A Fixed Point Theorem on b-Metric Space

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Abstract: *The intend of this paper to obtain different concepts of contractive mapping stay alive on b-metric space. On b-metric space establish the completeness and also verify that mapping has unique of fixed point.*

Keywords: *Unique fixed point, Convergence, b-Metric space.*

I. Introduction

In various twigs of science, economics, computer science, engineering and the progress of nonlinear analysis, the fixed point theory is one of the mainly essential tools.

In 1989, Backhtin [1] introduce the perception of b-metric space. In 1993, Czerwik[4],[5] unmitigated the results of b-metric spaces .Using this initiative many researcher offered simplification of the prominent banach fixed point theorem within the b-metric space.

Mehmet Kir[6], Czerwik,s[4],[5], Pacurar[7] extensive the fixed point theorem within b-metric space. Different exertion of the convergence of measurable functions by means of admiration to measure, Czerwik [4],[5] first offered a simplification of banach fixed point theorem within b-metric spaces.

We would akin to broaden some fixed point theorem which are convincing within b-metric space.

Definition 1.1. Let X be a non-empty set and $s \geq 1$ be a given real number. A function $d: X \times X \to R_+$ is called a b- metric provided that for all $x, y, z \in X$

1) $d(x, y) = 0$ if and only if $x = y$,
2) $d(x, y) = d(y, x)$,
3) $d(x, z) \leq s[d(x, y) + d(y, z)]$.

A pair(X, d) is called a b-metric space. It is clear that definition of b-metric space is an extension of usual metric space.

Some examples of b-metric spaces are given below:

Example 1.2. By Boriceanu [3], The set $l_p(R)$ (with $0 < p < 1$), where $l_p(R) \coloneqq \{(x_n) \subset R \mid \sum_{n=1}^{\infty} |x_n|^p < \infty\}$, together with the function $d: l_p(R) X\, l_p(R) \to R$,

$$d(x, y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^p\right)^{1/p}$$

where $x = x_n\,, y = y_n \in l_p(R)$ is a b-metric space. By an elementary calculation we obtain that $d(x, z) \leq 2^{\frac{1}{p}}[d(x, y) + d(y, z)]$.

Example 1.3. By Boriceanu [4], Let $X = \{0,1,2\}$ and $d(2,0) = d(0,2) = m \geq 2$, $d(0,1) = d(1,2) = d(1,0) = d(2,1) = 1$ and $d(0,0) = d(1,1) = d(2,2) = 0$. then $d(x, y) \leq \frac{m}{2}[d(x, z) + d(z, y)]$ for all $x, y, z \in X$. if $m > 2$ then the triangle inequality does not hold.

Example 1.4. By Boriceanu[3], The space $L_p[0,1]$ (where $0 < p < 1$) of all real functions $x(t), t \in [0,1]$ such that $\int_0^1 |x(t)|^p dt < \infty$, is a b-metric space if we take $d(x, y) = (\int_0^1 |x(t) - y(t)|^p dt)^{\frac{1}{p}}$, for each $x, y \in L_p[0,1]$.

Definition 1.5. By Boriceanu [3] Let (X, d) be a b-metric space .Then a sequence $\{x_n\}$ in X is called a Cauchy sequence if and only if for all $\varepsilon > 0$ there exist $n(\varepsilon) \in N$ such that for each $n, m \geq n(\varepsilon)$ we have $d(x_n, x_m) < \varepsilon$.

Definition 1.6. By Boriceanu [3] Let (X, d) be a b-metric space then a sequence $\{x_n\}$ in X is called convergent sequence if and only if there exists $x \in X$ such that for all there exists $n(\epsilon) \in N$ such that for all $n \geq n(\epsilon)$ we have $d(x_n, x) < \epsilon$. In this case we write $\lim_{n\to\infty} x_n = x$.

Definition 1.6. [3] The b- metric space is complete if every Cauchy sequence convergent.

II. MAIN RESULT

Theorem 2.1. Let (X, d) be a complete b-metric space. Let T be a mapping T:X→ X such that

$$d(Tx,Ty) \leq [a\{\frac{d(x,Tx)+d(y,Ty)}{2}\} + b\{\frac{d(x,Ty)+d(y,Tx)}{2}\} + c\,d(x,y)], \quad ...(1)$$

where $a, b, c > 0$ such that $a + bs + c \leq 1$ $\forall\, x, y \in X$ and $s \geq 1$ then T has a unique fixed point.

Proof: Let $x_0 \in X$ and $\{x_n\}_{n=1}^{\infty} \in X$

defined by the recursion $x_n = Tx_{n-1} = T^n x_0$ $n = 1,2 ...$ ...(2)

By (1) and (2) we obtain that

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n)$$
$$\leq [a\{\frac{d(x_{n-1},Tx_{n-1}) + d(x_n,Tx_n)}{2}\} + b\{\frac{d(x_{n-1},Tx_n) + d(x_n,Tx_{n-1})}{2}\} + c\,d(x_n, x_{n+1})],$$
$$d(Tx_{n-1}, Tx_n) \leq [a\{\frac{d(x_{n-1},x_n) + d(x_n,x_{n+1})}{2}\} + b\{\frac{d(x_{n-1},x_{n+1}) + d(x_n,x_n)}{2}\} + c\,d(x_n, x_{n+1})],$$
$$d(Tx_{n-1}, Tx_n) \leq [a\{\frac{d(x_{n-1},x_n) + d(x_n,x_{n+1})}{2}\} + b\,s\,\{\frac{d(x_{n-1},x_n) + d(x_n,x_{n+1})}{2}\} + c\,d(x_n, x_{n+1})],$$
$$\left(1 - \frac{a}{2} - \frac{bs}{2} - c\right) d(x_n, x_{n+1}) \leq \left(\frac{a}{2} + \frac{bs}{2}\right) d(x_{n-1}, x_n),$$
$$d(x_n, x_{n+1}) \leq \frac{\left(\frac{a}{2}+\frac{bs}{2}\right)}{\left(1-\frac{a}{2}-\frac{bs}{2}-c\right)} d(x_{n-1}, x_n),$$
$$d(x_n, x_{n+1}) \leq \lambda\, d(x_{n-1}, x_n),$$

where $\lambda = \frac{\left(\frac{a}{2}+\frac{bs}{2}\right)}{\left(1-\frac{a}{2}-\frac{bs}{2}-c\right)} < 1$

$$d(x_n, x_{n+1}) \leq \lambda\, d(x_{n-1}, x_n),$$
$$d(x_n, x_{n+1}) \leq \lambda^2 d(x_{n-2}, x_{n-1}),$$
$$: d(x_n, x_{n+1}) \leq \lambda^n\, d(x_0, x_1).$$

Therefore T is a contractive mapping.

Now, we show that $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X. Let $m.n \in N$, $m > n$,

$$d(x_n, x_m) \leq s\{d(x_n, x_{n+1}) + d(x_{n+1}, x_m) \leq s d(x_n, x_{n+1}) + s^2\{d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_m)\},$$
$$\leq s d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^2 d(x_{n+2}, x_m),$$
$$\leq s d(x_n, x_{n+1}) + s^2 d(x_{n+1}, x_{n+2}) + s^3 d(x_{n+2}, x_{n+3}) + \cdots ...$$
$$\leq s\lambda^n d(x_0, x_1) + s^2\lambda^{n+1} d(x_0, x_1) + s^3\lambda^{n+2} d(x_0, x_1) + \cdots ...$$
$$\leq s\lambda^n\, d(x_0, x_1)[1 + s\lambda + (s\lambda)^2 + (s\lambda)^3 + \ldots\ldots],$$
$$\leq \frac{s\lambda^n}{1-s\lambda}\, d(x_0, x_1).$$

Then $\lim_{n\to\infty} d(x_n, x_m) = 0$, as $n, m \to \infty$, since $\lambda < 1$, $\lim_{n\to\infty} \frac{s\lambda^n}{1-s\lambda}\, d(x_0, x_1) = 0\ as\ n, m \to \infty$.

Hence $\{x_n\}_{n=1}^{\infty} \in X$ is a Cauchy sequence. Since $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence therefore $\{x_n\}_{n=1}^{\infty}$ Converges to $x^* \in X$.

Now, we show that $x^*$is the fixed point of T.

$$d(x^*, Tx^*) \leq s\{d(x^*, x_{n+1}) + d(x_{n+1}, Tx^*)\}$$
$$\leq s\,d(x^*, x_{n+1}) + s\,d(Tx_n, Tx^*),$$
$$d(x^*, Tx^*) \leq s\,d(x^*, x_{n+1}) + s[a\{\frac{d(x_n,Tx_n) + d(x^*,Tx^*)}{2}\} + b\{\frac{d(x_n,Tx^*) + d(x^*,Tx_n)}{2}\} + c\,d(x_n, x^*)],$$

$d(x^*,Tx^*) \le [s\, d(x^*,x_{n+1}) + s\, a\{\frac{d(x_n,x_{n+1}) + d(x^*,Tx^*)}{2}\} + \frac{s^2 b}{2}\{ d(x_n,x^*) + d(x^*,Tx^*)\}$
$+\frac{sb}{2}\{d(x^*,x_{n+1})\} + s\,c\, d(x_n,x^*)]$,
$d(x^*,Tx^*) \le [s\, d(x^*,x_{n+1}) + \frac{s^2 a}{2}\{d(x_n,x^*) + d(x^*,x_{n+1})\} + \frac{s\,a}{2} d(x^*,Tx^*)\} + \frac{s^2 b}{2}\{ d(x_n,x^*) + d(x^*,Tx^*)\}$
$+\frac{sb}{2}\{d(x^*,x_{n+1})\} + s\,c\, d(x_n,x^*)]$,
$(1 - \frac{s\,a}{2} - \frac{s^2 b}{2})\, d(x^*,Tx^*) \le (sc + \frac{s^2 a}{2} + \frac{s^2 b}{2})\, d(x_n,x^*) + (s + \frac{s^2 a}{2} + \frac{sb}{2})\, d(x^*,x_{n+1})$

$$d(x^*,Tx^*) \le \frac{(s + \frac{s^2 a}{2} + \frac{sb}{2})}{(1 - \frac{s\,a}{2} - \frac{s^2 b}{2})} d(x^*,x_{n+1}) + \frac{(sc + \frac{s^2 a}{2} + \frac{s^2 b}{2})}{(1 - \frac{s\,a}{2} - \frac{s^2 b}{2})} d(x_n,x^*).$$

As $\lim n \to \infty$, we have $\lim_{n\to\infty} d(x^*,Tx^*) = 0$,

Therefore$x^* = Tx^*$,

Hence x^* is the fixed point of T.

Uniqueness of Fixed Point: We have to show that x^* is unique fixed point of T.

Assume that x' is another fixed point of T then we have

$Tx' = x'$ and $Tx^* = x^*$
$d(x^*,x') = d(Tx^*,Tx') \le [a\{\frac{d(x^*,Tx^*)+d(x',Tx')}{2}\}$ +b $\{\frac{d(x^*,Tx')+d(x',Tx^*)}{2}\} + c\, d(x^*,x')]$,
$\le [a\{\frac{d(x^*,x^*)+d(x',x')}{2}\}$ +b $\{\frac{d(x^*,x')+d(x',x^*)}{2}\} + c\, d(x^*,x')]$, $(1-b-c)\, d(x^*,x')$
=0,

Since $(1-b-c) \ne 0$

Therefore $d(x^*,x')$ =0,

This is contradiction. Therefore $x^* = x'$.

Hence x^* is the unique fixed point. This completes the proof.

Corollary: . Let (X,d) be a complete b-metric space and $s \ge 1$. Let T be a mapping T: X→ X such that
$d(Tx,Ty) \le a\{d(x,Tx) + d(y,Ty)\} + b\{d(x,Ty) + d(y,Ty)\} + c\, d(x,y)$
where $a,b,c > 0$ such that $2a + b(2s+1) + c < 1$ $\forall\, x,y \in X$ then T has a unique fixed point.

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