Multiplicities for Riemann ξ Function and its Derivatives in the Critical Strip and Analysis for Their Nontrivial Zeros

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Abstract: *The multiplicity for Riemann ξ function in the critical strip $0 < \sigma < 1$ is presented based on the intermediate value theorem, and, similarly, the result is also true for Riemann ξ function's derivatives, and then the nontrivial zeros of Riemann ξ function are analyzed. All the nontrivial zeros fall on the critical line with the real part of 1 / 2, So the Riemann hypothesis is proven true and all the propositions equivalent to Riemann hypothesis and the conclusions based on Riemann hypothesis are all true.*

Keywords: *Riemann ξ function, Euler production, analytical continuation, second mean-value theorem, nontrivial zeros, Riemann hypothesis (RH).*

I. Introduction

It is well known that Riemann Zeta function satisfies the Euler product formula. i.e.,

for $s = \sigma + it$, $Re\, s = \sigma > 1$,

$$\zeta(s) = \sum_1^\infty \frac{1}{n^s} = \prod_p (1 - p^{-s})^{-1} \quad (1.1)$$

where, p in the formula pervades all primes.

The Euler multiplication formula establishes a close relation between the distribution of prime numbers and natural numbers. It is another expression of the fundamental theorem of arithmetic, that is, the decomposition theorem of integer unique prime numbers.

If, $Res = \sigma \le 1$, the series (1.1) does not converge absolutely. In 1859, Riemann extended(1.1) to the analytical continuation to the whole complex plane with an exception of the point $s = 1$ [1].

Riemann stated the famous Riemann hypothesis (RH) : all nontrivial zeros lie on the critical line $Res = \sigma = \frac{1}{2}$. In 1895, that 36 years later after Riemann first mentioned the problem , Hadamard [2] proved that there are no zeros on the real lines $Res = \sigma = 0$;. $Res = \sigma = 1$, with this , the prime number theorem was proven. Now, we can say that the so-called nontrivial zeros, are the zeros for the Riemann zeta function in the strip $0 < \sigma < 1$.

There are various analytic continuation ways to extend Riemann zeta function, the following form in terms of parameter integral based on the property of Jacobi functional equation is rooted in Riemann's first work on the function[1][2].

$$\zeta(s) = \pi^{s/2}\Gamma^{-1}(\frac{s}{2})(\frac{1}{s(s-1)} + \int_1^{\infty} \omega(x)(x^{\frac{s}{2}} + x^{\frac{1-s}{2}})\frac{dx}{x}) \qquad (1.2)$$

where $\omega(x) = \sum_1^{\infty} exp - (n^2 \pi x)\ x > 0$

By introducing an auxiliary function, Riemann arrived at the following entire function

$$\xi(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma(\frac{s}{2})\zeta(s)$$

$$\xi(s) = \frac{1}{2} + s(s-1)\int_1^{\infty} \omega(x)(x^{\frac{s}{2}} + x^{\frac{1-s}{2}})\frac{dx}{x} \quad (1.3)$$

The Riemann hypothesis (RH) states that, $0 < \mathrm{Re}\, s = \sigma < 1$，$\xi(s) = 0 \rightarrow \sigma = \frac{1}{2}$.

More than 160 years have passed since the Riemann hypothesis was first proposed in 1859. Due to the importance of the problem, various approaches, including physical methods, have been attempted[2], but none has been proved true so far.

The Euler product formula not only establishes the close relation between the Riemann zeta function and the prime numbers in the range of $\mathrm{Re}\, s = \sigma > 1$, but according to the global nature of the entire function, it is reasonable to expect that the Riemann zeta function in the critical strip might also have some multiplicity. If it does, it will be convenient to analyze the non-trivial zeros. And it has found that there exists counterexample with the functional equation property such as Riemann zeta function, without the production form, and the RH analogs is false[3].

The mean value theorem of integrals based on the intermediate value theorem is often used as a tool for the detailed analysis of integral function to eliminate the integral sign. As we have all known that there is no intermediate value features in complex function analysis, to use the generalized second mean value theorem for integrals before the Riemann zeta function analytical continuation, and then analysis of Riemann zeta function nontrivial zero, may be an effective way.

II. Multiplicity for Riemann ξ function in the critical strip

According to (1.2), Riemann zeta function $\zeta(s) = \sum_1^{\infty} \frac{1}{n^s}$has its analytical

continuation to the whole plane except at s = 1.And now we are ready to have the similar production form for Riemann zeta function in the critical strip $0 < \sigma < 1$ or in the closed domain $\forall\ \delta > 0, \delta \le \mathrm{Res} = \sigma \le 1 - \delta$. Consider within the critical strip, we have the following expression for Riemann zeta function.

Theorem 2.1. $\forall\ \delta > 0, \delta \le \mathrm{Res} = \sigma \le 1 - \delta$ and for any real number $\alpha, -\frac{1}{2} < \alpha < \frac{1}{2}$, we have the following production property form for Riemann zeta function.

$$\zeta(s) = \pi^{s/2}\Gamma^{-1}(\frac{s}{2})\int_1^{\infty} (\omega(x) - \frac{1}{2}x^{\frac{-1}{2}})\ x^{\alpha}(x^{\frac{s}{2}-\alpha} + x^{\frac{1-s}{2}-\alpha})\frac{dx}{x} \qquad (2.1)$$

The proof is direct, it is just the case of (1.2) in the critical strip.

Although (2.1) has its productive expression, it is acceptable for the analysis for trivial zeros, but not enough for the analysis of non-trivial zeros. We manage to deal with the expression without integration symbol.

In order to transform the product function integral into the product form of the integral result, we use the integral mean value theorem to remove the integral symbol, and then consider the zero analysis of complex variables after its analytic extension. To use integral mean value theorem, we have initially to deal with the function in real case.

Theorem 2.2 On the closed region, $\delta \leq \mathrm{Re}s = \sigma \leq 1-\delta$, $-\frac{1}{2} < \alpha < \frac{1}{2}$, taking the auxiliary function,

$$\xi(s) = \int_1^{\infty}(\omega(x) - \frac{1}{2}x^{-\frac{1}{2}})(x^{\frac{s}{2}} + x^{\frac{1-s}{2}})\frac{dx}{x} \tag{2.2}$$

Then, there exists $c(x) > 1$ such that the corresponding analytic function is

$$\xi(s) = (1 - \int_1^{\infty}\frac{\omega(x)}{x}dx)(c(s)^{\frac{s}{2}} + c(s)^{\frac{1-s}{2}}) \tag{2.3}$$

Proof.

The complex function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ is written as ,i.e.,

$$\xi(z) = f(x+iy) = u(x+iy,0) + iv(x+iy,0)$$

In particular, $v(x,0) = 0$ and we have

$$\xi(z) = f(x+iy) = u(x+iy,0) \tag{2.4}$$

For the integral of two product functions, if the function g(x) is continuous in [a, b] keeping the same sign and the function h(x) is continuous, then there exists a point c in [a, b], such that

$$\int_a^b g(x)h(x)dx = h(c)[G(b) - G(a)] \tag{2.5}$$

where, $G'(x) = g(x)$.

In particular, if $h(x)$ is equal to 1, then (2.4) is reduced to the form of the fundamental theorem of calculus. If $G(x) = x$, then (2.4) is reduced to the classical mean value theorem of integrals.

Lets $G(x) = x^{-\frac{1}{2}} - \int_x^{\infty}\frac{\omega(t)}{t}dt$, then $G'(x) = g(x) = \frac{\omega(x)}{x} - \frac{1}{2}x^{-\frac{3}{2}}$ and it keeps same sign for $x > 1$, so there exists a real function $c(x) > 1$, such that

$$\xi(x) = (1 - \int_1^{\infty}\omega(y)dy)(c(x)^{\frac{x}{2}} + c(x)^{\frac{1-x}{2}})$$

Similarly,

$$\xi(\sigma)=\int_1^{\infty}(\frac{\omega(x)}{x}-\frac{1}{2}x^{-\frac{3}{2}})(x^{\frac{\sigma}{2}}+x^{\frac{1-\sigma}{2}})\,dx$$

$$\xi(\sigma)=\int_1^{\infty}(x^{\frac{\sigma}{2}}+x^{\frac{1-\sigma}{2}})(x^{-\frac{1}{2}}-\int_x^{\infty}\frac{\omega(y)}{y}dy)'\,dx$$

$$\xi(\sigma)=\lim_{N\to\infty}\int_1^{N}(x^{\frac{\sigma}{2}}+x^{\frac{1-\sigma}{2}})(x^{-\frac{1}{2}}-\int_x^{\infty}\frac{\omega(y)}{y}dy)'\,dx$$

Let $G(x)=x^{-\frac{1}{2}}-\int_x^{\infty}\frac{\omega(y)}{y}dy$, there exists

$\infty>c(\sigma)>1,\xi(\sigma)=(1-\int_1^{\infty}\frac{\omega(y)}{y}dy))(c(\sigma)^{\frac{\sigma}{2}}+c(\sigma)^{\frac{1-\sigma}{2}})$

replace σ by $s=\sigma+it$,we arrive at the corresponding analytic complex function

$$\xi\,(s)=(1-\int_1^{\infty}\frac{\omega(x)}{x}dx)(c(s)^{\frac{s}{2}}+c(s)^{\frac{1-s}{2}})$$

The proof is completed.

III. Riemann ξ function's derivatives

For the importance of Riemann ξ function, its derivatives have also attracted people's focus on them and we have know that if Riemann hypothesis is true then all nontrivial zeros of Riemann ξ function's derivatives' nontrivial zeros all fall on the critical line.

Theorem 3.1. If $0<\sigma<1$, then, Riemann ξ function's derivatives also have the multiplicities as followings

$$\xi^{(n)}(s)=\int_1^{\infty}[\omega(x)-\frac{1}{2}x^{-\frac{1}{2}}]2^{-n}(\ln x)^n(x^{\frac{s}{2}}+(-1)^n x^{\frac{1-s}{2}})\frac{dx}{x} \qquad (3.1)$$

and

$$\xi^{(n)}(s)=(1-\int_1^{\infty}\frac{\omega(x)}{x}dx)2^{-n}[(\ln c(s)]^n(c(s)^{\frac{s}{2}}+(-1)^n c(s)^{\frac{1-s}{2}}) \qquad (3.2)$$

The proof is similar to the proof of theorem 2.2.

Since we have had the productive forms of Riemann ξ function and its derivatives, just as the factorization form for algebra equation, it is easier to analyze its non trivial zeros for the functions now.

IV. Nontrivial zeros for Riemann ξ function and its derivatives

Theorem 4.1

For Riemann zeta function$\xi(s)$ and its derivatives $\xi^{(n)}(s)$, if $0<\sigma<1$, then we have the followings ,

$$\xi(s)=0\to\sigma=\frac{1}{2} \qquad (4.1)$$

and

$$\xi^{(n)}(s) = 0 \to \sigma = \frac{1}{2} \qquad (4.2)$$

Proof.

To prove (4.1), in the critical strip $0 < \sigma < 1$, since $(1 - \int_1^\infty \frac{\omega(x)}{x} dx) < 0$, so we have,

$$\xi(s) = 0 \leftrightarrow f(s) = 0 \leftrightarrow c(s)^{\frac{s}{2}} + c(s)^{\frac{1-s}{2}} = 0$$

We then notice that the point $s = \sigma + it$ satisfying

$$c(s) = c(\sigma, t) = 1$$

is not the nontrivial zeros of (3.2) for $1 + 1 \neq 0$.

we can rewrite $c(s)^{\frac{s}{2}} + c(s)^{\frac{1-s}{2}}$ as the following form

$$e^{(\ln r_c + \theta_c i + 2k\pi i)(\frac{\sigma+it}{2})} + e^{(\ln r_c + \theta_c i + 2k\pi i)(\frac{1-\sigma-it}{2})} = 0 \qquad (4.3)$$

$$e^{(\ln r_c + \theta_c i + 2k\pi i)(\frac{\sigma+it}{2})} = e^{\pi i} e^{(\ln r_c + \theta_c i + 2k\pi i)(\frac{1-\sigma-it}{2})} \qquad (4.4)$$

$$(2\sigma - 1)\ln r_c = 4t\theta_c \qquad (4.5)$$

$$t\ln r_c + \frac{(2\sigma-1)}{2}\theta_c = 0(\mathrm{mod}\pi) \qquad (4.6)$$

Since $\ln r_c = 0$, $\theta_c = 0$ are not the solutions for $c(s)^{\frac{s}{2}} + c(s)^{\frac{1-s}{2}} = 0$, so we have,

$$\theta_c = 0;\ t\ln r_c = \pi(\mathrm{mod}\pi)$$

and from (4.5), we can only have,

$$\sigma = \frac{1}{2}$$

The proof is completed.

Similarly we can prove (4.2).

And from theorem 3.2. and 4.1. we have the following corollary

Corollary 4.2

Riemann hypothesis is true, and all the propositions equivalent to Riemann hypothesis and the conclusions based on Riemann hypothesis are all true.

References

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