Three Phases of Service For A Single Server Queueing System Subject To Server Breakdown And Bernoulli Vacation

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Abstract *– This paper deals withan unreliable server having three phases of heterogeneous service on the basis of M/G/1 queueing system. We suppose that customers arrive and join the system according to a Poisson's process with arrival rate λ. After completion of three successive phases of service the server either goes for a vacation with probability p (0 ≤ p ≤ 1) or continue to serve the next units, if any, with probability q (=1 - p). Otherwise it remains in the system until a customer arrives. The server is supposed to be unreliable, hence when the server is working during any phase of service, it may breakdown at any instant and thus service facility will fail for a short interval of time. Firstly, now we derive the joint probability distribution for the server. Secondly, we derive the probability generating function of the stationary queue size distribution at a departure epoch. Third, we derive Laplace Stieltjes transform of busy period distribution and waiting time distribution. Finally, we obtain some important performance measures and reliability analysis of this model.*

Keywords - *First phase of service, Second phase of service, Third phase of service, Random breakdowns, Bernoulli vacation, M/G/1 queue, Stationary queue size distribution and reliability index.*

I. INTRODUCTION

It is not necessary that server is available for permanent basis because practically it looks to be unrealistic. That's why we consider a queueing system where the server may breakdown at any instant during any phase of service, while serving the customers. Such a system is also known as queue with service interruptions or queue with unreliable server. Most probably the study of service interruptions queueing models was starting at early 1950's. The fundamental work on these models was done by Gaver [1], Avi-ltzhak and Naor [2], Thirurengadan [3] and Mitrany and Avi-ltzhak [4]. Some queueing models with service interruptions was firstly studied by Li et al. [5], Sengupta [6], TakinandSengupta [7] and Tang [8] with a common feature that whenever the server goes to fails then service fails for a short interval of time and server go for repair instantly. Most recently Ke et al [9] discussed the vacation policies for governing the vacation mechanism. Some papers are also discussed about unreliable server queueing models in which concepts of different control operating politics along with vacations. Queueing models with vacations are more realistic and flexible in studying real word queueing situations. The applications arise naturally in call centres with multi task employees, customized manufacturing, telecommunication and computer networks, maintenance activities, and production quality control problems.

The condition of Bernoulli service discipline was first introduced significantly by Keilson and servi [17], then kella [18] represented a generalised Bernoulli scheme where a single server goes for a *k* consecutive vacations with probability p_i. Recently, there has been considerable attention paid to study M/G/1 type queueing system with two phases of service under Bernoulli vacation schedule under different vacation policies see [19-22], in which after two successive phases of service, the server may go for Bernoulli vacation. The purpose of studying such types of vacation models are more helpful in computer networks and telecommunication systems, where messages are processed in two stages by a single server. Here we are trying to study M/G/1 type queueing system with three phases of service under Bernoulli vacation. Now there are lots of possibility for the failure of service stations and it may be repair after a break. For this, Li et al [5] considered reliability analysis of such a model under Bernoulli vacation schedule with the assumption that the server is subject to breakdowns and repairs. We investigate M/G/1 unreliable server queue with three phases of service, Bernoulli vacation and server breakdown during the service together. The remaining overview of this paper is as follows –In point 2 we represent the description of mathematical model. Point 3 stands for derivations of the stationary distribution of the queue size for the server state at a random epoch. Point 4 stands for distribution of busy period and waiting time. Point 5 stands for reliability analysis of this model. Finally conclusion is drawn in last one. To derive the probability generating function for queue size distribution at different phases of service, we apply the supplementary variable technique by introducing one or more supplementary variables.

II. THE MATHEMATICAL MODEL

Let us suppose an M/G/1 retrial queueing system. In this system arrivals of customers are according to a Poisson process with arrival rate 'λ'. It is a single server model which provides its services in three phases of heterogeneous service in successive service: first phase of service (FPS) denoted by C_1, followed by asecond phase of service (SPS) denoted by C_2 which again followed by third phase of service (TPS) denoted by C_3. The system serves according to first come, first serve (FCFS) service discipline. The service time for ith phases are independent random variables follow general law of distribution with probability distribution function *d.f.*$C_i(x)$, i = 1, 2,3. Laplace–Stieltjes transform (LST) $\beta_i^*(\theta) = E[e^{-\theta C_i}]$and finite moments are $\beta_i^{(k)}$, $k \geq 1$ for i= 1, 2, 3. As soon as the third phase of service for a unit is completed, the server may go for a vacation of random length W with probability $p(0 \leq p \leq 1)$ or it may continue to serve the next unit, if any, with probability *q= (1 - p)*, otherwise, it remains in the system and wait for a new arrival i.e. the server takes a *Bernoulli vacation*. The random variable for vacation time of the server follows a general law of distribution with *d.f.* W(y), Laplace–Stieltjes transform (LST) $w^*(\theta) = E[e^{-\theta W}]$ and finite moments $\phi^{(k)}$, $k \geq 1$ independent of the service time random variables. With any phase of service, when the server serves their service, it may breakdown at any time/instant and the service channel will fail for a short interval of time. This breakdowns are generated by exogenous Poisson process with rates α_1 for FPS, α_2 for SPS and α_3 for TPS respectively. Now when this breakdown occurs, it is sent for repair during which the server stops providing service to the arriving customers till service channel is repaired. The customers which were just being served before server breakdown wait for the service to complete its remaining service. The repair time (denoted by S_1 for FPS,S_2 for SPS and S_3 for TPS) distributions of the server for three phases of service are assumed to be arbitrarily distributed with *d.f.* $G_1(y)$, $G_2(y)$ and $G_3(y)$, Laplace–Stieltjes transform (LST) $G_1^*(\theta) = E[e^{-\theta S_1}]$, $G_2^*(\theta) = E[e^{-\theta S_2}]$ and $G_3^*(\theta) = E[e^{-\theta S_3}]$ and also finite k^{th} moments $g_1^{(k)}$, $g_2^{(k)}$ and $g_3^{(k)}$ respectively. Immediately when the server is fixed i.e., repaired, the server is again ready to start its remaining service to customers in all the three phases of service and in this case the service times are cumulative, which may be referred to as *generalized service times*. In addition, we assume that input process, server's life time, server's repair time, service time and vacation time random variables are mutually independent of each other.

III. STATIONARY QUEUE SIZE DISTRIBUTION

In stationary queue size distribution, we derive the system state equation for its stationary queue size distribution by treating the elapsed FPS time, the elapsed SPS time, the elapsed TPS time, the elapsed vacation time and the elapsed repair time of the serveras supplementary variables. The Supplementary variable technique is a method which is used for solution of Non – Markovianqueueing problems. We can convert by this technique a Non – Markovianqueueing model in Markovianqueueing model by introducing one or more supplementary variables. These supplementary variables are introduced corresponding to either elapsed time or remaining time of the random variables. Now D. R. Cox [23] was first to study Non – Markovian stochastic process by the inclusion of supplementary variables.

$M_Q(t)$ the queue size (including the one being served, if any) at time t

$C_1^0(t)$ the elapsed FPS time at time t

$C_2^0(t)$ the elapsed SPS time at time t

$C_3^0(t)$ the elapsed TPS time at time t

$W^0(t)$ the elapsed vacation time at time t

$S_1^0(t)$ the elapsed repair time for FPS during which breakdown occurs time at time t

$S_2^0(t)$ the elapsed repair time for SPS during which breakdown occurs time at time t

$S_3^0(t)$ the elapsed repair time for TPS during which breakdown occurs time at time t

Further, let us introduce the following random variable :

$$Y(t)=\begin{cases} 0; & \textit{if the system is idle at time } t,\\ 1, & \textit{if the server is busy with FPS at time } t,\\ 2, & \textit{if the server is busy with SPS at time } t,\\ 3, & \textit{if the server is busy with TPS at time } t,\\ 4, & \textit{if the server is on vacation at time } t,\\ 5, & \textit{if the server is under repair during FPS at time } t,\\ 6, & \textit{if the server is under repair during SPS at time } t,\\ 7, & \textit{if the server is under repair during TPS at time } t. \end{cases}$$

In M/G/1 queue, we assume that arrival of customers take place according to Poisson's process with rate λ. Let Q represents probability when the server is in idle state. Let $P_{i,n}(x)$ represents the steady state probability that there are $n \geq 0$ customers in the queue excluding one in i^{th} $(=1,2,3)$ phase of service and the elapsed service time of this customer is x. Let $S_{i,n}(x)$ represents the steady state probability that there are $n \geq 0$ customers in the queue excluding one customer who is repeating the i^{th} $(=1,2,3)$ phase of service and the elapsed service time of this customer is x.

Now, the supplementary variable $W^0(t), C_i^0(t)$, and $S_i^0(t)$ for $i=1,2,3$ are introduced in order to obtain a bivariate Markov process $\{M_Q(t), X(t)\}$, where $X(t)=0$ if $Y(t)=0$, $X(t)=C_1^0(t)$ if $Y(t)=1$, $X(t)=C_2^0(t)$ if $Y(t)=2$, $X(t)=C_3^0(t)$ *if* $Y(t)=3$, $X(t)=W^0(t)$ if $Y(t)=4$, $X(t)=R_1^0(t)$ if $Y(t)=5$, $X(t)=R_2^0(t)$ if $Y(t)=6$, and $X(t)=R_3^0(t)$ if Y(t) = 7.

Further we define the following probabilities:

$U_0(t) = P_r\{M_Q(t) = 0, X(t) = 0\},$

$Q_n(y;t)dy = P_r\{M_Q(t) = n, X(t) = W^0(t); \quad y < W^0(t) \le y + dy\};\ y > 0, n \ge 0$

and for $i = 1,2,3$ and $n \ge 0,$

$P_{i,n}(x;t)dx = P_r\{M_Q(t) = n, X(t) = C_i^0(t); \quad x < C_i^0(t) \le x + dx\};\ x > 0,$

$S_{i,n}(x,y;t)dy = P_r\{M_Q(t) = n, X(t) = S_i^0(t); \quad y < S_i^0(t) \le y + dy \big| C_i^0(t) = x\};\ (x,y) > 0.$

For the analysis of the limiting behaviour of this queueing with the help of Kolmogorov forward equation provided limiting probabilities $U_0 = \lim_{t\to\infty} U_0(t),\ Q_n(y)dy = \lim_{t\to\infty} Q_n(y;t)dy,\ P_{i,n}(x)dx = \lim_{t\to\infty} P_n(x;t)dx,$ and $S_{i,n}(x,y)dy = \lim_{t\to\infty} S_{i,n}(x,y;t)dy$ for $i = 1,2,3$ and $n \ge 0$ exist and positive under the condition that they are independent of the initial state.

Further, it is assumed that $W(0) = 0, W(\infty) = 1, C_i(0) = 0, C_i(\infty) = 1,\ G_i(0) = 0, G_i(\infty) = 1$ for $i = 1,2,3$ and that $W(y)$ is continuous at $y = 0$ for $i = 1,2,3$; $C_i(x)$ is continuous at $x = 0$ and $G_i(y)$ is continuous at $y = 0$ for $i = 1,2,3$ respectively, so that

$$\phi(y)dy = \frac{dW(y)}{1 - W(y)},$$

$$\mu_i(x)dx = \frac{dC_i(x)}{1 - C_i(x)}$$

and $\zeta_i(y)dy = \dfrac{dG_i(y)}{1 - G_i(y)}$

are the first order differential (Hazard rate) functions of W, C_i and G_i respectively for $i = 1,2,3.$

A. The steady state equations

The Kolmogorov forward equation to govern the system under steady state conditions (e.g. see Cox [23]) can be written as follows:

$$\left.\begin{aligned}
&\frac{d}{dx}P_{1,n}(x) + [\lambda + \alpha_1 + \mu_1(x)]P_{1,n}(x) = \lambda(1 - \delta_{n,0})P_{1,n-1}(x) + \int_0^\infty \zeta_1(y)R_{1,n}(x,y)dy;\ n \ge 0,\\
&\frac{d}{dx}P_{2,n}(x) + [\lambda + \alpha_2 + \mu_2(x)]P_{2,n}(x) = \lambda(1 - \delta_{n,0})P_{2,n-1}(x) + \int_0^\infty \zeta_2(y)R_{2,n}(x,y)dy;\ n \ge 0,\\
&\frac{d}{dx}P_{3,n}(x) + [\lambda + \alpha_3 + \mu_3 x)]P_{3,n}(x) = \lambda(1 - \delta_{n,0})P_{3,n-1}(x) + \int_0^\infty \zeta_3(y)R_{3,n}(x,y)dy;\ n \ge 0,
\end{aligned}\right\} \quad (3.1)$$

$$\frac{d}{dy}Q_n(y)+[\lambda+\phi(y)]Q_n(y)=\lambda(1-\delta_{n,0})Q_{n-1}(y);\ n\geq 0, \tag{3.2}$$

$$\left.\begin{aligned}
&\frac{d}{dy}S_{1,n}(x,y)+[\lambda+\zeta_1(y)]S_{1,n}(x,y)=\lambda(1-\delta_{n,0})S_{1,n-1}(x,y);\ n\geq 0,\\
&\frac{d}{dy}S_{2,n}(x,y)+[\lambda+\zeta_2(y)]S_{2,n}(x,y)=\lambda(1-\delta_{n,0})S_{2,n-1}(x,y);\ n\geq 0,\\
&\frac{d}{dy}S_{3,n}(x,y)+[\lambda+\zeta_3(y)]S_{3,n}(x,y)=\lambda(1-\delta_{n,0})S_{3,n-1}(x,y);\ n\geq 0,
\end{aligned}\right\} \tag{3.3}$$

$$\lambda U_0=\int_0^\infty \phi(y)Q_0(y)dy+q\int_0^\infty \mu_3(x)P_{3,0}(x)dx; \tag{3.4}$$

Where $\delta_{n,m}$ denotes Kronecker's delta function.

These set of equations are to be solved under the following boundary condition at $x=0$;

$$P_{1,n}(0)=\lambda\delta_{n,0}U_0+\int_0^\infty \mu_2(x)P_{2,n}(x)dx+\int_0^\infty \mu_3(x)P_{3,n}(x)dx+q\int_0^\infty \phi(y)Q_n(y)dy;\ n\geq 0, \tag{3.5}$$

$$\left.\begin{aligned}
&P_{2,n}(0)=\int_0^\infty \mu_1(x)P_{1,n}(x)dx;\ n\geq 0\\
&P_{3,n}(0)=\int_0^\infty \mu_2(x)P_{2,n}(x)dx;\ n\geq 0
\end{aligned}\right\} \tag{3.6}$$

$at\ \ y=0:$

$$Q_n(0)=p\int_0^\infty \mu_3(x)P_{3,n}(x)dx;\ n\geq 0 \tag{3.7}$$

And at $y=0$ for $i=1,2,3$ and fixed values of x :

$$S_{i,n}(x;0)=\alpha_i P_{i,n}(x);\ n\geq 0. \tag{3.8}$$

With normalizing condition

$$U_0+\sum_{n=0}^{\infty}\left[\int_0^\infty Q_n(y)dy+\sum_{i=1}^{3}\left\{\int_0^\infty P_{i,n}(x)dx+\int_0^\infty\int_0^\infty S_{i,n}(x,y)dxdy\right\}\right]=1. \tag{3.9}$$

B. The model solution

To solve the system of eqs. (3.1)-(3.8),let us introduce the following PGFs for $i=1,2,3$ and $|z|<1$:

$$S_i(x,y;z)=\sum_{n=0}^{\infty} z^n S_{i,n}(x;y);\ S_i(x,0;z)=\sum_{n=0}^{\infty} z^n S_{i,n}(x;0),$$

$$Q(y;z)=\sum_{n=0}^{\infty} z^n Q_n(y);\ Q(0;z)=\sum_{n=0}^{\infty} z^n Q_n(0),$$

$$P_i(x,z)=\sum_{n=0}^{\infty} z^n P_{i,n}(x);\ P_i(0,z)=\sum_{n=0}^{\infty} z^n P_{i,n}(0)$$

Let $\lambda(z)=\lambda(1-z),$ then proceeding in usual manner with eqs. (3.2) and (3.3), we get a set of differential equation of Lagrangian type whose solutions are given by :

$$Q(y;z)=Q(0;z)[1-W(y)]\exp\{-\lambda(z)y\};\ y>0, \qquad (3.10)$$

$$S_i(x,y;z)=S_i(x,0;z)[1-G_i(y)]\exp\{-\lambda(z)y\};\ y>0 \text{ for } i=1,2,3 \qquad (3.11)$$

Where $S_i(x,0;z)$ can be obtained from eqs. (3.8), which after simplification yields

$$S_i(x,0;z)=\alpha_i P_i(x;z) \text{ for } i=1,2,3 \qquad (3.12)$$

Now solving the differential equation (3.1), we get

$$P_i(x;z)=P_i(0;z)[1-C_i(x)]\exp\{-D_i(z)x\};x>0 \text{ for } i=1,2,3 \qquad (3.13)$$

Where $D_i(z)=\lambda(z)+\alpha_i(1-G_i^*(\lambda(z)))$ for $i=1,2,3$

Utilizing eqs. (3.13) and (3.12) in (3.11), we get for $i=1,2,3$

$$S_i(x,y:z)=\alpha_i P_i(0;z)[1-C_i(x)]\exp\{-D_i(z)x\}\times[1-G_i(y)]\exp\{-\lambda(z)y\} \qquad (3.14)$$

Multiplying equation (3.5) by z^n and then taking summation over all possible values of $n\geq 0,$ we get on simplification

$$zP_1(0;z)=q[P_2(0;z)\beta_2^*(D_2(z))+P_3(0;z)\beta_3^*(D_3(z))]+Q(0;z)\vartheta^*(\lambda(z))-\lambda(z)U_0. \qquad (3.15)$$

Similarly from Eqs. (3.6) and (3.7), we have

$$\left.\begin{aligned} P_2(0;z)&=P_1(0;z)\beta_1^*(D_1(z)) \\ P_3(0;z)&=P_2(0;z)\beta_2^*(D_2(z)) \end{aligned}\right\} \qquad (3.16)$$

and $Q(0;z)=pP_3(0;z)\beta_3^*(D_3(z));$ respectively $\qquad (3.17)$

Now utilizing eqs. (3.16) and (3.17) in (3.15) and then simplifying, we get.

$$P_1(0;z)=\frac{\lambda(z)U_0}{[\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))-z]} \tag{3.18}$$

Let $z\to 1$ in eqs. (3.18), we obtain by the L'Hospital's rule

$$P_1(0;1)=\frac{\lambda U_0}{(1-\rho_H)};$$

Where $\rho_H=\rho_1(1+\alpha_1 g_1^{(1)})+\rho_2(1+\alpha_2 g_2^{(1)})+\rho_3(1+\alpha_3 g_3^{(1)})+p\rho_v$ is the utilizing factor of the system, $\rho_i=\lambda\beta_i^{(1)}$ for $i=1,2,3$ and $\rho_v=\lambda y^{(1)}$. This gives for $i=1,2,3$.

$$\left.\begin{aligned}
P_i(x;1)&=\frac{\lambda U_0[1-C_i(x)]}{(1-\rho_H)}\\
S_i(x,y;1)&=\frac{\alpha_i\lambda U_0[1-C_i(x)][1-G_i(y)]}{(1-\rho_H)}\\
\text{and } Q(y,1)&=\frac{p\lambda U_0[1-W(y)]}{(1-\rho_H)}
\end{aligned}\right\} \tag{3.19}$$

Now utilizing the normalizing condition (3.9). we get

$$U_0=(1-\rho_H); \tag{3.20}$$

Note that equation (3.19) represents steady-state probability that the server is idle but available in the system, Also, from equation (3.19). we have $\rho_H<1,$ which is the necessary and sufficient condition under which steady-state solution exists. Thus we summarize our results in the following Theorem 3.1.

Theorem 3.1. Under the stability condition $\rho_H<1,$ the joint distribution of the state of the server and the queue size has the following partial PGFs.

$$P_1(x,z)=\frac{(1-\rho_H)\lambda(z)[1-C_1(x)]\exp.\{-D_1(z)x\}}{[\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))-z]} \tag{3.21}$$

$$P_2(x,z)=\frac{(1-\rho_H)\lambda(z)\beta_1^*(D_1(z))[1-C_2(x)]\exp.\{-D_2(z)x\}}{[\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))-z]} \tag{3.22}$$

$$P_3(x,z)=\frac{(1-\rho_H)\lambda(z)\beta_1^*(D_1(z))\beta_2^*(D_2(z))[1-C_3(x)]\exp.\{-D_3(z)x\}}{[\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))-z]} \tag{3.23}$$

$$Q(y,z)=\frac{p(1-\rho_H)\lambda(z)\beta_1^*(A_1(z))\beta_2^*(A_2(z))[1-V(y)\exp.\{-\lambda(z)x]}{[\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))-z]} \tag{3.24}$$

$$S_1(x,y;z)=\frac{\alpha_1(1-\rho_H)\lambda(z)[1-C_1(x)]\exp\{-D_1(z)x\}\times[1-G_1(y)]\exp.\{-\lambda(z)y\}}{[\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))-z]} \quad (3.25)$$

$$S_2(x,y;z)=\frac{\alpha_2(1-\rho_H)\lambda(z)\beta_1^*(D_1(z)[1-B_2(x)]\exp\{-D_2(z)x\}\times[1-G_2(y)]\exp.\{-\lambda(z)y\}}{[\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))-z]} \quad (3.26)$$

and

$$S_3(x,y;z)=\frac{\alpha_3(1-\rho_H)\lambda(z)\beta_1^*(D_1(z)\beta_2^*(D_2(z))[1-B_3(x)]\exp\{-D_3(z)x\}\times[1-G_3(y)]\exp.\{-\lambda(z)y\}}{[\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))-z]}$$

(3.27)Where $\lambda(z)=\lambda(1-z)$ and $A_i(z)=\lambda(z)+\alpha_i(1-G_i^*(\lambda(z)))$; respectively for $i=1,2,3$

Remark 3.1 It is important to note here that such types of joint distributions are important to obtain the distribution of each state of the server in more comprehensive manner, which helps us to obtain marginal distributions of the server's states as well as stationary queue size distribution at a departure epoch.

Theorem 3.2 Under the stability condition $\rho_H<1$ the marginal PGFs of the server's state queue size distributions are given by

$$P_1(z)=\frac{(1-\rho_H)\lambda(z)[1-\beta_1^*(D_1(z))]}{D_1(z)[\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))-z]} \quad (3.28)$$

$$P_2(z)=\frac{(1-\rho_H)\lambda(z)\beta_1^*(D_1(z))[1-\beta_2^*(D_2(z))]}{D_2(z)[\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))-z]} \quad (3.29)$$

$$P_3(z)=\frac{(1-\rho_H)\lambda(z)\beta_1^*(D_1(z))\beta_2^*(D_2(z))[1-\beta_3^*(D_3(z))]}{D_3(z)[\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))-z]} \quad (3.30)$$

$$Q(z)=\frac{p(1-\rho_H)\beta_1^*(A_1(z)\beta_2^*(A_2(z))[1-\vartheta^*(\lambda(z))]}{[\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))-z]} \quad (3.31)$$

$$S_1(z)=\frac{\alpha_1(1-\rho_H)(1-G_1(\lambda(z)))[1-\beta_1^*(D_1(z))]}{D_1(z)[\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))-z]} \quad (3.32)$$

$$S_2(z)=\frac{\alpha_2(1-\rho_H)(1-G_2(\lambda(z)))\beta_1^*(D_1(z)))[1-\beta_2^*(D_2(z))]}{D_2(z)[\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))-z]} \quad (3.33)$$

$$S_3(z)=\frac{\alpha_3(1-\rho_H)(1-G_2(\lambda(z)))\beta_1^*(D_1(z)\beta_2^*(D_2(z))[1-\beta_3^*(D_3(z))]}{D_3(z)[\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))-z]} \quad (3.34)$$

Proof. Integrating equations (3.21) (3.22) (3.23) and (3.24) with respect to x and y respectively and then using the well known result of renewal theory.

$$\int_0^\infty e^{-\theta x}(1-C_i(x))dx = \frac{[1-\beta_i^*(\theta)]}{\theta} \text{ for } i=1,2,3$$

and $$\int_0^\infty e^{-\theta x}(1-W(y))dy = \frac{[1-\vartheta^*(\theta)]}{\theta};$$

we get formulae equations (3.28), (3.29) (3.30) and (3.31).

Similarly, integrating equation (3.14) with respect to y, get for $i=1,2,3$

$$S_i(x,z) = \int_0^\infty S_i(x,y;z)dy = \alpha_i[\lambda(z)]^{-1}[1-G_i^*(\lambda(z))]P_i(0;z)[1-C_i(x)]\exp\{-D_i(z)x\}. \quad (3.35)$$

Further integrating equations (3.35) with respect to x and utilizing equations (3.18) and (3.20), we claimed in formulae (3.32) (3.33) and (3.34). Next the system state probabilities are given in Corollary 3.1.

Corollary 3.1 If the system is in steady-state conditions, then

(i) The probability that the system is idle is

$$P_i = 1-\rho_1\{1+\alpha_1 g_1^{(1)}\} - \rho_2\{1+\alpha_2 g_2^{(1)}\} - \rho_3\{1+\alpha_3 g_3^{(1)}\} - \lambda p\gamma^{(1)}$$

(ii) the probability that the server is busy with FPS is $P_{C_1} = \rho_1$;

(iii) the probability that the server is busy with SPS is $P_{C_2} = \rho_2$;

(iv) the probability that the server is busy with TPS is $P_{C_3} = \rho_3$;

(v) the probability that the server is on vacation, $P_w = p\rho_w$;

(v) the probability that the server is under repair during FPS is, $P_{S_1} = \alpha_1\rho_1 g_1^{(1)}$.

(vi) the probability that the server is under repair during SPS is, $P_{S_2} = \alpha_2\rho_2 g_2^{(1)}$.

(vi) the probability that the server is under repair during TPS is, $P_{S_3} = \alpha_3\rho_3 g_3^{(1)}$.

Proof. Here we have

$$P_w = \lim_{z\to 1} Q(z),\ P_{C_i} = \lim_{z\to 1} P_i(z),\ P_{S_i} = \lim_{z\to 1} S_i(z), \text{ for } i=1,2,3 \text{ and } P_1 = 1-\sum_{i=1}^{3}\{P_{C_i}+P_{S_i}\} - P_{w.}$$

The stated formulae follow by direct calculation.

Finally, the derivation of the stationary queue size distribution at a departure epoch of this model is given in the proof of Theorem 3.3.

Theorem 3.3. Under the steady- state condition, the PGF of the stationary queue size at a departure epoch of this model is given by

$$\pi(z)=\frac{(1-\rho_H)(1-z)\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))}{[\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))-z]} \quad (3.36)$$

Proof. Following the argument of PASTA (see Wolf [24]). We state that a departing customer will see 'j' customer in the queue just after a departure if and only if there were 'j' customer in the queue TPS or a vacation just before the departure. Now denoting $\{\pi_j : j\geq 0\}$ as the probability that there are j units in the queue at a departure epoch, then for $j\geq 0$ we may write.

$$\pi_j=K_0 q\int_0^\infty \mu_3(x)P_{3,j}(x)dx+K_0\int_0^\infty \gamma(y)Q_j(y)dy; \quad (3.37)$$

Where K_0 is the normalizing constant.

Now multiplying both sides of Eq. (3.37) by z^j and then taking summation over $j\geq 0$ and utilizing equations (3.10) and (3.13), we get on simplification.

$$\pi(z)=\frac{K_0U_0\lambda(z)\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))}{[\{q+(q+p\vartheta^*(\lambda(z)))\beta_3^*(D_3(z))\}\beta_1^*(D_1(z))\beta_2^*(D_2(z))-z]} \quad (3.38)$$

Utilizing normalizing condition $\pi(1)=1,$ we get $K_0=\frac{(1-\rho_H)}{\lambda U_0}$

Hence formula (3.36) follows by inserting (3.39) in (3.38). Next the mean queue size of this model is given in Corollary 3.2

Corollary 3.2 Under the stability conditions, the mean number of customers in the system (i.e. mean queue length) $E[M_Q(t)]$ is given by

$$E[M_Q(t)]=\rho_H+\frac{\rho_1\rho_2\rho_3(1+\alpha_1g_1^{(1)})(1+\alpha_2g_2^{(1)})(1+\alpha_3g_3^{(1)})}{(1-\rho_H)}$$

$$+\frac{\lambda[\rho_1\{\alpha_1g_1^{(2)}+2(1+\alpha_1g_1^{(1)})^2\beta_S^{(1)}\}+\rho_2\{\alpha_2g_2^{(2)}+2(1+\alpha_2g_2^{(1)})^2\beta_S^{(2)}\}+\rho_3\{\alpha_3g_3^{(2)}+2(1+\alpha_3g_3^{(1)})^2\beta_S^{(3)}\}]}{2(1-\rho_H)}$$

$$+\frac{\lambda p[2\gamma^{(1)}\rho_1(1+\alpha_1g_1^{(1)})+2\gamma^{(1)}\rho_2(1+\alpha_2g_2^{(1)})+2\gamma^{(1)}\rho_3(1+\alpha_3g_3^{(1)})+\lambda\gamma^{(2)}]}{2(1-\rho_H)} \quad (3.39)$$

Where $\beta_S^{(i)}=\frac{\beta_i^{(2)}}{2\beta_i^{(1)}}$ is the residual service time of ith phase of service for $i=1,2,3$

Proof : The result follows directly by differentiating eqs(3.36) with respect to z and then taking limit z→1 by using the L-hospital rule.

IV. BUSY PERIOD DISTRIBUTION AND WAITING TIME DISTRIBUTION

In this section we provide main results for busy period distribution and waiting time distribution of this model. Since the derivation of busy period distribution is standard and it follows from existing literature of classical $M/G/1$ queue hence present the result without derivation in Theorem 4.1

Now we define 'T' as length of time interval that makes the server busy and this continues to the instant when the system becomes empty.

Theorem 4.1 Let $T_B^*(\theta)=E[e^{-\theta T_B}]$ be the LST of T_B. then Taka'cs functional equation under the steady state condition is given by

$$T_B^*(\theta)=H^*(\theta+\lambda(1-T_B^*(\theta)));$$

Where $H^*(\theta)=\{q+p\gamma^*(\theta)\}\beta_1^*(\theta+\alpha_1(1-G_1^*(\theta)))\beta_2^*(\theta+\alpha_2(1-G_2^*(\theta)))\beta_3^*(\theta+\alpha_3(1-G_3^*(\theta)))$.

The mean busy period is found to be

$$E(T_B)=\frac{\beta_1^{(1)}\{1+\alpha_1 g_1^{(1)}\}}{(1-\rho_H)}+\frac{\beta_2^{(2)}\{1+\alpha_2 g_2^{(1)}\}}{(1-\rho_H)}+\frac{\beta_3^{(3)}\{1+\alpha_3 g_3^{(1)}\}}{(1-\rho_H)}+\frac{p\gamma^{(1)}}{(1-\rho_H)}$$

Similarly, the waiting time distribution of a test customer for our model has the following LST.

Theorem 4.2 Let $W_Q^*(\theta)$ be the LST of the waiting time distribution of a test customer for this model under steady state condition, then

$$W_Q^*(\theta)=\frac{\theta(1-\rho_H)\{q+pW^*(\theta)\}\beta_1^*(\theta+\alpha_1(1-G_1^*(\theta)))\beta_2^*(\theta+\alpha_2(1-G_2^*(\theta)))\beta_3^*(\theta+\alpha_3(1-G_3^*(\theta)))}{\theta-\lambda[1-\{q+pW^*(\theta)\}\beta_1^*(\theta+\alpha_1(1-G_1^*(\theta)))\beta_2^*(\theta+\alpha_2(1-G_2^*(\theta)))\beta_3^*(\theta+\alpha_3(1-G_3^*(\theta)))]}$$

(4.1)

and $E[W_Q]=\dfrac{E[N_Q(t)]}{\lambda}$ (4.2)

Proof. The results follows directly from formula (3.36) by utilizing distributional form of Little's Law (e.g. see Keilson and Servi [26]);

$$W_Q^*(\lambda-\lambda z)=\pi(z). \quad (4.3)$$

Now setting $\lambda-\lambda z=\theta$ in eq. (4.3) and utilizing eq(3.35), we get (4.1). Similarly formula (4.2) follows directly by routine differentiation in (4.1) with respect to θ and then taking limit $\theta\to 0$ by using the L'Hospital's rule.

V. RELIABILITY ANALYSIS

Our final goal is to derive some reliability indices of this model. Now we will discuss two reliability indices of the system viz. – the system availability and failure frequency under the steady state conditions. Suppose that the system is initially empty. Let $A_E(t)$ be the point wise availability of the server at time $'t'$ that is, the probability that the server is either serving a customer or the server is available if the server is free and up during an idle period, such that the steady state availability of the server will be

$$A_E = \lim_{t\to\infty} A_E(t).$$

Theorem 5.1 The steady state availability of the server is given by.

$$A_E = 1 - \rho_1\alpha_1 g_1^{(1)} - \rho_2\alpha_2 g_2^{(1)} - \rho_3\alpha_3 g_3^{(1)} - \lambda\theta\gamma^{(1)} \qquad (5.1)$$

Proof. The result follows directly from Theorem (3.2) by considering the following equation.

$$A = U_0 + \sum_{i=1}^{3}\int_0^\infty P_i(x,1)dx = U_0 + \lim_{z\to 1}[P_1(z) + P_2(z) + P_3(z)].$$

By using (3.20), (3.28) (3.29) and (3.30) we get (5.1).

Theorem 5.2 The steady state failure frequency of the server is given by.

$$M_f = \alpha_1\rho_1 + \alpha_2\rho_2 + \alpha_3\rho_3 \qquad (5.2)$$

Proof. The result follows directly from equation (3.19) by utilizing the argument of Li et. Al. [5]

$$M_f = \alpha_1\int_0^\infty P_1(x,1)dx + \alpha_2\int_0^\infty P_2(x,1)dx + \alpha_3\int_0^\infty P_3(x,1)dx.$$

Now since $\int_0^\infty [1 - C_i(x)]dx = \int_0^\infty x dC_i(x) = \beta_i^{(1)}$; for $i = 1,2,3$; therefore from eq. (3.19) we have (5.2).

Next, we derive Laplace transform of reliability function and denote τ by the time to the first failure of the server, and then the reliability function of the server is $R(t) = P(\tau > t)$.

VI. CONCLUDING REMARKS

In this paper we are studing M/G/1 queueing system where arrival of customers are serving specific characteristic according to which – each customer requires three successive phases of service whereas the server is unreliable and it can may breakdown during any phase of serving service and after completion of three phases of service the server either go for a Bernoulli vacation or it can continuing its service to others customers if any. The obtained results are the following - the probability generating function of the joint distributions of the server state and queue size, the queue size distribution at the departure epoch, waiting time distribution, busy period distribution, the system availability, the failure frequency and the Laplace transform of the system reliability function.

This study can be complemented in various ways by introducing concepts of new vacation policies like modified vacation policy, work vacation policy etc. Further present model can be generalized for the arrival process to the case of a compound Poisson process.

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