Weighted dom-chromatic number of Type-I weighted complete caterpillars

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Abstract

A set D of vertices is a dominating set of G if every vertex not in D is adjacent to at least one member of D. A set D of vertices is said to be dom-chromatic if D is a dominating set and $\chi(\langle D\rangle) = \chi(G)$. A Weighted tree, (T, w) a tree together with a positive weight function on the vertex set $w : V(T) \longrightarrow R^+$. The weighted domination number $\gamma_w(T)$ of (T, w) is the minimum weight $w(D) = \sum_{v\in D} w(v)$ of a dominating set D of T. The weighted dom-chromatic number $\gamma_{wch}(T)$ of (T, w) is the minimum weight $w(D) = \sum_{v\in D} w(v)$ of a dom-chromatic set D of T. A caterpillar is a graph which can be obtained from the path on n vertices by appending x_i pendant vertices to the i^{th} vertex of the path, P_n. The caterpillar with parameters $n, x_1, x_2, \ldots, x_n$ will be denoted as $P_n(x_1, x_2, \ldots, x_n)$. In this paper, the weighted dom-chromatic numbers are determined for some Type-I weighted complete caterpillars.

Keywords: dominating set, dom-chromatic set, weighted domination, weighted dom-chromatic number, Type-I weighted labeling

MSC Subject Classification: 05C69

1 Introduction

A set S of vertices is a *dominating set* of G if every vertex not in S is adjacent to at least one member of S. The minimum cardinality of a dominating set in

G is called the *domination number* and is denoted by $\gamma(G)$. The set $\mathcal{D}(G)$ is the collection of all dominating sets of G. A subset S of V is said to be a *dom-chromatic set* (or *dc-set*) if S is a dominating set and $\chi(< S >) = \chi(G)$. The minimum cardinality of a dom-chromatic set in G is called the *dom-chromatic number* (or *dc-number*) and is denoted by $\gamma_{ch}(G)$. The set $\mathcal{D}_{ch}(G)$ is the collection of all dom-chromatic sets of G

A *Weighted tree*, (T, w) a tree together with a positive weight function on the vertex set $w : V(T) \longrightarrow R^+$. The *weighted domination number* $\gamma_w(T)$ of (T, w) is the minimum weight $w(D) = \sum_{v \in D} w(v)$ of a dominating set D of T. The *weighted dom-chromatic number* $\gamma_{wch}(T)$ of (T, w) is the minimum weight $w(D) = \sum_{v \in D} w(v)$ of a dom-chromatic set D of T.

P. Palanikumar and S. Balamurugan [13] has introduced the concept of *Type-I weighted labeling* and they study the weighted dom-chromatic number of a weighted tree and determine the weighted dom-chromatic number of a *Type I* weighted paths. Also, they found the Weighted dom-chromatic number of some classes of Type-I weighted caterpillars in [14]. In this paper, the weighted dom-chromatic numbers are determined for some Type-I weighted complete caterpillars.

Theorem 1.1. *[13] Let (T, w) be a weighted tree and $[1, 2, \ldots, n]$ be a leaf-first labeling of T Where $w(i) = w_i$ for $i = 1, 2, \ldots, n$. If i is a leaf of T then $\eta_{ch}(i) = w_i$; $\theta_{ch}(i) = 0$; $\lambda_{ch}(i) = w_i$; $\mu_{ch}(i) = 0$.*

Definition 1.2. *[13] Let (T, w) be a weighted tree and $[1, 2, \ldots, n]$ be a leaf-first labeling of (T, w). Then L is said to be of Type-I if i is the first leaf of $T - \{1, 2, \ldots, i-1\}$ from left.*

Theorem 1.3. *[13] For a path P_n,$(n \geq 3)$ of Type-I,*

$$\gamma_{wch}(P_n) = \begin{cases} \frac{1}{6}\left(n^2 + n + 6\right) & \text{if } n \equiv 0 (mod\ 3) \\ \frac{1}{6}\left(n^2 + n + 10\right) & \text{if } n \equiv 1 (mod\ 3) \\ \frac{1}{6}\left(n^2 + n + 12\right) & \text{if } n \equiv 2 (mod\ 3) \end{cases}$$

2 Complete Caterpillar

A caterpillar is a graph which can be obtained from the path on n vertices by appending x_i pendant vertices to the i^{th} vertex of the path, P_n. The caterpillar with parameters $n, x_1, x_2, \ldots, x_n$ will be denoted as $P_n\,(x_1, x_2, \ldots, x_n)$.

Note, this is a tree with the property that the removal of its leaves and incident edges results in a path P_n called the *spine* of the caterpillar. Let l denote the number of leaves, i.e., $l = \sum_{i=1}^{n} x_i$. We say a caterpillar is complete if every vertex on the spine of the caterpillar is adjacent to at least one leaf.

Also, a caterpillar $P_n\,(x_1, x_2, \ldots, x_n)$, is complete if the $x_i > 0$ for $1 \leq i \leq n$. First, let us consider the class of complete caterpillars where $x_i = 1$ for $1 \leq i \leq n$, that is , having only one leaf attached to each vertex on the spine of the caterpillar. Examples are given in Figure 2.1.

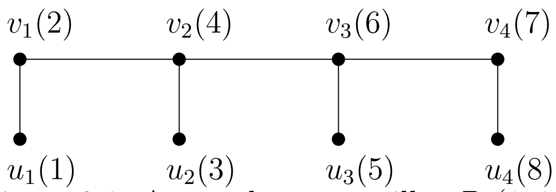

Figure 2.1: A complete caterpillar $P_4\,(1, 1, 1, 1)$

As an example of Type -I weighted complete caterpillar $P_4\,(1, 1, 1, 1)$, shown in Figure 2.1. Let $P_4\,(1, 1, 1, 1)$ have the vertices $\{v_1, v_2, v_3, v_4, u_1, u_2, u_3, u_4\}$ as shown in Figure 2.1. The minimum weighted dom-chromatic set is $D = \{v_1, u_1, u_2, u_3, v_4\}$ and $w(D) = 1{+}2{+}3{+}5{+}7 = 18$. Thus $\gamma_{wch}(P_4\,(1, 1, 1, 1)) = 18$.

Observe that the weighted dom chromatic number of the above caterpillars is the sum of the weight of the pendent vertices of P_4 and weight of the leaves except the maximum weight of the leaf.

Now, we are ready to prove Proposition.

Theorem 2.1. *For a caterpillar, $P_n\,(x_1, x_2, \ldots, x_n)$, where $x_i = 1$ for $1 \leq i \leq n$ of Type-I, then $\gamma_{wch}(P_n\,(x_1, x_2, \ldots, x_n)) = n^2 + 2$*

Proof. Let $G = (P_n(x_1, x_2, \ldots, x_n), w)$ be a weighted complete caterpillar with $x_i = 1$ for $1 \leq i \leq n$. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. Attach one pendent vertex u_j for $1 \leq j \leq n$ at each v_i for $1 \leq i \leq n$ as shown in Figure 3.2. Then $V((P_n(x_1, x_2, \ldots, x_n)) = \{v_1, v_2, \ldots, v_n, u_1, u_2, \ldots, u_n\}$. Thus $|V((P_n(x_1, x_2, \ldots, x_n))| = 2n$.

let $L = [1, 2, \ldots, n]$ be a leaf-first labeling of $(P_n(x_1, x_2, \ldots, x_n), w)$ and L is of Type-I.

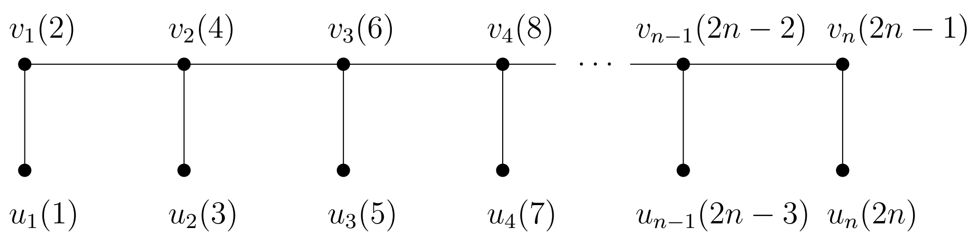

Figure 2.2: A complete caterpillar $P_n(1, 1, \ldots, 1)$

To dominate the maximum weighted vertex u_n, choose the minimum weighted vertex is v_n for the γ_w-set of G. A dominating set of G either contains a non-end vertex v or the end vertex of v. This implies that the weighted vertex v_n unioned with the set of leaves except u_n will be a minimum weighted dominating set of G. Thus, the minimum weighted dominating set of G is $\{u_1, u_2, \ldots, u_n, v_n\}$. For chromatic preservation, add a neighbor of least weight vertex to this set. Naturally, it is v_1. Therefore, the least weight dom-chromatic set is $D = \{v_1, u_1, u_2, \ldots, u_n, v_n\}$. Hence the minimum weight of a dom-chromatic set is, $\gamma_w(P_n(x_1, x_2, \ldots, x_n)) = w(D) = \sum w(v_i) = (1 + 3 + 5 + \ldots + 2n - 3 + 2n - 1) + 2 = n^2 + 2$. □

Our next result of interest gives the class of complete caterpillars with $x_i = l$ where $l > 1$ for $1 \leq i \leq n$, that is , having k leaves attached to each vertex on the spine of the caterpillar. We observe that the weighted dom-chromatic number of the above caterpillars is the sum of the weights of the vertices of P_n.

Now, we are ready to prove our theorem.

Theorem 2.2. *For a caterpillar, $P_n(x_1, x_2, \ldots, x_n)$, where $x_i = l$, where $l > 1$ for $1 \leq i \leq n$ of Type-I, then $\gamma_{wch}(P_n(x_1, x_2, \ldots, x_n)) = (n+1)\frac{|V(G)|}{2} - 1$*

Proof. Let $G = (P_n(x_1, x_2, \ldots, x_n), w)$ be a weighted complete caterpillar with $x_i = l$, where $l > 1$ for $1 \leq i \leq n$. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. Attach l pendent vertex $u_{i,1}, u_{i,2}, \ldots, u_{i,l}$ at each v_i for $1 \leq i \leq n$ as shown in Figure 2.3. Then $V((P_n(x_1, x_2, \ldots, x_n)) = \{v_i, u_{i,j} : 1 \leq i \leq n\,; 1 \leq j \leq l\}$. Thus $|V((P_n(x_1, x_2, \ldots, x_n))| = n(l+1)$.

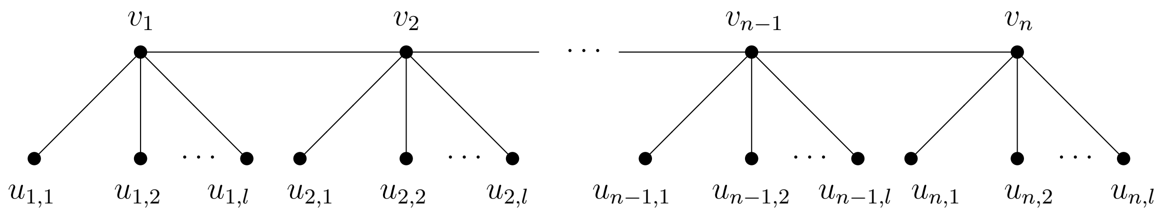

Figure 2.3: A complete caterpillar $P_n(l, l, \ldots, l)$

Let $L = [1, 2, \ldots, n]$ be a leaf-first labeling of $(P_n(x_1, x_2, \ldots, x_n), w)$ and L is of Type-I as shown in Figure 2.4

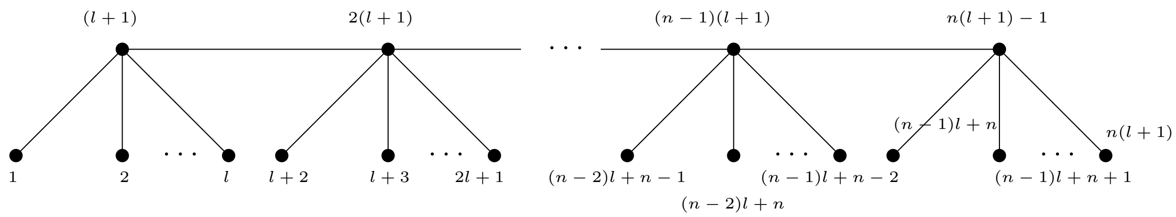

Figure 2.4: A weighted complete caterpillar $P_n(l, l, \ldots, l)$

A dominating set of G either contains a non-end vertex or the end vertex. It is easily observe that the set of vertices of P_n is a minimum weighted dominating set of G. Also, this set preserves the minimum weighed dom-chromatic in G. Thus, the minimum weighted dom-chromatic set of G is $D = \{v_1, v_2, \ldots, v_n\}$. Hence the minimum weight of a dom-chromatic set is, $\gamma_w(P_n(x_1, x_2, \ldots, x_n)) = w(D) = \sum w(v_i) = (l+1) + 2(l+1) + 3(l+1) + \ldots + (l+1)(n-1) + [n(l+1) - 1] = (n+1)\frac{|V(G)|}{2} - 1$. □

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