On the Linear Stability of Shear Flows in the β-plane

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Abstract: *The linear stability of parallel zonal flows of an incompressible, inviscid fluid on a β- plane is considered in this paper. For this problem, we derived two parabolic instability regionsfor a class of flows and which intersect with Howard semicircle instability region under some condition.*

Key words: *Shear flows, incompressible fluid, β-plane.*

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I. Introduction

We consider the linear stability of zonal flows of an inviscid, incompressible fluid on a β-plane. For this problem, [5] derived potential vorticity equation that governs the stability. [1] posed this problem and now it is known as Kuo problem. For this problem, [1] derived the inflexion point criterion namely $U'' - \beta$ vanishes somewhere in the flow domain which is the extension of Rayleigh inflexion point theorem for the case of shear flows. [3] shown the range of wave velocity of an unstable mode. The instabilityregion of [3] does not depend on $U'' - \beta$. [2] derived two parabolic instability region which includes $U'' - \beta$ and depends on functions which is either greater than zero or less than zero. [6] derived condition for temporal growth rate of an unstable mode and necessary condition for the non-existence for non-oscillatory unstable modes. For the case of homogeneous shear flows in sea straits of arbitrary cross section, [4] derived parabolic instability regions. Our present work is an extension of the work done by [4] for the case of zonal flows.

In this paper, we derived two parabolic instability region for a class of flows which does not depend function either greater than zero or less than zero as given in [2] and intersect with standard Howard semicircle under some condition.

II. Kuo Problem

The Kuo problem is given by

$$W'' - \left[k^2 + \frac{U'' - \beta}{U - c}\right]W = 0, \quad (1)$$

with boundary conditions

$$W(z_1) = 0 = W(z_2). \quad (2)$$

Here prime denotes differentiation with respect to z, $c = c_r + ic_i$ is the complex wave velocity of the disturbance, k is the wave number, U(z) is the basic velocity profile, β is Coriolis force in the latitudinal direction.

Introducing the transformation $W = (U - c)^{\frac{1}{2}} G$, we get

$$\left[(U - c)G'\right]' - k^2 (U - c)G - \frac{(U')^2}{4(U - c)} G - \left(\frac{U''}{2} - \beta\right)G = 0, \quad (3)$$

withboundary conditions

$$G(z_1) = 0 = G(z_2). \quad (4)$$

III. Main Results

Theorem 3.1:

If $c = c_r + ic_i$ with $c_i > 0$ then

(i) $$\int_{z_1}^{z_2} (U - c_r)\left[|G'|^2 + k^2 |G|^2\right] dz + \int_{z_1}^{z_2} \frac{(U')^2 (U - c_r)}{4|U - c|^2} |G|^2 dz + \int_{z_1}^{z_2} \left(\frac{U''}{2} - \beta\right)|G|^2 dz = 0.$$

(ii) $$\int_{z_1}^{z_2} \left[|G'|^2 + k^2 |G|^2\right] dz - \int_{z_1}^{z_2} \frac{(U')^2}{4|U - c|^2} |G|^2 dz = 0.$$

Proof:

Multiplying (3) by G^*, integrating over $[z_1, z_2]$ and using (4), we get

$$\int_{z_1}^{z_2}(U-c)\left[|G'|^2+k^2|G|^2\right]dz+\int_{z_1}^{z_2}\frac{(U')^2}{4(U-c)}|G|^2\,dz+\int_{z_1}^{z_2}\left(\frac{U''}{2}-\beta\right)|G|^2\,dz=0.$$

Equating real and imaginary parts, we get

$$\int_{z_1}^{z_2}(U-c_r)\left[|G'|^2+k^2|G|^2\right]dz+\int_{z_1}^{z_2}\frac{(U')^2(U-c_r)}{4|U-c|^2}|G|^2\,dz+\int_{z_1}^{z_2}\left(\frac{U''}{2}-\beta\right)|G|^2\,dz=0, \tag{5}$$

and

$$-c_i\int_{z_1}^{z_2}\left[|G'|^2+k^2|G|^2\right]dz+c_i\int_{z_1}^{z_2}\frac{(U')^2}{4|U-c|^2}|G|^2\,dz=0.$$

Since $c_i > 0$, we have

$$\int_{z_1}^{z_2}\left[|G'|^2+k^2|G|^2\right]dz-\int_{z_1}^{z_2}\frac{(U')^2}{4|U-c|^2}|G|^2\,dz=0. \tag{6}$$

Theorem 3.2:

If $U_{\min} > 0$ then

$$c_i^2 \le \lambda\left[c_r - U_{\min} + U_{\max}\right], \text{ where } \lambda = \frac{\dfrac{(U')^2_{\max}}{4}}{\dfrac{\pi^2}{(z_2-z_1)^2}+k^2}.$$

Proof:

Multiplying (6) by $(U_{\min} - U_{\max})$ and adding with (5), we get

$$\int_{z_1}^{z_2}(U-c_r+U_{\min}-U_{\max})\left[|G'|^2+k^2|G|^2\right]dz+\int_{z_1}^{z_2}\frac{(U')^2(U-c_r-U_{\min}+U_{\max})}{4|U-c|^2}|G|^2dz+\int_{z_1}^{z_2}\left(\frac{U''}{2}-\beta\right)|G|^2dz=0$$

Since $(U-c_r+U_{\min}-U_{\max})<0$, dropping this term from the above equation, we get

$$\int_{z_1}^{z_2}\left(\frac{U''}{2}-\beta\right)|G|^2dz>\int_{z_1}^{z_2}\frac{(U')^2(c_r-U+U_{\min}-U_{\max})}{4|U-c|^2}|G|^2dz\cdot \tag{7}$$

Multiplying (6) by (c_r) and adding with (5), we get

$$\int_{z_1}^{z_2}(U)\left[|G'|^2+k^2|G|^2\right]dz+\int_{z_1}^{z_2}\frac{(U')^2(U-2c_r)}{4|U-c|^2}|G|^2dz+\int_{z_1}^{z_2}\left(\frac{U''}{2}-\beta\right)|G|^2dz=0\cdot \tag{8}$$

Substituting ((7) in (8), we get

$$\int_{z_1}^{z_2}(U)\left[|G'|^2+k^2|G|^2\right]dz\le\int_{z_1}^{z_2}\frac{(U')^2(c_r-U_{\min}+U_{\max})}{4|U-c|^2}|G|^2dz\cdot$$

Since $\frac{1}{|U-c|^2}\le\frac{1}{c_i^2}$, and using Rayleigh-Ritz inequality, we have

$$U_{\min}\left[\frac{\pi^2}{(z_2-z_1)^2}+k^2\right]\int_{z_1}^{z_2}|G|^2dz\le\frac{\frac{(U')^2_{\max}}{4}}{c_i^2}\left[c_r-U_{\min}+U_{\max}\right]\int_{z_1}^{z_2}|G|^2dz;$$

That is

$$c_i^2\le\lambda\left[c_r-U_{\min}+U_{\max}\right], \tag{9}$$

where $\lambda=\dfrac{\frac{(U')^2_{\max}}{4}}{\frac{\pi^2}{(z_2-z_1)^2}+k^2}$.

Theorem 3.3:

If $\lambda < \lambda_c$ where $\lambda_c = (3U_{\max} - U_{\min}) - 2\sqrt{U_{\max}(2U_{\max} - U_{\min})}$ then $c_i^2 \le \lambda[c_r - U_{\min} + U_{\max}]$ intersect the Howard semicircle.

Proof:

The Howard semicircle is given by

$$\left[c_r - \frac{U_{\min} + U_{\max}}{2}\right] + c_i^2 \le \left[\frac{U_{\max} - U_{\min}}{2}\right]^2. \qquad (10)$$

Substituting (9) in (10), we get

$$c_r^2 + (\lambda - U_{\min} - U_{\max})c_r + (U_{\min}U_{\max} - \lambda U_{\min} + \lambda U_{\max}) \le 0.$$

The discriminant part of above equation is given by

$$\lambda^2 + [2U_{\min} - 6U_{\max}]\lambda + [U_{\max} - U_{\min}]^2 \ge 0.$$

Solving, we get

$$\lambda = (3U_{\max} - U_{\min}) \pm 2\sqrt{U_{\max}(2U_{\max} - U_{\min})}.$$

If $\lambda < \lambda_c$, where $\lambda_c = (3U_{\max} - U_{\min}) - 2\sqrt{U_{\max}(2U_{\max} - U_{\min})}$ then the parabola

$c_i^2 \le \lambda[c_r - U_{\min} + U_{\max}]$ intersect the Howard semicircle.

Theorem 3.4:

If $\left(\frac{U''}{2} - \beta\right)_{\min} > 0$ then

$$c_i^2 \le \lambda\left[c_r - 2U_{\min} + U_{\max}\right], \text{ where } \lambda = \left[\frac{\frac{(U')^2}{4}}{\left(\frac{U''}{2} - \beta\right)}\right]_{\max}.$$

Proof:

Multiplying (6) by $(U_{\max} - U_{\min})$ and adding with (5), we get

$$\int_{z_1}^{z_2} (U - c_r - U_{\min} + U_{\max})\left[|G'|^2 + k^2|G|^2\right]dz + \int_{z_1}^{z_2} \frac{(U')^2 (U - c_r + U_{\min} - U_{\max})}{4|U - c|^2}|G|^2 dz + \int_{z_1}^{z_2}\left(\frac{U''}{2} - \beta\right)|G|^2 dz = 0$$

Since $(U - c_r - U_{\min} + U_{\max}) > 0$, dropping this term from the above equation, we get

$$\int_{z_1}^{z_2} \frac{\left(\frac{U''}{2} - \beta\right)|U - c|^2 + \frac{(U')^2}{4}(U - c_r + U_{\min} - U_{\max})}{|U - c|^2}|G|^2 dz < 0.$$

Since $|U - c|^2 \ge c_i^2$, we get

$$c_i^2 \le \lambda\left[c_r - 2U_{\min} + U_{\max}\right], \tag{11}$$

where $\lambda = \left[\frac{\frac{(U')^2}{4}}{\left(\frac{U''}{2} - \beta\right)}\right]_{\max}$.

Theorem 3.5:

If $\lambda < \lambda_c$ where $\lambda_c = (3U_{\max} - 3U_{\min}) - 2\sqrt{2}(U_{\max} - U_{\min})$ then $c_i^2 \le \lambda\left[c_r - 2U_{\min} + U_{\max}\right]$ intersect the Howard semicircle.

Proof:

Substituting (11) in (10), we get

$$c_r^2 + (\lambda - U_{\min} - U_{\max})c_r + (U_{\min}U_{\max} - 2\lambda U_{\min} + \lambda U_{\max}) \le 0.$$

The discriminant part of above equation is given by

$$\lambda^2 + [6U_{\min} - 6U_{\max}]\lambda + [U_{\max} - U_{\min}]^2 \ge 0.$$

Solving, we get

$$\lambda = (3U_{\max} - 3U_{\min}) \pm 2\sqrt{2}(U_{\max} - U_{\min}).$$

If $\lambda < \lambda_c$, where $\lambda_c = (3U_{\max} - 3U_{\min}) - 2\sqrt{2}(U_{\max} - U_{\min})$ then the parabola $c_i^2 \le \lambda[c_r - 2U_{\min} + U_{\max}]$ intersect the Howard semicircle.

IV. Concluding Remarks

In this present paper, we derived analytical results on Kuo problem, which deals with linear stability of zonal flows of an inviscid, incompressible fluid on a β–plane. We obtained two parabolic instability region which intersect with standard Howard semicircle for a class of flows under some condition.

References

[1] H.L. Kuo, Dynamic instability of two-dimensional non-divergent flow in a barotropic atmosphere, Journal of Meteorol. vol. 6(1949), 105-122.

[2] M. Padmini,M. Subbiah, Note on Kuo's problem, J. Math. Anal. Appl., vol. 173(1993), 659-665.

[3] J. Pedlosky, Geophysical Fluid Dynamics, Springer-Verlag, New York/Berlin, 1979.

[4] K. Reena Priya, V.Ganesh,On the instability region for the extended Rayleigh problem of hydrodynamic stability, Applied Mathematical Sciences, vol. 9(2015), 2265-2272.

[5] C.G. .Rossby, Relation between variation in the intensity of the zonal circulation of the atmosphere and the displacement of the semi-permanent centers of action, Journal of Marine Res., vol.7 (1939), 38-59.

[6] Subodh Kumar Rana, Ruchi Goel, S.C. Agarwal,On the shear flow instability in the β-plane, Journal of International Academy of Physical Sciences, vol.13 (2009), 63-72.