On unified subclass of univalent functions of complex order using the Frasin operator

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Abstract — *In the present investigation, we consider a unified class of univalent functions of complex order defined in the open unit disk $\mathbb{U}$ involving the Frusin operator. Some known consequences of the results are also derived.*

Keywords — *Analytic functions, Univalent functions, Starlike functions, Complex order and Subordination.*

I. INTRODUCTION

Let $\mathcal{A}$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (1.1)$$

which are analytic in the open unit disk $\mathbb{U} = \{z;\ |z| < 1\}$

Let f and g are analytic in $\mathbb{U}$. We say that the function f is subordinate to g and we write $f \prec g$ or $f(z) \prec g(z) (z \in \mathbb{U})$, if there exists an analytic function ω in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$, such that $f(z) = g(\omega(z))\ (z \in \mathbb{U})$. if g is univalent in $\mathbb{U}$, then the following equivalence relationship holds good see(9):

$$f(z) \prec g(z) \Leftrightarrow f(0) \prec g(0), f(\mathbb{U}) \subset g(\mathbb{U}) \qquad (1.2)$$

$\mathcal{S}$ (say) be the subclass of $\mathcal{A}$ consisting of univalent functions. For the function $f \in \mathcal{A}$ and making use of the binomial series

$$(1-\vartheta)^k = \sum_{i=0}^{k} \binom{k}{i} (-1)^i (\vartheta)^i, \text{ where } i = \mathbb{N} \cup \{0\} \text{ and } k \in \mathbb{N}.$$

Now we define the differential operator $\mathcal{D}_{k,\vartheta}^{\xi} f(z)$ as follows:

$$\mathcal{D}^0 f(z) = f(z). \qquad (1.3)$$

$$\mathcal{D}_{k,\vartheta}^{1} f(z) = (1-\vartheta)^k f(z) + (1-(1-\vartheta)^k) z f'(z). \qquad (1.4)$$

$$= \mathcal{D}_{k,\vartheta} f(z),\ \ \vartheta > 0; k \in \mathbb{N}. \qquad (1.5)$$

$$\mathcal{D}_{k,\vartheta}^{\xi} f(z) = \mathcal{D}_{k,\vartheta}\left(\mathcal{D}^{\xi-1} f(z)\right), (\xi \in \mathbb{N}). \qquad (1.6)$$

If f is given by (1.1), then from (1.5) and (1.6), we see that

$$\mathcal{D}_{k,\vartheta}^{\xi} f(z) = z + \sum_{n=2}^{\infty}\left(1 + (n-1)\sum_{i=1}^{k}\binom{k}{i}(-1)^{i+1}\vartheta^i\right)^{\xi} a_n z^n, \xi \in \mathbb{N} \cup \{0\}. \qquad (1.7)$$

Using the relation (1.7), it is easily verified that

$$\mathcal{C}_i^k(\vartheta)\, z\left(\mathcal{D}_{k,\vartheta}^{\xi} f(z)\right)' = \mathcal{D}_{k,\vartheta}^{\xi+1} f(z) - \left(1 - \mathcal{C}_i^k(\vartheta)\right)\mathcal{D}_{k,\vartheta}^{\xi} f(z). \qquad (1.8)$$

where, $\mathcal{C}_i^k(\vartheta) = \sum_{i=1}^{k}\binom{k}{i}(-1)^{i+1}\vartheta^i$.

We observe that for $k = 1$, we obtain the differential operator $\mathcal{D}_{1,\vartheta}^{\xi}$ defined by Al-Oboudi [11] and for $k = \vartheta = 1$, we get Salagean differential operator $\mathcal{D}^{\xi}$ [5].

The main aim of the present investigation is to apply a method based on the differential subordination in order to derive many subordination results involving the operator $\mathcal{D}_{k,\vartheta}^{\xi}$. Furthermore, we get the previous results of Srivastava and Lashin [14] as special cases of some of the results presented here.

II. DEFINITIONS

Let $\phi(z)$ be an analytic function with positive real part of ϕ with $\phi(0) = 1, \phi'(0) > 0$ which maps $\mathbb{U}$ onto a region Starlike with respect to $(1-b)$.

Then the class $A^p(\xi, m, \lambda)$ consists of all analytic functions $f \in \mathcal{A}_{H(\mu)}$ satisfying

$$\frac{1}{b}\left[(1-\beta)\frac{z\left(D_{m,\lambda}^{\xi} f(z)\right)'}{D_{m,\lambda}^{\xi} f(z)} + \beta\frac{z\left(D_{m,\lambda}^{\xi+1}\right)'}{D^{\xi+1}} - (1-b)\right] \prec \phi(z)$$

$(b \in \mathbb{C}^*, \beta \geq 0, m, \xi \in N \cup \{0\})$

III. MAIN RESULTS

Unless otherwise mentioned, we assume throughout the sequel that $b \in \mathbb{C}^*, \beta \geq 0, m, \xi$ ...and all powers are understood as principle values.

To prove our main results, we need the following Lemma.

Lemma 3.1: Let ϕ be the a convex function defined on, $\phi(0) = 1$, define $F(z)$ by

$$F(z) = z\, exp\left(\int_0^z \frac{\phi(t)-1}{t}dt\right) \tag{3.1}$$

Let $p(z) = 1 + p_1 z + p_2 z^2 + \cdots$ be analytic in, then,

$$1 + \frac{zq'(z)}{q(z)} \prec \phi(z) \tag{3.2}$$

If and only if for all $|s| \leq 1$ and $|t| \leq 1$, we have

$$\frac{p|tz|}{P|sz|} \prec \frac{SF|tz|}{tF|sz|} \tag{3.3}$$

Lemma 3.2: Let $q(z)$ be univalent in $\mathbb{U}$ and let $\phi(z)$ be analytic in domain containing $q'(z)$ if $\frac{zq'(z)}{q(z)}$ is starlike, then

$$zp'(z)\,\phi(p(z)) \prec zq'(z)\phi(q(z))$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

Theorem 3.1: Let $\phi(z)$ and $F(z)$ be as lemma 3.1. The function $f \in A^p(\xi, m, \lambda, \phi)$ if and only if $|s| \leq 1$, and $|t| \leq 1$, we have

$$\left[\frac{S}{t}\left(\left(\frac{D_{m,\lambda}^{\xi} f(tz)}{D_{m,\lambda}^{\xi} f(sz)}\right)^{1-\beta}\right)\left(\frac{D_{m,\lambda}^{\xi+1} f(tz)}{D_{m,\lambda}^{\xi+1} f(sz)}\right)^{\beta}\right]^{\frac{1}{b}} \prec \frac{S}{t}\frac{F(tz)}{F(sz)}$$

Proof: let $p(z) = \left[\frac{D_{m,\lambda}^{\xi} f(z)}{z}\left(\frac{D_{m,\lambda}^{\xi+1} f(z)}{D_{m,\lambda}^{\xi} f(z)}\right)^{\beta}\right]^{\frac{1}{b}}$

Taking logarithm derivative, we have

$$1 + \frac{zp'(z)}{p(z)} = \frac{1}{b}\left[(1-\beta)\frac{z\left(D_{m,\lambda}^{\xi} f(z)\right)'}{D_{m,\lambda}^{\xi} f(z)} + \beta z\frac{\left(D^{\xi+1} f(z)\right)'}{D^{\xi+1} f(z)} - (1-\beta)\right]$$

Since $f \in A^p(\xi, m, \lambda, \phi), \in A^p(\xi, m, \lambda, \phi)$, then,

$$1 + \frac{zp'(z)}{p(z)} \prec \phi(z)$$

and the result now follows from Lemma 3.1. Putting $m = 0$ and $\beta = 0$

In theorem 3.1, we obtain the following corollary

Corollary 3.1: Let $\phi(z)$ and $F(z)$ be as lemma 3.1. The function $f \in A(\bar{S}, \phi)$ if and only if for all $|S| \leq 1$and $|t| \leq 1$, we have

$$\left[\frac{S}{t}\frac{f(tz)}{f(sz)}\right]^{\frac{1}{b}} \prec \frac{S}{t}\frac{F(tz)}{F(sz)} \tag{3.4}$$

For $m = 0$ and $\beta = 0$ in theorem 3.1, we have the following result.

Corollary 3.2: Let $\phi(z)$ and $F(z)$ be as lemma 3.1. The function $f \in A(\bar{S}, \phi)$ if and only if for all $|s| \leq 1$and $|t| \leq 1$, we have

$$\left[\frac{f'(tz)}{f'(sz)}\right]^{\frac{1}{b}} \prec \frac{S}{t}\frac{F(tz)}{F(sz)} \tag{3.5}$$

Theorem 3.2: Let $\phi(z)$ be starlike with respect to 1 and $F(z)$ given by (3.1) be starlike if $f \in A^p(\xi, m, \lambda, \phi)$, then,

$$\left[\frac{D_{m,\lambda}^{\xi}f(z)}{z}\left(\frac{D_{m,\lambda}^{\xi+1}f(z)}{D_{m,\lambda}^{\xi}f(z)}\right)^{\beta}\right] \prec \left[\frac{F(z)}{z}\right]^{b}$$

Proof: let $q(z) = \frac{F(z)}{z}$

On solving, we obtain

$$1 + \frac{zp'(z)}{p(z)} = \frac{1}{b}\left[(1-\beta)\frac{z\left(D_{m,\lambda}^{\xi}f(z)\right)'}{D_{m,\lambda}^{\xi}f(z)} + \beta z\frac{\left(D_{m,1}^{\xi+1}f(z)\right)'}{D_{m,1}^{\xi+1}f(z)} - (1-b)\right]$$

Since $f \in A^{p}(\xi, m, \lambda,)$, we have $\frac{zp'(z)}{p(z)} \prec \frac{zq'(z)}{q(z)}$

The result now follows by an application of Lemma 3.2. Taking $\xi = 0$ and $\beta = 0$ in theorem 3.2, we get the following corollary.

Corollary 3.3: Let $\phi(z)$ be starlike with respect to (1-b) and $F(z)$ given by (3.1) be starlike iff $f \in A^{p}(\xi)$ then $\frac{f(z)}{z} \prec \left(\frac{F(z)}{b}\right)^{b}$.

Taking $\xi = 0$ and $\beta = 1$in theorem 3.2, we have the following corollary

Corollary 3.4: If $f \in A^{p}(0,1)$, let $\phi(z)$ be starlike with respect to (1-b) and $F(z)$ given by (3.1) be starlike. If $f \in A^{m}(0,1)$, then we have

$$\frac{f(z)}{z}\left[\frac{(1-\lambda)^{m}f(z)zf'(z)}{f(z)}\right] \prec \left(\frac{F(z)}{b}\right)^{b}.$$

Taking $\xi = 0, \lambda = 1$ we get following corollary

Corollary 3.5: $\frac{f(z)}{z}\left(\frac{zf'(z)}{f(z)}\right)^{\beta} \prec \left(\frac{F(z)}{b}\right)^{b}$.

Taking $\xi = 0$ and $\phi(z) = \frac{1+Az}{1+Bz}$; $(-1 \le B \le A \le 1)$

In theorem 3.2, we get the following corollary.

Corollary 3.6: $f \in A^{m}(0,\beta,b), f \in A^{p}\left(\frac{1+Az}{1+Bz}, 0, \beta, b\right)$; $(-1 \le B \le A \le 1)$.

Then we have

$$\frac{f(z)}{z}\left(\frac{zf'(z)}{f(z)}\right)^{p} \prec (1+Bz)^{\frac{A-B}{B}} (B \neq 0)$$

For $e^{-i\gamma}\cos\gamma$ in theorem 3.2. We get the following result.

$$\frac{f(z)}{z} \prec (1-z)^{-2b}$$

For $\xi = 0, \phi(z) = \frac{1+z}{1-z}$, $\beta = 0, \lambda = 1$ in theorem 3.2. We get the following corollary

Corollary 3.7: $f \in A^{m}(b)$, then we have

$f'(z) \prec (1-z)^{-2b}$

For $\xi = 0, \phi(z) = \frac{1+z}{1-z}$, $\beta = 0$ and replacing b by $be^{-i\gamma}\cos\gamma$ $(|\gamma| < \frac{\pi}{2}, b \in \mathbb{C}^{*}$ in theorem 3.2, we get the following result of Aouf et. Al [11].

Corollary 3.8: $f \in A^{\gamma}(b)$, then we have

$$\xi = 0, \phi(z) = \frac{1+z}{1-z}, \quad \beta = 0, \lambda = 1$$

then,

$$\frac{f(z)}{z} \prec (1-z)^{-2be^{-i\gamma}\cos\gamma}$$

For $m = 0, \phi(z) = \frac{1+z}{1-z}$, $\beta = 1, b \to be^{-i\gamma}\cos\gamma$ $\left(|\gamma| < \frac{\pi}{2}, b \in \mathbb{C}^{*}\right)$, $\lambda = 1$ in theorem 3.2 we get following result.

Corollary 3.9: If $f \in A^{\gamma,m}(b)$ then we have

$$f'(z) \prec (1-z)^{-2be^{-i\gamma}\cos\gamma}$$

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