Hypersurface of special Finsler space

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Abstract

In 1985 Matsumoto [1] discussed the properties of special hypersurface of Rander space with $b_i(x)$ being gradient of scalar function b(x).He has considered a hypersurface which is given by b(x)=constant. In 2009 Prasad and Shukla [2] have considered the hypersurface of generalized Matsumoto space with the same equation b(x) =constant.

In this paper our study confines to the hypersurface of a Finsler space with special (α, β) metric

$\alpha \cosh\frac{\beta}{\alpha} + \beta$ given by b(x) =constant. We will find out the conditions under which the hypersurface is a hyperplane of first or second kind have been obtained. This hyperplane is not a hyperplane of third kind.

I. Introduction

Let $F^n= (M^n, L)$ be an n dimensional Finsler space, where M^n is an n-dimensional differentiable Manifold and L(x, y) is the fundamental function. The concept of an (α, β) metric was introduced in 1972 by Matsumoto [3]. A Finsler space L(x, y) is called an (α, β) metric if L is positively homogeneous function of α and β of degree one, where $\alpha^2=a_{ij}(x)y^iy^i$ and $\beta=b_i(x)y^i$ is one form onM^n. We have some interesting examples of an (α,β) metric ,for instance L=α+β (Randers metric)[4] , L= $\frac{\alpha^2}{\beta}$ (Kropina metric)[5]. In 1989 M.Matsumoto, while studying the slope of mountain, introduced a (α, β) metric, given by $L = \frac{\alpha^2}{\alpha-\beta}$ which has been called Matsumoto Space [4].

The purpose of the present paper is to study the properties of hypersurface of special Finsler space whose metric is given by

$$L(x, y) = \alpha \cosh\frac{\beta}{\alpha} + \beta \qquad (1)$$

Where $\alpha^2=a_{ij}(x)y^iy^i$ and $\beta=b_i(x)y^i$

II. Fundamental quantities of special Finsler space

The derivative of the metric (1) with respect to α & β are given by

$$L_\alpha = \cosh\frac{\beta}{\alpha} - \frac{\beta}{\alpha}\sinh\frac{\beta}{\alpha} \qquad (2.1)$$

$$L_\beta = \sinh\frac{\beta}{\alpha} + 1$$

(2.2)

$$L_{\alpha\alpha} = \frac{\beta^2}{\alpha^3}\cosh\frac{\beta}{\alpha} \tag{2.3}$$

$$L_{\beta\beta} = \frac{1}{\alpha}\cosh\frac{\beta}{\alpha} \tag{2.4}$$

$$L_{\alpha\beta} = -\frac{\beta}{\alpha^2}\cosh\frac{\beta}{\alpha} \tag{2.5}$$

Where $L_\alpha = \frac{\partial L}{\partial\alpha}$, $L_\beta = \frac{\partial L}{\partial\beta}$, $L_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha^2}$,$L_{\beta\beta} = \frac{\partial^2 L}{\partial\beta^2}$ and $L_{\alpha\beta} = \frac{\partial^2 L}{\partial\alpha\partial\beta}$.

The normalized element of support $l_i = \frac{\partial L}{\partial y^i}$ is given by [8]

$$l_i = \alpha^{-1}L_\alpha Y_i + L_\beta b_i \tag{2.6}$$

Where $Y_i = a_{ij}y^j$

The angular metric tensor $h_{ij} = L\frac{\partial^2 L}{\partial y^i\partial y^j}$ is given by [8]

$$h_{ij} = Pa_{ij} + q_0 b_i b_j + q_{-1}(b_i Y_j + b_j Y_i) + q_{-2}Y_i Y_j \tag{2.7}$$

Where
$$P = LL_\alpha\alpha^{-1} = \frac{L}{\alpha}[\cosh\frac{\beta}{\alpha} - \frac{\beta}{\alpha}\sinh\frac{\beta}{\alpha}] \tag{2.8}$$

$$q_0 = LL_{\beta\beta} = \frac{L}{\alpha}\cosh\frac{\beta}{\alpha} \tag{2.9}$$

$$q_{-1} = LL_{\alpha\beta}\alpha^{-1} = -\frac{L\beta}{\alpha^3}\cosh\frac{\beta}{\alpha} \tag{2.10}$$

$$q_{-2} = L\alpha^{-2}(L_{\alpha\alpha} - L_\alpha\alpha^{-1}) = \frac{L}{\alpha^2}[\frac{\beta^2}{\alpha^3}\cosh\frac{\beta}{\alpha} - \frac{1}{\alpha}\cosh\frac{\beta}{\alpha} + \frac{\beta}{\alpha^2}\sinh\frac{\beta}{\alpha}] \tag{2.11}$$

Now the fundamental metric tensor $g_{ij} = \frac{1}{2}\frac{\partial^2 L^2}{\partial y^i\partial y^j}$ is given by [8, 9]

$$g_{ij} = Pa_{ij} + P_0 b_i b_j + P_{-1}(b_i Y_j + b_j Y_i) + P_{-2}Y_i Y_j \tag{2.12}$$

Where
$$P_0 = q_0 + L_\beta^2 = \frac{L}{\alpha}\cosh\frac{\beta}{\alpha} + [\sinh\frac{\beta}{\alpha} + 1]^2 \tag{2.13}$$

$$P_{-1} = q_{-1} + L^{-1}PL_\beta = -\frac{L\beta}{\alpha^3}\cosh\frac{\beta}{\alpha} + \frac{1}{\alpha}[\cosh\frac{\beta}{\alpha} - \frac{\beta}{\alpha}\sinh\frac{\beta}{\alpha}][\sinh\frac{\beta}{\alpha} + 1] \tag{2.14}$$

$$P_{-2} = q_{-2} + P^2L^{-2} = q_{-2} + \frac{1}{\alpha^2}[\cosh\frac{\beta}{\alpha} - (\frac{\beta}{\alpha})\sinh\frac{\beta}{\alpha}]^2 \tag{2.15}$$

The reciprocal tensor g^{ij} of g_{ij} is given by [9]

$$g^{ij} = P^{-1}a^{ij} - s_0 b^i b^j - s_{-1}(b^i y^j + b^j y^i) - s_{-2}y^i y^j \tag{2.16}$$

Where

$$\left\{\begin{array}{c} b^i = a^{ij}b_j \quad \text{and} \quad b^2 = a_{ij}b^ib^j \\ s_0 = \frac{1}{\tau P}[PP_0 + (P_0P_{-2} - P_{-1}^2)\alpha^2] \\ s_{-1} = \frac{1}{\tau P}[PP_{-1} + (P_0P_{-2} - P_{-1}^2)\beta] \\ s_{-2} = \frac{1}{\tau P}[PP_{-2} + (P_0P_{-2} - P_{-1}^2)b^2] \\ \tau = P(P + P_0b^2 + P_{-1}\beta) + (P_0P_{-2} - P_{-1}^2)(\alpha^2b^2 - \beta^2) \end{array}\right\} \quad (2.17)$$

The hv –torsion tensor $C_{ijk} = \frac{1}{2}\frac{\partial g_{ij}}{\partial y^k}$ is given by [10]

$$2PC_{ijk} = P_{-1}\left(h_{ij}m_k + h_{jk}m_i + h_{ki}m_j\right) + \gamma_1 m_i m_j m_k \quad (2.18)$$

Where $\left\{\gamma_1 = P\frac{\partial P_0}{\partial \beta} - 3P_{-1}q_0 \quad \text{and} \quad m_i = b_i - \alpha^{-2}\beta Y_i\right.$ (2.19)

Obviously the covariant vector m_i is non-vanishing and orthogonal to the element of support y^i.

Let $\left\{\begin{matrix} i \\ j\ k \end{matrix}\right\}$ be the component of christoffel symbol of associated Riemannian space R^n and ∇_k denote the covariant differentiation with respect to x^k relative to christoffel symbol. We shall consider the following tensors:

$$2E_{ij} = b_{ij} + b_{ji}, \qquad 2F_{ij} = b_{ij} - b_{ji} \quad (2.20)$$

Where $b_{ij} = \nabla_j b_i$

If we denote the Cartan's connection CΓ as $(\Gamma_{jk}^{*i}, \Gamma_{0k}^{*i}, C_{jk}^i)$then the difference tensor

$D_{jk}^i = \Gamma_{jk}^{*i} - \left\{\begin{matrix} i \\ j\ k \end{matrix}\right\}$ Of special Finsler space with (α, β) metric L= $\alpha \cosh\frac{\beta}{\alpha} + \beta$ is given by

$$D_{jk}^i = B^iE_{jk} + F_j^iB_k + F_k^iB_j + B_j^ib_{0k} + B_k^ib_{oj} - b_{0m}g^{im}B_{jk} - C_{jm}^iA_k^m - C_{km}^iA_j^m + C_{jkm}A_s^mg^{is} + \lambda^s(C_{jk}^iC_{sk}^m + C_{km}^iC_{sj}^m - C_{ms}^iC_{kj}^m)$$

(2.21)

Where

$$\left\{\begin{array}{c} B_k = P_0b_k + P_{-1}Y_k \\ B^i = g^{ij}B_j \\ F_i^k = g^{kj}F_{ji} \\ B_{ij} = \frac{\left\{P_{-1}(a_{ij} - \alpha^{-2}Y_iY_j) + \frac{\partial P_0}{\partial \beta}m_im_j\right\}}{2} \\ B_i^k = g^{kj}B_{ji} \\ A_k^m = B_k^mE_{00} + B^mE_{k0} + B_kF_0^m + B_0F_k^m \\ \lambda^m = B^mE_{00} + 2B_0F_0^m, \ B_0 = B_iy^i \end{array}\right\} \quad (2.22)$$

Where '0' denote the contraction with y^iexcept for the quantities P_0, q_0 & s_0.

III. Induced Cartan connection

Let F^{n-1} be a hypersurface of F^n , given by the equation $x^i = x^i(u^\alpha)$, α=1, 2…n-1.

Suppose that the matrix of projection factor $B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}$ is of rank (n-1). Then $B^i_\alpha(u)$ may be regarded as (n-1) linearly independent vectors tangential to F^{n-1} at the point (u^α) and the vector X^i tangential to F^{n-1} at the point may be expressed in the form of

$$X^i = B^i_\alpha X^\alpha,$$

Where X^α are the components of vector with respect to co-ordinate system (u^α). The element of support $y^i$of F^n is taken to tangential to F^{n-1},

i.e. $$y^i = B^i_\alpha(u)v^\alpha \tag{3.1}$$

Thus v^α is the element of support of F^{n-1}at the point (u^α).

The metric tensor $g_{\alpha\beta}$and hv-torsion tensor $C_{\alpha\beta\gamma}$of $F^{n-1}$is given by

$$g_{\alpha\beta} = g_{ij}B^i_\alpha B^J_\beta, \quad = C_{ijk}B^i_\alpha B^j_\beta B^k_\gamma \tag{3.2}$$

At each point $(u^\alpha)$of F^{n-1},the unit normal vector N^i(u, v) is defined by

$$\left.\begin{aligned} g_{ij}\big(x(u), y(u,v)\big)B^i_\alpha N^j = 0 \\ g_{ij}\big(x(u), y(u,v)\big)N^i N^j = 1 \end{aligned}\right\} \tag{3.3}$$

As for the angular metric tensor

$$h_{ij} = g_{ij} - l_i l_j, \qquad \text{we have} \qquad h_{\alpha\beta} = h_{ij} - B^i_\alpha B^j_\beta$$

$$h_{ij}B^i_\alpha N^j = 0 \qquad , \qquad h_{ij}N^i N^j = 1 \tag{3.4}$$

If (B^i_α, N_i) *is the inverse of matrix of* (B^i_α, N^i), *we have* ,

$$\left.\begin{aligned} B^\alpha_i = g^{\alpha\beta}g_{ij}B^j_\beta, \qquad B^i_\alpha B^\beta_i = \delta^\beta_\alpha, \quad B^i_\alpha N_i = 0. \\ N^i N_j = 1 \qquad and \qquad B^i_\alpha B^\alpha_j + N^i N_j = \delta^i_j \end{aligned}\right\} \tag{3.5}$$

The induced Cartan's connection ICΓ= $(\Gamma^{*i}_{jk}, \Gamma^{*i}_{0k}, C^i_{jk})$ is given by [1]

$$\Gamma^{*\alpha}_{\beta\gamma} = B^\alpha_i\left(B^i_{\beta\gamma} + \Gamma^{*i}_{jk}B^j_\beta B^k_\gamma\right) + M^\alpha_\beta H_\gamma \tag{3.6}$$

$$G^\alpha_\beta = B^\alpha_i\left(B^i_{0\beta} + \Gamma^{*i}_{0j}B^j_\beta\right), \tag{3.7}$$

$$C^\alpha_{\beta\gamma} = B^\alpha_i C^i_{jk} B^j_\beta B^k_\gamma$$

(3.8)

Where $$M_{\alpha\beta} = N_i C^i_{jk} B^j_\alpha B^k_\beta \,, \qquad M^\alpha_\beta = M_{\gamma\beta} g^{\alpha\gamma} \tag{3.9}$$

$$H_\alpha = N_i (B^i_{0\alpha} + \Gamma^{*i}_{0j} B^j_\alpha) \tag{3.10}$$

$$B^i_{\alpha\beta} = \frac{\partial B^i_\alpha}{\partial u^\beta} \,, \qquad B^i_{0\beta} = B^i_{\alpha\beta} v^\alpha \quad . \tag{3.11}$$

The quantities $M_{\alpha\beta}$ & H_α are called second fundamental v-tensor and normal curvature vector respectively [1]. The second fundamental h-tensor $H_{\alpha\beta}$ is defined as [1]

$$H_{\alpha\beta} = N_k \left(B^k_{\alpha\beta} + \Gamma^{*k}_{ij} B^i_\alpha B^j_\beta \right) + M_\alpha H_\beta \tag{3.12}$$

Where $$M_\beta = N_i C^i_{jk} B^j_\beta N^k \tag{3.13}$$

The relative h-and v- covariant derivatives of projection factor B^i_α with respect to ICΓ are given by

$$B^i_{\alpha|\beta} = H_{\alpha\beta} N^i, \qquad B^i_\alpha|_\beta = M_{\alpha\beta} N^i \tag{3.14}$$

From (3.12) it is clear that the second fundamental h-tensor $H_{\alpha\beta}$ is not symmetric and

$$H_{\alpha\beta} - H_{\beta\alpha} = M_\alpha M_\beta - M_\beta M_\alpha . \tag{3.15}$$

From (3.10), (3.12) and (3.13) it is clear that

$$H_{0\alpha} = H_\alpha \quad , H_{\alpha 0} = H_\alpha + M_\alpha H_0 . \tag{3.16}$$

We quote the following lemmas due to Matsumoto [1].

Lemma 1:- The normal curvature $H_0 = H_\alpha v^\alpha$ vanishes iff the normal curvature vector H_α $vanishes$.

Lemma 2:- A hypersurface F^{n-1} is a hyper plane of first kind iff $H_\alpha = 0$.

Lemma 3:- A hypersurface F^{n-1} is a hyperplane of second kind iff $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.

Lemma 4:- A hypersurface F^{n-1} is a hyperplane of third kind with respect to connection CΓ iff $H_\alpha = 0, M_{\alpha\beta} = 0$ and $H_{\alpha\beta} = 0$.

IV. Hypersurface F^{n-1}(c) of special Finsler space L= $\alpha \cosh\frac{\beta}{\alpha} + \beta$

Let us consider a special Finsler space with metric L= $\alpha \cosh\frac{\beta}{\alpha} + \beta$ with a gradient $b_i(x) = \frac{\partial b}{\partial x^i}$, for a scalar function b(x) = c (constant) ,from parametric equations $x^i = x^i(u^\alpha)$ of F^{n-1} we get $\frac{\partial b(x(u))}{\partial u^\alpha} = 0$,which implies that $b_i B^i_\alpha = 0$.This shows that b_i(x) are covariant component of a normal vector field of F^{n-1}(c) . Therefore along F^{n-1}(c), we have

$$b_i B^i_\alpha = 0 \qquad and \qquad b_i y^i = 0 \tag{4.1}$$

The induced metric L (u, v) of F^{n-1}(c) is given by

$$L(u,v) = \sqrt{a_{\alpha\beta}(u)v^{\alpha}v^{\beta}} \quad , \qquad a_{\alpha\beta} = a_{ij}B^i_{\alpha}B^j_{\beta} \tag{4.2}$$

Which is a Riemannian metric.

At the points of F^{n-1}(c), the quantities given in (2.8), (2.9), (2.10) and (2.11) become

$$P=1, \quad q_0 = 1\,, \qquad q_{-1} = 0, \qquad q_{-2}= \frac{1}{\alpha^2} \tag{4.3}$$

The quantities in the equations (2.13), (2.14) and (2.15) reduces to

$$P_0 = 2 \qquad p_{-1} = \frac{1}{\alpha} \quad , \quad P_{-2} = 0 \tag{4.4}$$

Whereas (2.17) reduces to

$$\left.\begin{aligned} \tau = 1 + b^2 \quad &, \qquad s_0 = \frac{1}{1+b^2} \\ s_{-1} = \frac{1}{\alpha(1+b^2)} \quad &, \qquad s_{-2} = -\frac{b^2}{\alpha^2(1+b^2)} \end{aligned}\right\} \tag{4.5}$$

Therefore equation (2.16) becomes.

$$g^{ij} = a^{ij} - \frac{1}{1+b^2}b^ib^j - \frac{1}{\alpha(1+b^2)}(b^iy^j + b^jy^i) + \frac{b^2}{\alpha^2(1+b^2)}y^iy^j \tag{4.6}$$

Thus along F^{n-1}(c), (4.1) and (4.6) gives

$$g^{ij}b_ib_j = \frac{b^2}{1+b^2} \tag{4.7}$$

Hence we have $$b_i(x(u)) = N_i\sqrt{\frac{b^2}{1+b^2}} \tag{4.8}$$

Where $\quad b^2 = a^{ij}b_ib_j \quad$ and b is the length of the vector b^i.

Again from (4.6) and (4.8) we have $\quad b^i = a^{ij}b_j$

$$b^i = \sqrt{b^2(1+b^2)}N^i + \frac{b^2}{\alpha}y^i$$

(4.9)

Theorem 4.1: Let F^n= (M^n, L) be special Finsler space with metric L= $\alpha \cosh\frac{\beta}{\alpha} + \beta$ with gradient $b_i(x) = \frac{\partial b}{\partial x^i}$ and let $F^{n-1}(c)$be a hypersurface of F^n given by b(x) =constant. Suppose that the Riemannian metric $a_{ij}(x)dx^idx^j$ is positive definite and $b_i$are component of a non-zero vector field, then the induced metric on $F^{n-1}(c)$ is a Riemannian metric given by (4.2) and the relations (4.8) and (4.9) hold.

Using (4.3) in (2.7) we get angular metric tensor along $F^{n-1}(c)$, given by

$$h_{ij} = a_{ij} + b_i b_j - \frac{1}{\alpha^2} Y_i Y_j \quad (4.10)$$

Again using (4.3) and (4.4) in (2.12) we get metric tensor along $F^{n-1}(c)$ given by

$$g_{ij} = a_{ij} + 2b_i b_j + \frac{1}{\alpha}(b_i Y_j + b_j Y_i) \quad (4.11)$$

If $h^{(a)}_{\alpha\beta}$ denote the angular metric tensor of the Riemannian $a_{ij}(x)$,then using (4.1) in (4.10), we have along $F^{n-1}(c)$ $\quad h_{\alpha\beta} = h^{(a)}_{\alpha\beta}$ (4.12)

From (2.13), we have $\quad \frac{\partial P_0}{\partial \beta} = \frac{2}{\alpha} \quad (along\ \ F^{n-1}(c))$ (4.13)

Therefore (2.19) gives

$$\gamma_1 = -\frac{1}{\alpha} \quad \text{and} \quad m_i = b_i \quad (4.14)$$

In view of (4.13) and (4.14) the hv –torsion tensor given by (2.18)

$$C_{ijk} = \frac{1}{2\alpha}[h_{ij} b_k + h_{jk} b_i + h_{ki} b_j - b_i b_j b_k] \quad (4.15)$$

Hence from (3.9), (4.1), (4.8) and (4.15) we have

$$M_{\alpha\beta} = \frac{1}{2\alpha}\sqrt{\frac{b^2}{1+b^2}}\, h_{\alpha\beta} \quad (4.16)$$

And from (3.4), (3.13), (4.1) and (4.15) we have

$$M_\alpha = 0 \quad (4.17)$$

On using (4.17) in (3.15) we have

$$H_{\alpha\beta} = H_{\beta\alpha} \quad (4.18)$$

Theorem (4.1): - The second fundamental v-tensor $M_{\alpha\beta}\ of\ \ F^{n-1}(c)$ is given by (4.16) and the second fundamental h-tensor $h_{\alpha\beta}$ is symmetric.

Now from (4.1) we get $\quad b_{i|\beta} b^i_\alpha + b_i B^i_{\alpha|\beta} = 0$

Using (3.14) and the fact that

$$b_{i|\beta} = b_{i|j} B^j_\beta + b_i|_j N^j H_\beta, \text{ we have}$$

$$[b_{i|j}\, B^j_\beta + b_i|_j N^j H_\beta] B^i_\alpha \;+\; b_i H_{\alpha\beta} N^i = 0. \quad (4.19)$$

Since $\quad b_i|_j = -b_h C^h_{ij},$

From (3.13), (4.8) and (4.17) we have

$$b_{i|j}B^i_\alpha B^j_\beta + H_{\alpha\beta}\sqrt{\tfrac{b^2}{1+b^2}} = 0 \tag{4.20}$$

Since $b_{i|j}$is symmetric tensor,

Contracting (4.20) with respect to v^β and using (3.16), we get

$$b_{i|j}B^i_\alpha y^j + H_\alpha\sqrt{\frac{b^2}{1+b^2}} = 0 \tag{4.21}$$

Further contracting (4.21) with v^α, we get

$$b_{i|j}y^i y^j + H_0\sqrt{\tfrac{b^2}{1+b^2}} = 0 \tag{4.22}$$

From lemma (1) and (2), it is clear that hypersurface $F^{n-1}(c)$ is a hypersurface of first kind iff $H_0 = 0$.

Thus from (4.22), it is obvious that hypersurface $F^{n-1}(c)$ is a hypersurface of first kind iff $b_{i|j}y^i y^j$=0.

This $b_{i|j}$ being covariant derivative with respect to CΓ of F^n, may depend ony^i. But $b_{ij} = \nabla_j b_i$is covariant derivative with respect to Riemannian connection $\left\{ {i \atop j\,k} \right\}$, constructed from $a_{ij}(x)$. Hence b_{ij} does not depend on y^i .

Since b_i is gradient vector, from (2.20) we have

$$E_{ij} = b_{ij}\ ,\ F_{ij}=0 \qquad \text{And}\quad F^i_j = 0 \tag{4.23}$$

Using (4.23) in (2.21) we get $D^i_{jk} = B^i b_{jk} + B^i_j b_{0k} + B^i_k b_{0j} - b_{0m}g^{im}B_{jk} - C^i_{jm}A^m_k - C_{jkm}A^m_s g^{is} + \lambda^s(C^i_{jm}C^m_{sk} + C^i_{km}C^m_{sj} - C^i_{ms}C^m_{kj})$

(4.24)

Now using (4.3), (4.4) and (4.6) in (2.22) we have

$$\left.\begin{aligned} B_i &= 2b_i + \alpha^{-1}Y_i \\ B^i &= g^{ij}B_j = \tfrac{1}{1+b^2}b^i + \tfrac{1}{\alpha(1+b^2)}y^i \\ B_{ij} &= \tfrac{1}{2\alpha}\left(a_{ij} - \tfrac{1}{\alpha^2}Y_iY_j\right) + \tfrac{1}{\alpha}b_ib_j \\ B^i_j &= \tfrac{1}{2\alpha}\left(\delta^i_j - \tfrac{y^iY_j}{\alpha^2}\right) + \tfrac{1}{2\alpha(1+b^2)}b^ib^j - \tfrac{b^2}{\alpha^2(1+b^2)}b_jy^i \\ \lambda^m &= B^m b_{00} \qquad \text{and} \qquad A^r_i = B^r_i b_{00} + B^r b_{i0} \end{aligned}\right\} \tag{4.25}$$

In the view of (4.1) we have $B^i_0 = 0$, which on using (4.25) gives $A^r_i = B^r b_{00}$ .therefore the contraction of (4.24) with y^k gives

$$D^i_{j0} = B^i b_{j0} + B^i_j b_{00} - B^r C^i_{jr} b_{00.} \qquad (4.26)$$

Again contracting (4.26) withy^j, we get

$$D^i_{00} = B^i b_{00}$$

i.e. $$D^i_{00} = [\tfrac{1}{1+b^2} b^i + \tfrac{1}{\alpha(1+b^2)} y^i] b_{00} \qquad (4.27)$$

Paying attention to (4.1), along $F^{n-1}(c)$ we finally get

$$b_i D^i_{j0} = \tfrac{b^2}{1+b^2} b_{j0} + \tfrac{(1+2b^2)}{2\alpha(1+b^2)} b_j b_{00} - \tfrac{1}{1+b^2} b_i b^m b_{00} C^i_{jm} \qquad (4.28)$$

On contracting by y^j we have

$$b_i D^i_{00} = \tfrac{b^2}{1+b^2} b_{00} \qquad (4.29)$$

From (3.9), (4.8), (4.9), (4.16) and (4.17) we have

$$b_i b^m C^i_{jm} B^j_\alpha = b^2 M_\alpha = 0$$

Thus the relation $b_{i|j} = b_{ij} - b_r D^r_{ij}$ and the equations (4.28) and (4.29) give

$$b_{i|j} y^i y^j = \tfrac{1}{1+b^2} b_{00} \qquad (4.30)$$

Hence (4.21) and (4.22) respectively becomes

$$\sqrt{\tfrac{b^2}{1+b^2}} H_\alpha + \tfrac{1}{1+b^2} b_{i0} B^i_\alpha = 0 \qquad (4.31)$$

$$\sqrt{\tfrac{b^2}{1+b^2}} H_0 + \tfrac{1}{1+b^2} b_{00} = 0 \qquad (4.32)$$

From equation (4.32) and lemmas 1 and 2, it is clear that the necessary and sufficient condition for $F^{n-1}(c)$ to be hyperplane of first kind is that $b_{00} = 0$.

Since b_{ij} does not depend on y^i satisfying (4.1), this condition may be written as

$$b_{ij} y^i y^j = (b_i y^i)(c_j y^j) = 0, \qquad \text{for some } C_j(x)$$

Therefore $$2b_{ij} = b_i C_j + b_j C_i \qquad (4.33)$$

From (4.1) and (4.33) it follows that

$$b_{00} = 0 \quad , \quad b_{ij} B^i_\alpha B^j_\beta = 0 \quad , \quad b_{ij} B^i_\alpha y^j = 0 \qquad (4.34)$$

Hence (4.32) gives $H_0 = 0$. Again from (4.33), (4.24) and (4.25) we have

$$\left.\begin{array}{ll} b_{i0}b^i = \frac{1}{2}c_0 b^2 \quad , & \lambda^r = 0 \\ A^i_j B^j_\beta = 0, & B_{ij}B^i_\alpha B^j_\beta = \frac{1}{2\alpha}h_{\alpha\beta} \end{array}\right\} \quad (4.35)$$

Thus from (4.23) we have

$$b_r D^r_{ij} B^i_\alpha B^j_\beta = -\frac{1}{4\alpha}(b_r b_s g^{rs}) C_0 h_{\alpha\beta} + b_r C_{ijm} A^m_s g^{rs} B^i_\alpha B^j_\beta \quad (4.36)$$

Also from (4.6) we find

$$b_r b_s g^{rs} = \frac{b^2}{1+b^2} \quad (4.37)$$

With the help of (3.9),(4.9),(4.16),(4.24) and (4.25) we get

$$b_r C_{ijm} A^m_s g^{rs} B^i_\alpha B^j_\beta = (b_r b_s g^{rs}) \frac{1}{4\alpha} \frac{C_0 b^2}{1+b^2} h_{\alpha\beta} \quad (4.38)$$

Substituting (4.37) and (4.38) in (4.36), we get

$$b_r D^r_{ij} B^i_\alpha B^j_\beta = -\frac{1}{4\alpha}\frac{C_0 b^2}{(1+b^2)^2} h_{\alpha\beta}. \quad (4.39)$$

Therefore (4.20) reduces to

$$\sqrt{\frac{b^2}{1+b^2}} H_{\alpha\beta} + \frac{C_0 b^2}{4\alpha(1+b^2)^2} h_{\alpha\beta} = 0 \quad (4.40)$$

Hence the hyprsurface $F^{n-1}(c)$ is Umbilic.

Theorem4.3: The necessary and sufficient condition for hypersurface $F^{n-1}(c)$ to be a hyperplane of the first kind is (4.33) and in this case the second fundamental tensor is proposnal to its angular metric tensor.

Now from lemma (3), the hypersurface $F^{n-1}(c)$ is a hyperplane of second kind iff $H_\alpha = 0$ and $H_{\alpha\beta} = 0$.Thus (4.40) gives $C_0 = C_i(x)y^i$ =0.Therefore there exist a function η(x) such that

$$C_i(x) = \eta(x) b_i(x) \quad (4.41)$$

Hence (4.33) reduces to $b_{ij} = \eta b_i b_j$.

Theorem 4.4: The necessary and sufficient condition for hypersurface $F^{n-1}(c)$ to be a hyperplane of second kind is $b_{ij} = \eta b_i b_{j.}$

Again lemma (4) together with (4.16) and (4.17) shows that $F^{n-1}(c)$ does not become a hyperplane of third kind.

Theorem 4.5: The hypersurface $F^{n-1}(c)$ is not a hyperplane of third kind.

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