with a multi-zero element of the same is again a multi zero of the multi-ring. Clearly, the zero element in a multi-ring is a multi-zero element.

Remark 2.53. [7] In the multi-ring $(m(\hat{Z}), \widehat{\oplus}, \widehat{\odot})$ induced by the ring $(Z, +, \cdot)$, the non-zero multi-zeros are only divisors of zero.

Theorem .2.54. [7] In the multi-ring, the non-zero multi-zero elements are divisors of zero.
Definition 2.55. [7] A multi-ring is said to have no non-multi-zero divisors of zero if its non-zero multi-zero elements are the only divisors of zero.

Example 2.56. [7] The multi-ring $(m(\hat{Z}), \widehat{\oplus}, \widehat{\odot})$ induced by the ring $(Z, +, \cdot)$ has no non-multi-zero divisors of zero.

Remark .2.57. [7] Let $(M,\oplus,\odot)$ be a multi-ring induced by the ring $(X,+,\cdot)$ with divisors of zero. Let θ be the zero element of the ring $(X,+,\cdot)$. As $(X,+,\cdot)$ is a ring with divisors of zero, so, there exists two non zero elements a and b in the ring $(X,+,\cdot)$ such that $a \cdot b = \theta$. Now, for some $r,s \in R-\{0\}$, let $X_a^r, X_b^s \in M$. Then, $X_a^r \odot X_b^s = X_{ab}^{rs} = X_\theta^{rs} \in M$ (since M is closed under $\odot$). Again, X_a^r and X_b^s both are non-zero non-multi-zero elements of $(M,\oplus,\odot)$. So, X_a^r and X_b^s are non-multi-zero divisors of the multi-ring $(M,\oplus,\odot)$.

Example 2.58. [7] Consider the multi-ring $(M,\oplus,\odot)$ induced by the ring $(X,+,\cdot)$ as mentioned in (i) of Example 2.51. where $X = Z_4$. Then, for $X_{[2]}^2, X_{[2]}^{\frac{1}{2}} \in M$, $X_{[2]}^2 \odot X_{[2]}^{\frac{1}{2}} = X_{[0]}^1$ which is the zero element of the multi-ring $(M,\oplus,\odot)$. Also, $X_{[2]}^2$ and $X_{[2]}^{\frac{1}{2}}$ are the non-zero non-multi-zero elements of the multi-ring $(M,\oplus,\odot)$. So, the multi-ring $(M,\oplus,\odot)$ contains multi-divisors of zero.

Definition 2.59. [7] (Multi-integral domain) Let M be a general mset drawn from a ring $(X,+,\cdot)$ (or, an integral domain $(X,+,\cdot)$) . Let $\oplus$ and $\odot$ are m-addition and m-multiplication respectively defined on M induced by the ring $(X,+,\cdot)$ (or, an integral domain $(X,+,\cdot)$). If the structure $(M,\oplus,\odot)$ satisfies the followings:

(1) $(M,\oplus)$ is a commutative group
(2) $(M,\odot)$ is a commutative monoid
(3) $\odot$ is distributive over $\oplus$ and
(4) M has no non-multi-zero divisors of zero, then $(M,\oplus,\odot)$ is called a multi-integral domain induced by the ring $(X,+,\cdot)$ (or, the integral domain $(X,+,\cdot)$).
It is worth noting that if M be a general mset drawn from an integral domain $(X,+,\cdot)$ which is closed under $\oplus$ and $\odot$, then immediately $(M,\oplus,\odot)$ has no non-multi-zero divisors of zero.

Example 2.60. [7] $(m(\hat{Z}),\widehat{\oplus},\widehat{\odot})$ is a multi-integral domain induced by the integral domain $(Z,+,\cdot)$.
Example 2.61. [7] The multi-ring $(M,\oplus,\odot)$ induced by the ring $(X,+,\cdot)$ as mentioned in Example 2.51. and Example 2.58. where $X = Z_4$ is not a multi-integral domain.
Remark 2.62. [7] $(m(\hat{Z}),\widehat{\oplus},\widehat{\odot},\widehat{\geq})$ is a partially ordered multi-integral domain induced by the integral domain $(Z,+,\cdot)$.

Definition 2.63. [7] (Definition of Multi-integer system) A partially ordered multi-integral domain $(M,\oplus,\odot,\geq)$ is called a multi-integer system if there exists a subset N_M of M such that

(1) Both $(N_M,\oplus)$ and $(N_M,\odot)$ are semigroups and under the same isomorphism $\emptyset: N_M \to N$ we have $(N_M,\oplus) \cong (\mathrm{m(N)},+)$ and $(N_M,\odot) \cong (\mathrm{m(N)},\cdot)$ as semi group. Furthermore, for every $x,y \in N_M$, we have $x > y \Rightarrow \emptyset(x) > \emptyset(y)$.
(2) For every $x \in M$, there exists $y,z \in N_M$ such that $x = y \oplus (-z)$.

Theorem 2.64. [7] (Existence and uniqueness of multi-integer system) Multi-integer system exists and any two multi-integer systems are isomorphic.
***Remark 2.65.* [7]** $(m(\hat{Z}),\widehat{\oplus},\widehat{\odot},\widehat{\geq})$ multi-integer system. Also, multi-integer system is unique.
So, $(m(\hat{Z}),\widehat{\oplus},\widehat{\odot},\widehat{\geq})$ can be considered as the multi-integer system. Any multi-integer system is afterwards denoted by $(m(Z),\oplus,\odot)$ where $m(Z)$ is the general mset drawn from Z characterized by the universal relation $Z \times Q^+$, i.e., $Z_p^q \in m(Z)$ if and only if $p \in Z$ and $q \in Q^+$. Binary operations $\oplus$ and $\odot$ are defined on $m(Z)$ as follows: for $Z_p^q, Z_r^s \in m(Z)$, $Z_p^q \oplus Z_r^s = Z_{p+r}^{qs}$ and $Z_p^q \odot Z_r^s = Z_{pr}^{qs}$. $>$ is defined on $m(Z)$ as follows: for $Z_p^q, Z_r^s \in m(Z)$, $Z_p^q > Z_r^s$ if and only if there exists $Z_a^b \in m(Z)$ with $a,b \in N$ such that $Z_p^q = Z_r^s \oplus Z_a^b$. Also, for $Z_p^q, Z_r^s \in m(Z)$, $Z_p^q \geq Z_r^s$ if and only if $Z_p^q > Z_r^s$ or $Z_p^q = Z_r^s$. The copy of the multi-natural numbers embedded in $m(Z)$ is still denoted by $m(N)$ and it has all the properties that we have proven in paper [6] if we consider it in isolation.

Example 2.66. Consider three multi-integers $Z_5^3, Z_3^4, Z_{-3}^{\frac{3}{5}}$. Then $Z_5^3 \oplus Z_3^4 = Z_{5+3}^{3\cdot 4} = Z_8^{12}$ and $Z_3^4 \odot Z_{-3}^{\frac{3}{5}} = Z_{3\cdot(-3)}^{4\cdot\frac{3}{5}} = Z_{-9}^{\frac{12}{5}}$.

III. The Multi-Fractional System

Here we shall represent multi-rational number system in terms of multi-integers that we have already constructed in a previous paper [7]. First of all, we shall introduce the concept of Multi-Fractional System together with some binary operations and order relations. Let us now introduce the following binary operation on m(Z)$\times$ $(m(Z) -$

$\{Z_0^q : q \in Q^+\}$). Let us rename the set $(m(Z) - \{Z_0^q : q \in Q^+\})$ as $m(Z_0)$ and it is the set of all non-multi-zero multi-integers.

Definition 3.1. For $(Z_a^b, Z_c^d), (Z_p^q, Z_r^s) \in m(Z) \times m(Z_0)$, we say (Z_a^b, Z_c^d) is equivalent to (Z_p^q, Z_r^s) and we write $(Z_a^b, Z_c^d) \sim (Z_p^q, Z_r^s)$ if and only if $Z_a^b \odot Z_r^s = Z_c^d \odot Z_p^q$.

Theorem 3.2. For $Z_a^b \in m(Z_0)$ and $Z_p^q, Z_r^s \in m(Z)$, $Z_a^b \odot Z_p^q = Z_a^b \odot Z_r^s \Rightarrow Z_p^q = Z_r^s$, $m(Z_0)$ being the set of all non-multi-zero multi-integers.

Proof: $Z_a^b \odot Z_p^q = Z_a^b \odot Z_r^s \Rightarrow Z_{ap}^{bq} = Z_{ar}^{bs} \Rightarrow ap = ar$ and $bq = bs \Rightarrow p = r$ and $q = s$ (Since, $Z_a^b \in m(Z_0)$, so, $a \neq 0$ and $b \neq 0$) $\Rightarrow Z_p^q = Z_r^s$. We can prove the second part in a similar argument.

Theorem 3.3. The relation $\sim$ is an equivalence relation defined on $m(Z) \times m(Z_0)$.

Proof: Since for all $(Z_a^b, Z_c^d) \in m(Z) \times m(Z_0)$, we have $Z_a^b \odot Z_c^d = Z_c^d \odot Z_a^b$. So, for all $(Z_a^b, Z_c^d) \in m(Z) \times m(Z_0)$, $(Z_a^b, Z_c^d) \sim (Z_a^b, Z_c^d)$. Therefore, $\sim$ is a reflexive relation on $m(Z) \times m(Z_0)$.

Next, for $(Z_a^b, Z_c^d), (Z_p^q, Z_r^s) \in m(Z) \times m(Z_0)$, let $(Z_a^b, Z_c^d) \sim (Z_p^q, Z_r^s)$. Then, $Z_a^b \odot Z_r^s = Z_c^d \odot Z_p^q \Rightarrow Z_r^s \odot Z_a^b = Z_p^q \odot Z_c^d \Rightarrow (Z_p^q, Z_r^s) \sim (Z_a^b, Z_c^d)$. Therefore, $\sim$ is a symmetric relation on $m(Z) \times m(Z_0)$.

Finally, for $(Z_a^b, Z_c^d), (Z_p^q, Z_r^s), (Z_u^v, Z_w^x) \in m(Z) \times m(Z_0)$, let $(Z_a^b, Z_c^d) \sim (Z_p^q, Z_r^s)$ also $(Z_p^q, Z_r^s) \sim (Z_u^v, Z_w^x)$. Then $Z_a^b \odot Z_r^s = Z_c^d \odot Z_p^q$ as well as $Z_p^q \odot Z_w^x = Z_r^s \odot Z_u^v$. Therefore, $(Z_a^b \odot Z_r^s) \odot Z_w^x = (Z_c^d \odot Z_p^q) \odot Z_w^x$ and $(Z_p^q \odot Z_w^x) \odot Z_c^d = (Z_r^s \odot Z_u^v) \odot Z_c^d$ so that $(Z_a^b \odot Z_r^s) \odot Z_w^x = (Z_c^d \odot Z_p^q) \odot Z_w^x = Z_c^d \odot (Z_p^q \odot Z_w^x) = (Z_p^q \odot Z_w^x) \odot Z_c^d = (Z_r^s \odot Z_u^v) \odot Z_c^d \Rightarrow (Z_r^s \odot Z_a^b) \odot Z_w^x = (Z_r^s \odot Z_u^v) \odot Z_c^d \Rightarrow Z_r^s \odot (Z_a^b \odot Z_w^x) = Z_r^s \odot (Z_u^v \odot Z_c^d) \Rightarrow Z_a^b \odot Z_w^x = Z_u^v \odot Z_c^d$ (By theorem 3.2., since $Z_r^s \in m(Z_0)$) $\Rightarrow Z_a^b \odot Z_w^x = Z_c^d \odot Z_u^v$.

Thus, $(Z_a^b, Z_c^d) \sim (Z_u^v, Z_w^x)$. Therefore, $\sim$ is a transitive relation on $m(Z) \times m(Z_0)$.

Therefore, $\sim$ is a equivalence relation on $m(Z) \times m(Z_0)$.

Remark 3.4. Let us denote the set of all equivalence classes of $m(Z) \times m(Z_0)$ by $m_f(Q)$ and we call it as multi-fractional system. An element $[(Z_a^b, Z_c^d)]$ on $m_f(Q)$ will now be simply be denoted by $[Z_a^b, Z_c^d]$ and accordingly $[Z_a^b, Z_c^d] = [Z_p^q, Z_r^s]$ if and only if $Z_a^b \odot Z_r^s = Z_c^d \odot Z_p^q$. Now we have only produced the elements of $m_f(Q)$. A bunch of elements can hardly be a system. We still need to define appropriate binary operations and order relations on it just as we did for $m_d(Z)$ [9]. Before we do so, let us note the following elementary properties of $m_f(Q)$.

Remark 3.5. For $(Z_a^b, Z_c^d), (Z_p^q, Z_r^s) \in m_f(Q)$, $[Z_a^b, Z_c^d] = [Z_p^q, Z_r^s] \Leftrightarrow Z_a^b \odot Z_r^s = Z_c^d \odot Z_p^q \Leftrightarrow Z_{ar}^{bs} = Z_{cp}^{dq} \Leftrightarrow ar = cp$ and $bs = dq \Leftrightarrow \frac{a}{c} = \frac{p}{r}$ and $\frac{b}{d} = \frac{q}{s}$.

Lemma 3.6. For $[Z_a^b, Z_c^d] \in m_f(Q)$, and for all $Z_q^p \in m(Z_0)$, $[Z_a^b, Z_c^d] = [Z_p^q \odot Z_a^b, Z_p^q \odot Z_c^d] = [Z_a^b \odot Z_p^q, Z_c^d \odot Z_p^q]$.

Proof: $[Z_a^b, Z_c^d] = [Z_p^q \odot Z_a^b, Z_p^q \odot Z_c^d] \Leftrightarrow Z_a^b \odot (Z_p^q \odot Z_c^d) = Z_c^d \odot (Z_p^q \odot Z_a^b) \Leftrightarrow Z_a^b \odot (Z_c^d \odot Z_p^q) = Z_c^d \odot (Z_a^b \odot Z_p^q) \Leftrightarrow (Z_a^b \odot Z_c^d) \odot Z_p^q = (Z_c^d \odot Z_a^b) \odot Z_p^q \Leftrightarrow (Z_a^b \odot Z_c^d) \odot Z_p^q = (Z_a^b \odot Z_c^d) \odot Z_p^q$ which is a tautology. Also, a similar tautology can be established for the second part. Hence the result.

Lemma 3.7. $[Z_a^b, Z_c^d] = [Z_0^1, Z_1^1]$ if and only if $a = 0$ and $b = d$.

Lemma 3.8. $[Z_a^b, Z_c^d] = [Z_1^1, Z_1^1]$ if and only if $a = c$ and $b = d$.

Definition 3.9. (Addition on $m_f(Q)$) There exists a well-defined binary operation $\boxplus$ on $m_f(Q)$ defined by $[Z_a^b, Z_c^d] \boxplus [Z_p^q, Z_r^s] = [(Z_a^b \odot Z_r^1) \oplus (Z_c^1 \odot Z_p^q), Z_c^d \odot Z_r^s] = [Z_{ar+cp}^{bq}, Z_{cr}^{ds}]$, $[Z_a^b, Z_c^d], [Z_p^q, Z_r^s] \in m_f(Q)$.

Proof: To show that $\boxplus$ is well-defined, we need to show that for any $[Z_a^b, Z_c^d], [Z_p^q, Z_r^s] \in m_f(Q)$, there is one and only one image under $\boxplus$.

Hence let, $[Z_a^b, Z_c^d] = [Z_{a'}^{b'}, Z_{c'}^{d'}]$ and $[Z_p^q, Z_r^s] = [Z_{p'}^{q'}, Z_{r'}^{s'}]$.

Now, $[Z_a^b, Z_c^d] = [Z_{a'}^{b'}, Z_{c'}^{d'}] \Rightarrow ac' = ca'$ and $bd' = db'$, also, $[Z_p^q, Z_r^s] = [Z_{p'}^{q'}, Z_{r'}^{s'}] \Rightarrow pr' = rp'$ and $qs' = sq'$.

Then $[Z_a^b, Z_c^d] \boxplus [Z_p^q, Z_r^s] = [Z_{ar+cp}^{bq}, Z_{cr}^{ds}]$ and $[Z_{a'}^{b'}, Z_{c'}^{d'}] \boxplus [Z_{p'}^{q'}, Z_{r'}^{s'}] = [Z_{a'r'+c'p'}^{b'q'}, Z_{c'r'}^{d's'}]$.

Also, $(ar + cp)c'r' = cr(a'r' + c'p')$ and $bqd's' = dsb'q'$.

Therefore, $[Z_{ar+cp}^{bq}, Z_{cr}^{ds}] = [Z_{a'r'+c'p'}^{b'q'}, Z_{c'r'}^{d's'}] \Rightarrow [Z_a^b, Z_c^d] \boxplus [Z_p^q, Z_r^s] = [Z_p^q, Z_r^s] = [Z_{p'}^{q'}, Z_{r'}^{s'}]$.

Therefore, $\boxplus$ is well-defined.

Proposition 3.10. (Properties of addition on $m_f(Q)$) Following properties of addition can be deduced:

(i) $\boxplus$ is commutative on $m_f(Q)$.

(ii) $\boxplus$ is associative on $m_f(Q)$.

(iii) $[Z_0^1, Z_1^1]$ is the identity element in $m_f(Q)$ for $\boxplus$.

(iv) For each $[Z_a^b, Z_c^d] \in m_f(Q)$, its $\boxplus$- inverse exists and is given by $[Z_{-a}^{\frac{1}{b}}, Z_c^{\frac{1}{d}}] \in m_f(Q)$ denoted by $([Z_a^b, Z_c^d])$.

(v) $(m_f(Q), \boxplus)$ is a commutative group.

Proof: The proof is immediate.

Definition 3.11. (Multiplication on $m_f(Q)$) There exists a well-defined binary operation $\boxdot$ on $m_f(Q)$ defined by $[Z_a^b, Z_c^d] \boxdot [Z_p^q, Z_r^s] = [Z_a^b \odot Z_p^q, Z_c^d \odot Z_r^s] = [Z_{ap}^{bq}, Z_{cr}^{ds}]$, $[Z_a^b, Z_c^d], [Z_p^q, Z_r^s] \in m_f(Q)$.

To show that $\boxdot$ is well defined, we need to show that for any $[Z_a^b, Z_c^d], [Z_p^q, Z_r^s] \in m_f(Q)$, there is one and only one image under $\boxdot$.

Hence let, $[Z_a^b, Z_c^d] = [Z_{a'}^{b'}, Z_{c'}^{d'}]$ and $[Z_p^q, Z_r^s] = [Z_{p'}^{q'}, Z_{r'}^{s'}]$.

Now, $[Z_a^b, Z_c^d] = [Z_{a'}^{b'}, Z_{c'}^{d'}] \Rightarrow ac' = ca'$ and $bd' = db'$, also, $[Z_p^q, Z_r^s] = [Z_{p'}^{q'}, Z_{r'}^{s'}] \Rightarrow pr' = rp'$ and $qs' = sq'$.

Then $[Z_a^b, Z_c^d] \boxdot [Z_p^q, Z_r^s] = [Z_{ap}^{bq}, Z_{cr}^{ds}]$ and $[Z_{a'}^{b'}, Z_{c'}^{d'}] \boxdot [Z_{p'}^{q'}, Z_{r'}^{s'}] = [Z_{a'p'}^{b'q'}, Z_{c'r'}^{d's'}]$.

Also, $ac'pr' = ca'rp'$ and $bqd's' = dsb'q'$.

Therefore, $Z_{ac'pr'}^{bqd's'} = Z_{ca'rp'}^{dsb'q'} \Rightarrow Z_{ap}^{bq} \odot Z_{c'r'}^{d's'} = Z_{cr}^{ds} \odot Z_{a'p'}^{b'q'}$

$\Rightarrow [Z_{ap}^{bq}, Z_{cr}^{ds}] = [Z_{a'p'}^{b'q'}, Z_{c'r'}^{d's'}] \Rightarrow [Z_a^b, Z_c^d] \boxdot [Z_p^q, Z_r^s] = [Z_{a'}^{b'}, Z_{c'}^{d'}] \boxdot [Z_{p'}^{q'}, Z_{r'}^{s'}]$

Therefore, $\boxdot$ is well-defined.

Proposition 3.12. (Properties of multiplication on $m_f(Q)$)

(i) $\boxdot$ is commutative on $m_f(Q)$.

(ii) $\boxdot$ is associative on $m_f(Q)$.

(iii) $[Z_1^1, Z_1^1]$ is the identity element in $m_f(Q)$ for $\boxdot$.

Proof: For all $[Z_a^b, Z_c^d] \in m_f(Q)$, $[Z_a^b, Z_c^d] \boxdot [Z_1^1, Z_1^1] = [Z_a^b \odot Z_1^1, Z_c^d \odot Z_1^1] = [Z_a^b, Z_c^d]$.

(iv) For any element $[Z_a^b, Z_c^d] \in m(Z_0) \times m(Z_0)$, its $\boxdot$-inverse exists and is given by $[Z_c^d, Z_a^b]$ denoted by $[Z_a^b, Z_c^d]^{-1}$.

(v) Let us denote $m(Z_0) \times m(Z_0)$ as $m_f(Q_0)$, in fact $(m_f(Q_0), \boxdot)$ is a commutative group.

(vi) (Remark on distributive property) $[Z_a^b, Z_c^d] \boxdot ([Z_p^q, Z_r^s] \boxplus [Z_u^v, Z_x^y]) \neq ([Z_a^b, Z_c^d] \boxdot [Z_p^q, Z_r^s]) \boxplus ([Z_a^b, Z_c^d] \boxdot [Z_u^v, Z_x^y])$ in general.

Actually, $[Z_a^b, Z_c^d] \boxdot ([Z_p^q, Z_r^s] \boxplus [Z_u^v, Z_x^y]) = [Z_a^b, Z_c^d] \boxdot [Z_{px+ru}^{qv}, Z_{rx}^{sy}] = [Z_{a(px+ru)}^{bqv}, Z_{crx}^{dsy}]$.

But, $([Z_a^b, Z_c^d] \boxdot [Z_p^q, Z_r^s]) \boxplus ([Z_a^b, Z_c^d] \boxdot [Z_u^v, Z_x^y]) = [Z_{ap}^{bq}, Z_{cr}^{ds}] \boxplus [Z_{au}^{bv}, Z_{cx}^{dy}] = [Z_{apcx+crau}^{b^2qv}, Z_{c^2rx}^{d^2sy}]$ $= [Z_{ac(px+ru)}^{b^2qv}, Z_{c^2rx}^{d^2sy}]$.

(vii) (Multi-distributive property) For all $[Z_a^b, Z_c^d], [Z_p^q, Z_r^s], [Z_u^v, Z_x^y] \in m_f(Q)$, $[Z_1^b, Z_1^d] \boxdot ([Z_a^b, Z_c^d] \boxdot ([Z_p^q, Z_r^s] \boxplus [Z_u^v, Z_x^y])) = ([Z_a^b, Z_c^d] \boxdot [Z_p^q, Z_r^s]) \boxplus ([Z_a^b, Z_c^d] \boxdot [Z_u^v, Z_x^y])$. Let us define the above property to be the multi-distributive property of $\boxdot$ over $\boxplus$ on $m_f(Q)$.

Remark 3.13. (Order on $m_f(Q)$) After defining two binary operations on $m_f(Q)$, the next natural thing is to order the elements of $m_f(Q)$. Our aim is to define an order that will make $m_f(Q)$ a partially ordered multi-field. In this connection, we shall first define subsets of $m_f(Q)$ that serves as the set of multi-natural numbers and multi-integers. Intuitively, these sets should turn out eventually to resemble $m(N)$ and $m(Z)$. Also, to define an appropriate order, the main job is to identify the subsets of $m_f(Q)$ that will serve as the set of positive elements. So, we are representing the following notation:

Proposition 3.14. The subset $m_f^+(Q)$ of $m_f(Q_0)$ defined by $m_f^+(Q) = \{[Z_a^b, Z_c^d] \in m_f(Q) : \frac{a}{c} > 0$ and $\frac{b}{d} \in N\}$ is well defined.

Proof: To show that $m_f^+(Q)$ is well-defined, we need to show that any $[Z_a^b, Z_c^d]$ cannot be both in and out of the set.
Hence let, $[Z_a^b, Z_c^d] = [Z_p^q, Z_r^s]$ and suppose that $[Z_a^b, Z_c^d] \in m_f^+(Q)$.
Then, $Z_a^b \odot Z_r^s = Z_c^d \odot Z_p^q \Rightarrow ar = cp$ and $bs = dq$.
Also, $\frac{a}{c} > 0$ and $\frac{b}{d} \in N$.
So, $\frac{p}{r} > 0$ and $\frac{q}{s} \in N$.
Therefore, $[Z_p^q, Z_r^s] \in m_f^+(Q)$.
Hence, $m_f^+(Q)$ is well-defined subset of $m_f(Q)$.

Proposition 3.15. The subset $m_f^-(Q)$ of $m_f(Q)$ defined by $m_f^-(Q) = \{-[Z_a^b, Z_c^d] \in m_f(Q): [Z_a^b, Z_c^d] \in m_f^+(Q)\}$ is well-defined.
Proof: The proof is immediate.

Definition 3.16. We define the subset $m_f(Z_Q) = \{[Z_a^b, Z_1^1] \in m_f(Q): Z_a^b \in m(Z)\}$. The following theorem tells us that $m_f(Z_Q)$ appears to be indeed a very good model of $m(Z)$.
Proposition3.17. $[Z_u^v, Z_w^x] \in m_f(Z_Q) \Leftrightarrow w|u$ and $x|v$.
Proof: $[Z_u^v, Z_w^x] \in m_f(Z_Q) \Leftrightarrow$ there exists $Z_a^b \in m(Z)$ such that $[Z_u^v, Z_w^x] = [Z_a^b, Z_1^1] \Leftrightarrow Z_u^v \odot Z_1^1 = Z_w^x \odot Z_a^b \Leftrightarrow Z_u^v = Z_{wa}^{xb} \Leftrightarrow u = wa$ and $v = xb \Leftrightarrow w|u$ and $x|v$ as $u, v, w, a, b \in Z$.

Theorem 3.18. For the set $\in m_f(Z_Q)$ the following hold:
(i) $(m_f(Z_Q), \boxplus)$ is a subgroup of $(m_f(Q), \boxplus)$.
(ii) $(m_f(Z_Q), \boxdot)$ is a subgroup of $(m_f(Q), \boxdot)$.
(iii) $(m_f(Z_Q), \boxplus)$ is isomorphic to $(m(Z), \oplus)$ as group and $(m_f(Z_Q), \boxdot)$ is isomorphic to $(m(Z), \odot)$ as semi group under the same isomorphism.
(iv) For every $x \in m_f(Q)$, there exists $y, z \in (m_f(Z_Q)$ such that $x = y^{-1} \boxdot z$.
Proof: Clearly, $(m_f(Z_Q))$ is a non-empty subset of $m_f(Q)$.
(i) Let $[Z_a^b, Z_1^1]$, $[Z_c^d, Z_1^1] \in m_f(Z_Q)$.
Then $[Z_a^b, Z_1^1] \boxplus (-[Z_c^d, Z_1^1]) = [Z_a^b, Z_1^1] \boxplus \left[Z_{-c}^{\frac{1}{d}}, Z_1^1\right] = \left[Z_{a-c}^{\frac{b}{d}}, Z_1^1\right] \in m_f(Z_Q)$.
Therefore, $(m_f(Z_Q), \boxplus)$ is a subgroup of $(m_f(Q), \boxplus)$.
(ii) $[Z_a^b, Z_1^1] \boxplus [Z_c^d, Z_1^1] = [Z_{ac}^{bd}, Z_1^1] \in m_f(Z_Q)$.
Therefore, $m_f(Z_Q)$ is closed under $\boxdot$.
$(m_f(Z_Q), \boxdot)$ is a subgroup of $(m_f(Q), \boxdot)$.
(iii) Define $\psi: m_f(Z_Q) \to m(Z)$ by $\psi([Z_a^b, Z_1^1]) = Z_a^b$, $Z_a^b \in m(Z)$.
We shall first show that ψ is a well-defined function.
So let, $[Z_p^q, Z_1^1] = [Z_r^s, Z_1^1]$.
$[Z_p^q, Z_1^1] = [Z_r^s, Z_1^1] \Leftrightarrow Z_p^q \odot Z_1^1 = Z_1^1 \odot Z_r^s \Leftrightarrow Z_p^q = Z_r^s \Leftrightarrow \psi([Z_p^q, Z_1^1]) = \psi([Z_r^s, Z_1^1])$.
So, ψ is well-defined.
Immediately, ψ is a bijection.
Now for any $[Z_p^q, Z_1^1], [Z_r^s, Z_1^1] \in m_f(Z_Q)$,
$\psi([Z_p^q, Z_1^1] \boxplus [Z_r^s, Z_1^1]) = \psi([Z_{p+r}^{qs}, Z_1^1]) = Z_{p+r}^{qs} = Z_p^q \odot Z_r^s = \psi([Z_p^q, Z_1^1]) \oplus \psi([Z_r^s, Z_1^1])$.
Hence, $(m_f(Z_Q), \boxplus)$ is isomorphic to $(m(Z), \oplus)$.
Similarly, we can show that $(m_f(Z_Q), \boxdot)$ is isomorphic to $(m(Z), \odot)$.
(iv) Let $x = [Z_a^b, Z_c^d] \in m_f(Q)$, then there exists $y = [Z_c^d, Z_1^1]$, $z = [Z_a^b, Z_1^1] \in m_f(Z_Q)$ such that $y^{-1} \boxdot z = [Z_c^d, Z_1^1]^{-1} \boxdot [Z_a^b, Z_1^1] = [Z_1^1, Z_c^d] \boxdot [Z_a^b, Z_1^1] = [Z_a^b, Z_c^d] = x$
Hence the theorem.

Definition 3.19. Let us define each member of $m_f(Q)$ as a multi-rational number. Let us also define each member of $m_f^+(Q)$ as positive multi-rational number and each member of $m_f^-(Q)$ as negative multi-rational number.

Definition 3.20. (Positive multi-rational number, Negative multi-rational number, Zero, Special multi-rational number and Multi-zero)
Define $m_f(Q_S) = m_f(Q) - (m_f^+(Q) \cup m_f^-(Q) \cup \{[Z_0^1, Z_1^1]\})$.
We have defined every member of $m_f^+(Q)$ as a positive multi-rational number, every member of $m_f^-(Q)$ as a negative multi-rational number, $[Z_0^1, Z_1^1]$ is the zero and every member of $m_f(Q_S)$ as special multi-rational number. Also any multi-rational number of the form $[Z_0^a, Z_c^d]$ is a multi-zero which is obviously either a special multi-rational number or zero.

Theorem 3.21. If the product of two multi-rational numbers be zero, then at least one of them must be a multi-zero.
Proof: For $[Z_a^b, Z_c^d], [Z_p^q, Z_r^s] \in m_f(Q)$, let, $[Z_a^b, Z_c^d] \boxdot [Z_p^q, Z_r^s] = [Z_0^1, Z_1^1]$, then $[Z_a^b \odot Z_p^q, Z_c^d \odot Z_r^s] = [Z_0^1, Z_1^1] \Rightarrow [Z_{ap}^{bq}, Z_{cr}^{ds}] = [Z_0^1, Z_1^1] \Rightarrow Z_{ap}^{bq} \odot Z_1^1 = Z_{cr}^{ds} \odot Z_0^1 \Rightarrow Z_{ap}^{bq} = Z_0^{ds} \Rightarrow ap = 0$ and $bq = ds \Rightarrow$ (either $a = 0$ or $p = 0$) $\Rightarrow$ $[Z_a^b, Z_c^d]$ or $[Z_p^q, Z_r^s]$ must be a multi-zero.

Definition 3.22. (Order on $m_f(Q)$) Let $[Z_a^b, Z_c^d], [Z_p^q, Z_r^s] \in m_f(Q)$. We define $[Z_a^b, Z_c^d] > [Z_p^q, Z_r^s]$ if $[Z_a^b, Z_c^d] \boxplus (-[Z_p^q, Z_r^s]) \in m_f^+(Q)$ i.e., if $[Z_a^b, Z_c^d] \boxplus ([Z_{-p}^{\frac{1}{q}}, Z_r^{\frac{1}{s}}]) \in m_f^+(Q)$, i.e., if $[Z_{ar-cp}^{\frac{b}{q}}, Z_{cr}^{\frac{d}{s}}]) \in m_f^+(Q)$, i.e., if $\frac{ar-cp}{cr} > 0$ and $\frac{\frac{b}{q}}{\frac{d}{s}} \in N$ i.e., $\frac{a}{c} > \frac{p}{r}$ and $\frac{bs}{dq} \in N$. Also, we define $[Z_a^b, Z_c^d] \geq [Z_p^q, Z_r^s]$ if $[Z_a^b, Z_c^d] > [Z_p^q, Z_r^s]$ or $[Z_a^b, Z_c^d] = [Z_p^q, Z_r^s]$.
Remark 3.23. Let us denote $[Z_a^b, Z_c^d] \boxplus (-[Z_p^q, Z_r^s])$ as $[Z_a^b, Z_c^d] - [Z_p^q, Z_r^s]$.
Theorem 3.24. (Partial order relation) $\geq$ defined on $m_f(Q)$ is a partial order relation.
Proof: Proof is immediate.
Remark 3.25. $(m_f(Q), \geq)$ is a poset but not a chain. e.g., $[Z_2^3, Z_5^7]$ and $[Z_3^4, Z_6^2]$ are two incomparable elements of $m_f(Q)$.
Since, $[Z_2^3, Z_5^7] \boxplus (-[Z_3^4, Z_6^2]) = [Z_2^3, Z_5^7] \boxplus \left[Z_{-3}^{\frac{1}{4}}, Z_6^2\right] = \left[Z_{-3}^{\frac{3}{4}}, Z_{30}^{14}\right] \notin m_f^+(Q)$ (because $-\frac{3}{30} = -\frac{1}{10} < 0$) and

$[Z_3^4, Z_6^2] \boxplus (-[Z_2^3, Z_5^7]) = [Z_3^4, Z_6^2] \boxplus \left[Z_{-2}^{\frac{1}{3}}, Z_5^7\right] = [Z_3^{\frac{4}{3}}, Z_{30}^{14}] \notin m_f^+(Q)$ (because $\frac{\frac{4}{3}}{14} = \frac{2}{21} \notin N$).
Proposition 3.26. For all $[Z_a^b, Z_c^d] \in m_f(Q)$, $[Z_a^b, Z_c^d] \ngtr [Z_a^b, Z_c^d]$.
Proof: $a - c \ngtr a - c$ for all $a, c \in N$ with $a \neq c$, so from Proposition 3.24., the above proposition immediately follows.
Proposition 3.27. For all $[Z_a^b, Z_c^d], [Z_e^f, Z_g^h] \in m_f(Q)$, $[Z_a^b, Z_c^d] > [Z_e^f, Z_g^h] \Leftrightarrow [Z_a^b, Z_c^d] \boxplus [Z_u^v, Z_w^x] > [Z_e^f, Z_g^h] \boxplus [Z_u^v, Z_w^x]$ for all $[Z_u^v, Z_w^x] \in m_f(Q)$.
Proof: Proof is immediate.
Proposition 3.28. For all $[Z_a^b, Z_c^d], [Z_e^f, Z_g^h], [Z_u^v, Z_w^x], [Z_p^q, Z_r^s] \in m_f(Q)$, $[Z_a^b, Z_c^d] > [Z_e^f, Z_g^h]$ and $[Z_u^v, Z_w^x] > [Z_p^q, Z_r^s] \Rightarrow [Z_a^b, Z_c^d] \boxplus [Z_u^v, Z_w^x] > [Z_e^f, Z_g^h] \boxplus [Z_p^q, Z_r^s]$.
Proof: Proof is immediate.
Proposition 3.29. For $[Z_a^b, Z_c^d], [Z_e^f, Z_g^h] \in m_f(Q)$, $[Z_a^b, Z_c^d] \geq [Z_e^f, Z_g^h] \Rightarrow [Z_a^b, Z_c^d] \boxplus [Z_1^1, Z_1^1] > [Z_e^f, Z_g^h]$.
Proposition 3.30. For all $[Z_a^b, Z_c^d] \in m_f(Q)$, $[Z_a^b, Z_c^d] \boxplus [Z_e^f, Z_g^h] > [Z_a^b, Z_c^d]$ for all $[Z_e^f, Z_g^h] \in m_f(Z_Q)$.
Proposition 3.31. For $[Z_a^b, Z_c^d], [Z_e^f, Z_g^h], [Z_u^v, Z_w^x] \in m_f(Q)$, $[Z_a^b, Z_c^d] \boxplus [Z_u^v, Z_w^x] = [Z_e^f, Z_g^h] \boxplus [Z_u^v, Z_w^x] \Rightarrow [Z_a^b, Z_c^d] = [Z_e^f, Z_g^h]$.
Proposition 3.32. For $[Z_a^b, Z_c^d], [Z_e^f, Z_g^h] \in m_f(Q)$, $[Z_a^b, Z_c^d] > [Z_e^f, Z_g^h] \Leftrightarrow [Z_a^b, Z_c^d] \boxdot [Z_u^v, Z_w^x] > [Z_e^f, Z_g^h] \boxdot [Z_u^v, Z_w^x]$ for all $[Z_u^v, Z_w^x] \in m_f^+(Q)$.
Proof: Proof is immediate.

Theorem 3.33. (Isomorphism theorem) Let us consider the general mset $m(\hat{Q})$drawn from Q characterized by the universal relation $\rho_{m(\hat{Q})} = Q \times Q^+$ (Q^+ being the set of all positive rational numbers) i.e., $Q_p^q \in m(\hat{Q})$ if and only if $p \in Q$ and $q \in Q^+$. Let us define two binary operations $\widehat{\boxplus}$ and $\widehat{\boxdot}$ as follows:
For $Q_p^q, Q_r^s \in m(\hat{Q})$, $Q_p^q \,\widehat{\boxplus}\, Q_r^s = Q_{p+r}^{qs}$ and $Q_p^q \,\widehat{\boxdot}\, Q_r^s = Q_{pr}^{qs}$.

Also, define $\widehat{>}$ on $m(\hat{Q})$ as follows: $Q_p^q, Q_r^s \in m(\hat{Q})$, $Q_p^q \widehat{>} Q_r^s$ if and only if there exists $Q_a^b \in m(\hat{Q})$ with $a \in Q^+$ and $b \in N$ such that $Q_p^q = Q_r^s \widehat{\oplus} Q_a^b$.
For $Q_p^q, Q_r^s \in m(\hat{Q})$, we define $Q_p^q = Q_r^s$ if and only if $p = r$ and $q = s$.
Also, for $Q_p^q, Q_r^s \in m(\hat{Q})$, we define $Q_p^q \widehat{\geqq} Q_r^s$ if and only if $Q_p^q \widehat{>} Q_r^s$ or $Q_p^q = Q_r^s$.
Then $(m_f(Q),\boxplus,\boxdot,\geq)$ and $(m(\hat{Q}),\widehat{\boxplus},\widehat{\boxdot},\widehat{\geqq})$ are isomorphic.
Proof: Let us now define a function $\tau: m_f(Q) \to m(\hat{Q})$ as follows:
$\tau([Z_a^b, Z_c^d]) = Q_{\frac{a}{c}}^{\frac{b}{d}}$, $[Z_a^b, Z_c^d] \in m_f(Q)$.

$[Z_a^b, Z_c^d], \left[Z_{a'}^{b'}, Z_{c'}^{d'}\right] \in m_f(Q)$, $[Z_a^b, Z_c^d] = \left[Z_{a'}^{b'}, Z_{c'}^{d'}\right] \Leftrightarrow \frac{a}{c} = \frac{a'}{c'}$ and $\frac{b}{d} = \frac{b'}{d'} \Leftrightarrow \left[Z_{\frac{a}{c}}^{\frac{b}{d}}, Z_{\frac{a'}{c'}}^{\frac{b'}{d'}}\right] \Leftrightarrow \tau([Z_a^b, Z_c^d]) = \tau(\left[Z_{a'}^{b'}, Z_{c'}^{d'}\right])$.
So, τ is well-defined and one to one.
Next, let $Q_p^q \in m(\hat{Q})$, then $p \in Q$ and $q \in Q^+$.
Therefore, there exists $a, b, c, d \in Z$ with $a > 0$, $d > 0$ such that $p = \frac{a}{c}$ and $q = \frac{b}{d}$. So, $b > 0$ and consequently, $[Z_a^b, Z_c^d] \in m_f(Q)$.
Also, $\tau([Z_a^b, Z_c^d]) = Z_{\frac{a}{c}}^{\frac{b}{d}} = Q_p^q$.
Therefore, τ is onto.
Therefore, τ is a bijection.
Now let $[Z_a^b, Z_c^d], \left[Z_{a'}^{b'}, Z_{c'}^{d'}\right] \in m_f(Q)$, then $\tau\left([Z_a^b, Z_c^d] \boxplus \left[Z_{a'}^{b'}, Z_{c'}^{d'}\right]\right) = \tau\left(\left[Z_{ac'+ca'}^{bb'}, Z_{cc'}^{dd'}\right]\right) = Q_{\frac{ac'+ca'}{cc'}}^{\frac{bb'}{dd'}} = Q_{\frac{a}{c}}^{\frac{b}{d}} \widehat{\boxplus}\, Q_{\frac{a'}{c'}}^{\frac{b'}{d'}}$
$= \tau([Z_a^b, Z_c^d]) \widehat{\boxplus}\, \tau(\left[Z_{a'}^{b'}, Z_{c'}^{d'}\right])$.
Also, $\tau\left([Z_a^b, Z_c^d] \boxdot \left[Z_{a'}^{b'}, Z_{c'}^{d'}\right]\right) = \tau\left(\left[Z_{aa'}^{bb'}, Z_{cc'}^{dd'}\right]\right) = Q_{\frac{aa'}{cc'}}^{\frac{bb'}{dd'}} = Q_{\frac{a}{c}}^{\frac{b}{d}} \widehat{\boxdot}\, Q_{\frac{a'}{c'}}^{\frac{b'}{d'}}$
$= \tau([Z_a^b, Z_c^d]) \widehat{\boxdot}\, \tau(\left[Z_{a'}^{b'}, Z_{c'}^{d'}\right])$.
Next for, $[Z_a^b, Z_c^d], [Z_p^q, Z_r^s] \in m_f(Q)$, let $[Z_a^b, Z_c^d] > [Z_p^q, Z_r^s]$.
Then, $[Z_a^b, Z_c^d] \boxplus \left(-[Z_p^q, Z_r^s]\right) \in m_f^+(Q) \Rightarrow \frac{ar-cp}{cr} > 0$ and $\frac{bs}{qd} \in N \Rightarrow$ there exist $s \in Q^+$ and $t \in N$ such that $\frac{ar-cp}{cr} = s$ and $\frac{bs}{qd} = t \Rightarrow \frac{a}{c} = \frac{p}{r} + s$ and $\frac{b}{d} = \frac{q}{s}t \Rightarrow Q_{\frac{a}{c}}^{\frac{b}{d}} = Q_{\frac{p}{r}+s}^{\frac{q}{s}t} \Rightarrow Q_{\frac{a}{c}}^{\frac{b}{d}} = Q_{\frac{p}{r}}^{\frac{q}{s}} \widehat{\boxplus}\, Q_s^t \Rightarrow$ there exist $Q_s^t \in m(\hat{Q})$ with $s \in Q^+$ and $t \in N$ such that $Q_{\frac{a}{c}}^{\frac{b}{d}} = Q_{\frac{p}{r}}^{\frac{q}{s}} \widehat{\boxplus}\, Q_s^t \Rightarrow Q_{\frac{a}{c}}^{\frac{b}{d}} \widehat{\geqq} Q_{\frac{p}{r}}^{\frac{q}{s}} \Rightarrow \tau([Z_a^b, Z_c^d]) \widehat{\geqq} \tau([Z_p^q, Z_r^s])$.
Therefore, $(m_f(Q),\boxplus,\boxdot,\geq)$ and $(m(\hat{Q}),\widehat{\boxplus},\widehat{\boxdot},\widehat{\geqq})$ are isomorphic.

Remark 3.34. (Properties of $(m(\hat{Q}),\widehat{\boxplus},\widehat{\boxdot},\widehat{\geqq})$)
Since $(m_f(Q),\boxplus,\boxdot,\geq)$ and $(m(\hat{Q}),\widehat{\boxplus},\widehat{\boxdot},\widehat{\geqq})$ are isomorphic, so $(m(\hat{Q}),\widehat{\boxplus})$ is a commutative group, $(m(\hat{Q}),\widehat{\boxdot})$ is a commutative monoid and $\widehat{\boxdot}$ obey multi-distributive property over $\widehat{\boxplus}$. $(m(\widehat{Q_0}),\widehat{\boxdot})$ is a commutative group where $m(\widehat{Q_0}) = [Q - \{0\}] \times Q^+$. Also, $(m(\hat{Q}),\widehat{\geqq})$ is a poset. Moreover, $\widehat{\geqq}$ defined on $m(\hat{Q})$ is an extension of $\geq$ defined on $m(Z)$.

Remark 3.35. $(m(\hat{Q}),\widehat{\boxplus})$ is a commutative group and $(m(\hat{Q}),\widehat{\boxdot})$ is a commutative monoid but $(m(\hat{Q}),\widehat{\boxplus},\widehat{\boxdot})$ is not a ring, since $\widehat{\boxdot}$ cannot be distributed over $\widehat{\boxplus}$. But $\widehat{\boxdot}$ obeys multi-distributive property over $\widehat{\boxplus}$. Let us now introduce a new concept of multi-field and $(m(\hat{Q}),\widehat{\boxplus},\widehat{\boxdot})$ to be such a multi-field.

Definition 3.36. (General mset drawn from a ring) Let $(X,+,\cdot)$ be a ring. Let M be a general mset drawn from X. Cosider two functions $\oplus: M \times M \to \pi(X)$ and $\odot: M \times M \to \pi(X)$ defined as follows:
For $X_a^r, X_b^s \in M$, $X_a^r \oplus X_b^s = X_{a+b}^{rs}$ and $X_a^r \odot X_b^s = X_{ab}^{rs}$.
Let us call $\oplus$ and $\odot$ respectively as m-addition and m-multiplication defined on M induced by the ring $(X,+,\cdot)$. Also let M be closed under $\oplus$ and $\odot$. Then immediately $\oplus$ obey commutative property and associative property on M. So, $(M,\oplus)$ is then a commutative semi group. Also, immediately $\odot$ obey associative property on M. So, $(M,\odot)$ is a semi group. We define M to be a general mset drawn from the ring $(X,+,\cdot)$.

Definition 3.37. Let M be a general mset drawn from a ring $(X,+,\cdot)$.Then $\odot$ obey multi-distributive property over $\oplus$.

Definition 3.38. (Multi-ring) Let M be a general mset drawn from a ring $(X,+,\cdot)$. Let $\oplus$ and $\odot$ are m-addition and m-multiplication respectively defined on M induced by the ring $(X,+,\cdot)$. If the structure $(M,\oplus,\odot)$ satisfies the followings:

(1) $(M,\oplus)$ is an abelian group
(2) $(M,\odot)$ is a semi group and
(3) $\odot$ obey multi-distributive property over $\oplus$
then we define $(M,\oplus,\odot)$ to be a multi-ring induced by the ring $(X,+,\cdot)$.

Theorem 3.39. Let M be a general mset drawn from a ring $(X,+,\cdot)$. Let $\oplus$ and $\odot$ are m-addition and m-multiplication respectively defined on M induced by the ring $(X,+,\cdot)$. Then $(M,\oplus,\odot)$ will be multi-ring induced by the ring $(X,+,\cdot)$ if and only if the following conditions are satisfied:

(1) There exists $X_\theta^1 \in M$ (θ being the zero element in the ring $(X,+,\cdot)$.
(2) For $a \in X$ and $r \in [R=\{0\}]$, $X_a^r \in M \Rightarrow X_{(-a)}^{\frac{1}{r}} \in M$.

Remark 3.40. (i) Let us consider the ring $(X,+,\cdot)$ where $X = Z_4$, the set of all residue classes modulo 4, also, $+$ and $\cdot$ are respectively addition and multiplication modulo 4. Consider the general mset M characterized by the relation $\rho_M = X \times G$ where $G = \{2^n : n \in Z\}$ between X and G. Then for all $a \in X$ and for all $r \in G$, $X_a^r \in M$. Let $\oplus$ and $\odot$ are m-addition and m-multiplication respectively defined on M induced by the ring $(X,+,\cdot)$. Then $(M,\oplus,\odot)$ forms a multi-ring induced by the ring $(X,+,\cdot)$.
(iii) $(m(\hat{Z}),\widehat{\oplus},\widehat{\odot})$ is a multi-ring induced by the ring $(Z,+,\cdot)$.

Remark 3.41. Let $(M,\oplus,\odot)$ to be a multi-ring induced by the ring $(X,+,\cdot)$ where M is a general mset drawn from the ring $(X,+,\cdot)$. Let θ be the zero element in $(X,+,\cdot)$. Let X_θ^1 must be the zero element in $(M,\oplus,\odot)$. Let us also define any element in M of the form X_θ^r for some $r \in R-\{0\}$ to be the multi-zero elements of M such that the product of any element of the multi-ring with a multi-zero element of the same is again a multi-zero of the multi-ring. Clearly, the zero element in a multi-ring is a multi-zero element.

Remark 3.42. In a multi-ring $(m(\hat{Z}),\widehat{\oplus},\widehat{\odot})$ induced by the ring $(Z,+,\cdot)$, the non-zero multi-zeros are only divisors of zero.

Theorem 3.43. In a multi-ring, non-zero multi-zero elements are divisors of zero.
Definition 3.44. A multi-ring is said to have no non-multi-zero divisors of zero if its non-zero multi-zero elements are the only divisors of zero.

Example 3.45. The multi-ring $(m(\hat{Z}),\widehat{\oplus},\widehat{\odot})$ induced by the ring $(Z,+,\cdot)$, has no non-multi-zeros divisors of zero.
Remark 3.46. Let $(M,\oplus,\odot)$ be a multi-ring induced by the ring $(X,+,\cdot)$ with divisors of zero. Let θ be the zero element in $(X,+,\cdot)$. As $(X,+,\cdot)$ is a ring with divisors of zero, so, there exists two non zero elements a and b in the ring $(X,+,\cdot)$ such that $a \cdot b = \theta$.
Now, for some $r,s \in R-\{0\}$, let $X_a^r, X_b^s \in M$.
Then $X_a^r \odot X_b^s = X_{ab}^{rs} = X_\theta^{rs} \in M$ (since M is closed under $\odot$).
Again, X_a^r and X_b^s are divisors of zero in the multi-ring $(M,\oplus,\odot)$. So, X_a^r and X_b^s are non-multi-zero divisors of zero in the multi-ring $(M,\oplus,\odot)$.

Example 3.47. Consider the multi-ring $(M,\oplus,\odot)$ induced by the ring $(X,+,\cdot)$ as mentioned in (i) of Example 3.40. where $X = Z_4$.

Then, for $X_{[2]}^2, X_{[2]}^{\frac{1}{2}} \in M$, $X_{[2]}^2 \odot X_{[2]}^{\frac{1}{2}} = X_{[0]}^1$ which is the zero element of the multi-ring $(M,\oplus,\odot)$ induced by the ring $(X,+,\cdot)$. Also, $X_{[2]}^2$ and $X_{[2]}^{\frac{1}{2}}$ are the non-zero non-multi-zero elements of the multi-ring $(M,\oplus,\odot)$ induced by the ring $(X,+,\cdot)$. So, the multi-ring $(M,\oplus,\odot)$ induced by the ring $(X,+,\cdot)$ contains multi-divisors of zero.

Definition 3.48. (Multi-field) Let M be a general mset drawn from a ring $(X,+,\cdot)$ (or, a field Let $\boxplus$ and $\boxdot$ are m-addition and m-multiplication respectively induced by the ring $(X,+,\cdot)$ (or, a field $(X,+,\cdot)$). The structure $(M,\boxplus,\boxdot)$ satisfies the followings:

1. $(M,\boxplus)$ is an commutative group
2. $(M,\boxdot)$ is a commutative monoid
3. Every non-zero non-multi-zero element of M has god its inverse in M with respect to $\boxdot$.
4. $\boxdot$ obeys multi-distributive over $\boxplus$

   Then we define $(M,\boxplus,\boxdot)$ to be a multi-field induced by the ring (or, a field) $(X,+,\cdot)$.

Example 3.49. $(m(\hat{Q}),\widehat{\boxplus},\widehat{\boxdot})$ is a multi-field induced by the field $(Q,+,\cdot)$.

Example 3.50. Consider the field $(X,+,\cdot)$ where $X = Z_4$, the set of all residue classes modulo 3, also, $+$ and $\cdot$ respectively addition and multiplication modulo 3. Consider the general mset M characterized by the relation $\rho_M = X \times G$ where $G = \{2^n : n \in Z\}$ between X and G. Then for all $a \in X$ and for all $r \in G$, $X_a^r \in M$. Let $\boxplus$ and $\boxdot$ are m-addition and m-multiplication respectively defined on M induced by the ring $(X,+,\cdot)$. Let $(M,\boxplus,\boxdot)$ forms a multi-field induced by the field $(X,+,\cdot)$.

Remark 3.51. $(m(\hat{Q}),\widehat{\boxplus},\widehat{\boxdot},\widehat{\geqq})$ is a partially ordered multi-field induced by the field $(Q,+,\cdot)$.

Definition 3.52. (Definition of multi-rational number system) A partially ordered multi-field $(F,\boxplus,\boxdot,\geq)$ is called a multi-rational number system if there exists a partially ordered sub domain $(Z_F,\boxplus,\boxdot,\geq)$ such that

1. $(Z_F,\boxplus,\boxdot,\geq) \cong (m(Z),\oplus,\odot,\geq)$
2. For every $x \in F$, there exists $y, z \in Z_F$ such that $x = y^{-1} \boxdot z$.

Theorem 3.53. (Existence and uniqueness of multi-rational number system) Multi-rational number system exists and any two multi-rational number systems are isomorphic.

Proof: We have previously shown that the system $(m(\hat{Q}),\widehat{\boxplus},\widehat{\boxdot},\widehat{\geqq})$ is a partially ordered multi-field drawn from the field $(Q,+,\cdot)$.

Now consider the submset $m(Z_{\hat{Q}}) = \{Q_a^b : a \in Z, b \in Q^+\}$ of $m(\hat{Q})$.

Again, $a \in Z, b \in Q^+$ implies $Q_a^b = Z_a^b$.

So, $m(Z_{\hat{Q}}) = m(Z)$.

Also consider the restrictions of $\widehat{\boxplus}$ and $\widehat{\boxdot}$ defined on $m(Z_{\hat{Q}})$. Immediately they are $\oplus$ and $\odot$ defined on $m(Z)$.

So, $(m(Z_{\hat{Q}}),\widehat{\boxplus},\widehat{\boxdot},\widehat{\geqq})$ is an ordered sub domain of $(m(\hat{Q}),\widehat{\boxplus},\widehat{\boxdot},\widehat{\geqq})$ and they are isomorphic under the isomorphism $\emptyset : m(Z_{\hat{Q}}) \to m(Z)$ defined by $\emptyset(Q_p^q) = Z_p^q, Q_p^q \in m(Z_{\hat{Q}})$.

Now let $Q_p^q, Q_m^n \in m(Z_{\hat{Q}})$ such that $Q_p^q \widehat{>} Q_m^n$.

As, $p.m \in Z$ and $q, n \in Q^+$ so that $Q_p^q = Z_p^q$ and $Q_m^n = Z_m^n$.

Now $Q_p^q \widehat{>} Q_m^n \Rightarrow$ there exists $Q_a^b \in m(\hat{Q})$ with $a > 0$ and $b \in N$ such that $Q_p^q = Q_m^n \widehat{\boxplus} Q_a^b$.

i.e., $Q_p^q = Q_{m+a}^{nb} \Rightarrow p = m + a \Rightarrow a = p - m \in Z \Rightarrow Q_a^b = Z_a^b$.

So, $Z_p^q = Z_m^n \widehat{\boxplus} Z_a^b$ and accordingly, $Z_p^q = Z_m^n \widehat{\oplus} Z_a^b$.

i.e., $\emptyset(Q_p^q) > \emptyset(Q_m^n)$.

Therefore, for all $Q_p^q, Q_m^n \in m(Z_{\hat{Q}})$, $Q_p^q \widehat{>} Q_m^n \Rightarrow \emptyset(Q_p^q) > \emptyset(Q_m^n)$.

Finally let, $x = Q_a^b \in m(\hat{Q})$, then $a \in Q$ and $b \in Q^+$.

So, there exists $m.n \in Z, n > 0, p, q \in N$ such that $a = \frac{m}{n}$ and $b = \frac{p}{r}$.

Then, $x = Q_a^b = Q_{\frac{m}{n}}^{\frac{p}{q}} = Q_{\frac{1}{n}}^{\frac{1}{q}} \widehat{\boxdot}\, Q_m^p = (Z_n^q)^{-1} \widehat{\boxdot}\, Z_m^p = y^{-1} \widehat{\boxdot}\, z$, say, where $y = Z_n^q$, $z = Z_m^p \in m(Z_{\hat{Q}})$ since $m, n \in Z; p, q \in N$.

Hence, $(m(\hat{Q}),\widehat{\boxplus},\widehat{\boxdot},\widehat{\geq})$ is a multi-rational number system and so multi-rational number system exists.
Next, let, $(m(Q),\boxplus,\boxdot,\geq)$ and $(m(Q'),\boxplus',\boxdot',\geq')$ be any two multi-rational number systems ($m(Q)$ and $m(Q')$ being two general msets).
Then by transitivity of isomorphism there exists an isomorphism $\emptyset: m(Z_Q) \to m(Z_{Q'})$ such that
For all $y, z \in m(Z_Q)$, $\emptyset(y \boxplus z) = \emptyset(y) \boxplus' \emptyset(z)$, $\emptyset(y \boxdot z) = \emptyset(y) \boxdot' \emptyset(z)$ and $y > z \Rightarrow \emptyset(y) \geq' \emptyset(z)$.
Also, for any $x \in m(Q)$, there exists $y_x, z_x \in m(Z_Q)$ such that $x = (y_x)^{-1} \boxplus z_x$.
Define $\psi: m(Q) \to m(Q')$ by $\psi(x) = (\emptyset(y_x))^{-1} \boxdot' \emptyset(z_x)$.
Then we can show that ψ is well defined.
Also we can show that ψ is bijective.
Again, for any $u, v \in m(Q)$, $\psi(u \boxplus v)$
$= \psi[((y_u)^{-1} \boxdot (z_u)) \boxplus ((y_v)^{-1} \boxdot (z_v))]$
$= \psi[((y_u)^{-1} \boxdot (y_v)^{-1}) \boxdot ((y_v \boxdot z_u) \boxplus (y_u \boxdot z_v))]$
$= \psi[(y_v \boxdot y_u)^{-1} \boxdot ((y_v \boxdot z_u) \boxplus (y_u \boxdot z_v))]$
$= \left(\emptyset(y_v) \boxdot' \emptyset(y_u)\right)^{-1} \boxdot' ((\emptyset(y_v \boxdot z_u) \boxplus' \emptyset(y_u \boxdot z_v))$
$= ((\emptyset(y_u))^{-1} \boxdot' (\emptyset(y_v))^{-1}) \boxdot' \left(\left((\emptyset(y_v) \boxdot' \emptyset(z_u)\right) \boxplus' ((\emptyset(y_u) \boxdot' \emptyset(z_v))\right)$
$= (((\emptyset(y_u))^{-1} \boxdot' (\emptyset(y_v))^{-1}) \boxdot' \left((\emptyset(y_v) \boxdot' \emptyset(z_u)\right)) \boxplus' (((\emptyset(y_u))^{-1} \boxdot' (\emptyset(y_v))^{-1}) \boxdot' \left((\emptyset(y_u) \boxdot' \emptyset(z_v)\right))$
$= \left((\emptyset(y_u))^{-1} \boxdot' \emptyset(z_u)\right) \boxplus' ((\emptyset(y_v))^{-1} \boxdot' \emptyset(z_v))$
$= \psi(u) \boxplus' \psi(v)$.
Similarly, we can show that $\psi(u \boxdot v) = \psi(u) \boxdot' \psi(v)$.
Again, for any $u, v \in m(Q), u > v \Rightarrow (y_u)^{-1} \boxdot z_u > (y_v)^{-1} \boxdot z_v \Rightarrow y_v \boxdot z_u > y_u \boxdot z_v$ $\Rightarrow \emptyset(y_v \boxdot z_u) >' \emptyset(y_u \boxdot z_v) \Rightarrow \emptyset(y_v) \boxdot' \emptyset(z_u) >' \emptyset(y_u) \boxdot' \emptyset(z_v)$
$\Rightarrow \left(\emptyset(y_u)\right)^{-1} \boxdot' \emptyset(z_u) >' \left(\emptyset(y_v)\right)^{-1} \boxdot' \emptyset(z_v) \Rightarrow \psi((y_u)^{-1} \boxdot z_u) >' \psi((y_v)^{-1} \boxdot z_v) \Rightarrow \psi(u) >' \psi(v)$.
Hence, $(m(Q),\boxplus,\boxdot,\geq) \cong (m(Q'),\boxplus',\boxdot',\geq')$.
Hence, the uniqueness of multi-rational number system.

Remark 3.54. Therefore, $(m(\hat{Q}),\widehat{\boxplus},\widehat{\boxdot},\widehat{\geq})$ is a multi-rational number system. Also, multi-rational number system is unique. So, from now on we shall abandon our multi-fractional system and consider instead the multi-rational number system $(m(\hat{Q}),\widehat{\boxplus},\widehat{\boxdot},\widehat{\geq})$. Any multi-rational number system is afterwards denoted by $(m(Q),\boxplus,\boxdot,\geq)$ where $m(Q)$ is the general mset drawn from Q characterized by the universal relation $Q \times Q^+$i.e., $Q_p^q \in m(Q)$ if and only if $p \in Q$ and $q \in Q^+$. Binary operations $\boxplus$ and $\boxdot$ are defined on $m(Q)$ as follows: for $Q_p^q, Q_r^s \in m(Q)$, $Q_p^q \boxplus Q_r^s = Q_{P+r}^{qs}$ and $Q_p^q \boxdot Q_r^s = Q_{pr}^{qs}$. $>$ is defined on $m(Q)$ as follows: for $Q_p^q, Q_r^s \in m(Q)$, $Q_p^q > Q_r^s$ if and only if there exists $Q_a^b \in m(Q)$ with $a > 0$, $b \in N$ such that $Q_p^q = Q_r^s \boxplus Q_a^b$. Also, for $Q_p^q, Q_r^s \in m(Q)$, $Q_p^q \geq Q_r^s$ if and only if $Q_p^q > Q_r^s$ or $Q_p^q = Q_r^s$. The copy of the multi-integers embedded in $m(Q)$ will still denoted by $m(Z)$ and it has all the properties that we have proven in paper [9] if we consider it in isolation.

Remark 3.55. Consider three multi-rational numbers $Q_{\frac{2}{3}}^{3}, Q_3^{\frac{1}{2}}$

Then, $Q_{\frac{2}{3}}^{3} \boxplus Q_3^{\frac{1}{2}} = Q_{\frac{2}{3}+3}^{3\cdot\frac{1}{2}} = Q_{3\frac{2}{3}}^{\frac{3}{2}}$ and $Q_{\frac{2}{3}}^{3} \boxdot Q_3^{\frac{1}{2}} = Q_{\frac{2}{3}\cdot 3}^{3\cdot\frac{1}{2}} = Q_2^{\frac{3}{2}}$.

IV. CONCLUSION

In this paper, we have defined and studied multi-rational number system as an extension of multi-integer system. There is a huge scope of future research words in the field of multiset. Especially further study can be carried out in the following directions.
To study extension of multi-rational number system towards multi-real number system.
To study thoroughly the properties of algebraic operations and order relations defined on it.
Also, to study the properties of general mset and multi-field.

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