

On New Mappings In Fuzzy Topological Spaces

Sanjivappa K Dembare

Assistant professor in Mathematics and HOD

Government First grade Degree college Sector no.43, Navangar, Bagalkote-587103, Karnataka, India.

Abstract: In this paper we introduce and study two new fuzzy homeomorphisms, namely fuzzy weakly-homeomorphism and fuzzy weakly-closed homeomorphism. We prove that every fuzzy homeomorphism is fuzzy weakly-homeomorphism and we prove that the composition of two fuzzy weakly closed homeomorphism is a weakly-homeomorphism.

2010 Mathematics Subject Classification: 54A40

Keywords and phrases: fuzzy weakly-homeomorphism, fuzzy weakly-closed set, fuzzy weakly-open set.

Introduction

The concept of a fuzzy subset was introduced and studied by L.A.Zadeh in the year 1965. The subsequent research activities in this area and related areas have found applications in many branches of science and engineering. In the year 1965, L.A.Zadeh [1] introduced the concept of fuzzy subset as a generalization of that of an ordinary subset. The introduction of fuzzy subsets paved the way for rapid research work in many areas of mathematical science. In the year 1968, C.L.Chang [2] introduced the concept of fuzzy topological spaces as an application of fuzzy sets to topological spaces. Subsequently several researchers contributed to the development of the theory and applications of fuzzy topology. The theory of fuzzy topological spaces can be regarded as a generalization theory of topological spaces. An ordinary subset A or a set X can be characterized by a function called characteristic function

$$\begin{aligned} \mu_A : X &\longrightarrow [0,1] \text{ of } A, \text{ defined by} \\ \mu_A(x) &= 1, \text{ if } x \in A. \\ &= 0, \text{ if } x \notin A. \end{aligned}$$

Thus an element $x \in X$ is in A if $\mu_A(x) = 1$ and is not in A if $\mu_A(x) = 0$. In general if X is a set and A is a subset of X then A has the following representation. $A = \{ (x, \mu_A(x)) : x \in X \}$, here $\mu_A(x)$ may be regarded as the degree of belongingness of x to A , which is either 0 or 1. Hence A is the class of objects with degree of belongingness either 0 or 1 only. Prof. L.A.Zadeh [1] introduced a class of objects with continuous grades of belongingness ranging between 0 and 1; he called such a class as fuzzy subset. A fuzzy subset A in X is characterized as a membership function $\mu_A : X \longrightarrow [0,1]$, which associates with each point in x a real number $\mu_A(x)$ between 0 and 1 which represents the degree or grade membership of belongingness of x to A .

The purpose of this paper is to introduce a new class of fuzzy sets called fuzzy weakly-closed sets in fuzzy topological spaces and investigate certain basic properties of these fuzzy sets. Among many other results it is observed that every fuzzy closed set is fuzzy weakly-closed but not conversely. Also we introduce fuzzy weakly-open sets in fuzzy topological spaces and study some of their properties.

1. Preliminaries

1.1 Definition:[1] A fuzzy subset A in a set X is a function $A : X \rightarrow [0, 1]$. A fuzzy subset in X is empty iff its membership function is identically 0 on X and is denoted by 0 or μ_ϕ . The set X can be considered as a fuzzy subset of X whose membership function is identically 1 on X and is denoted by μ_X or I_X . In fact every subset of X is a fuzzy subset of X but not conversely. Hence the concept of a fuzzy subset is a generalization of the concept of a subset.

1.2 Definition :[1] If A and B are any two fuzzy subsets of a set X , then A is said to be included in B or A is contained in B iff $A(x) \leq B(x)$ for all x in X . Equivalently, $A \leq B$ iff $A(x) \leq B(x)$ for all x in X .

1.3 Definition: [1] Two fuzzy subsets A and B are said to be equal if $A(x) = B(x)$ for every x in X . Equivalently $A = B$ if $A(x) = B(x)$ for every x in X .

1.4 Definition:[1] The complement of a fuzzy subset A in a set X , denoted by A' or $1 - A$, is the fuzzy subset of X defined by $A'(x) = 1 - A(x)$ for all x in X . Note that $(A')' = A$.

1.5 Definition:[1] The union of two fuzzy subsets A and B in X , denoted by $A \vee B$, is a fuzzy subset in X defined by $(A \vee B)(x) = \text{Max}\{\mu_A(x), \mu_B(x)\}$ for all x in X .

1.6 Definition:[1] The intersection of two fuzzy subsets A and B in X , denoted by $A \wedge B$, is a fuzzy subset in X defined by $(A \wedge B)(x) = \text{Min}\{A(x), B(x)\}$ for all x in X .

1.7 Definition:[1] A fuzzy set on X is 'Crisp' if it takes only the values 0 and 1 on X .

1.8 Definition :[2] Let X be a set and τ be a family of fuzzy subsets of (X, τ) is called a fuzzy topology on X iff τ satisfies the following conditions.

(i) $\mu_\phi; \mu_X \in \tau$: That is 0 and 1 $\in \tau$

(ii) If $G_i \in \tau$ for $i \in I$ then $\bigvee_{i \in I} G_i \in \tau$

(iii) If $G, H \in \tau$ then $G \wedge H \in \tau$

The pair (X, τ) is called a fuzzy topological space (abbreviated as fts). The members of τ are called fuzzy open sets and a fuzzy set A in X is said to be closed iff $1 - A$ is a fuzzy open set in X .

1.9 Remark :[2] Every topological space is a fuzzy topological space but not conversely.

1.10 Definition:[2] Let X be a fts and A be a fuzzy subset in X . Then $\bigwedge \{B : B \text{ is a closed fuzzy set in } X \text{ and } B \geq A\}$ is called the closure of A and is denoted by $\text{cl}(A)$.

1.11 Definition:[2] Let A and B be two fuzzy sets in a fuzzy topological space (X, τ) and let $A \geq B$. Then B is called an interior fuzzy set of A if there exists $G \in \tau$ such that $A \geq G \geq B$, the least upper bound of all interior fuzzy sets of A is called the interior of A and is denoted by A^0 .

1.12 Definition[3] A fuzzy set A in a fts X is said to be fuzzy semi open if and only if there exists a fuzzy open set V in X such that $V \leq A \leq \text{cl}(V)$.

1.13 Definition[3] A fuzzy set A in a fts X is said to be fuzzy semi-closed if and only if there exists a fuzzy closed set V in X such that $\text{int}(V) \leq A \leq V$. It is seen that a fuzzy set A is fuzzy semiopen if and only if $1-A$ is a fuzzy semi-closed.

1.14 Theorem:[3] The following are equivalent:

- (a) μ is a fuzzy semi closed set,
- (b) μ^c is a fuzzy semi open set,
- (c) $\text{int}(\text{cl}(\mu)) \leq \mu$.
- (b) $\text{int}(\text{cl}(\mu)) \geq \mu^c$

1.15 Theorem [3] Any union of fuzzy semi open sets is a fuzzy semi open set and (b) any intersection of fuzzy semi closed sets is a fuzzy semi closed.

1.16 Remark[3]

- (i) Every fuzzy open set is a fuzzy semi open but not conversely.
- (ii) Every fuzzy closed set is a fuzzy semi-closed set but not conversely.
- (iii) The closure of a fuzzy open set is fuzzy semi -open set
- (iv) The interior of a fuzzy closed set is fuzzy semi-closed set

1.17 Definition:[3] A fuzzy set μ of a fts X is called a fuzzy regular open set of X if $\text{int}(\text{cl}(\mu)) = \mu$.

1.18 Definition:[3] A fuzzy set μ of fts X is called a fuzzy regular closed set of X if $\text{cl}(\text{int}(\mu)) = \mu$.

1.19 Theorem:[3] A fuzzy set μ of a fts X is a fuzzy regular open if and only if μ^c fuzzy regular closed set.

1.20 Remark:[3]

- (i) Every fuzzy regular open set is a fuzzy open set but not conversely.
- (ii) Every fuzzy regular closed set is a fuzzy closed set but not conversely.

1.21 Theorem:[3]

- (i) The closure of a fuzzy open set is a fuzzy regular closed.
- (ii) The interior of a fuzzy closed set is a fuzzy regular open set.

1.22 Definition:[4] A fuzzy set α in fts X is called fuzzy rw closed if $\text{cl}(\alpha) \leq \mu$ whenever $\alpha \leq \mu$ and μ is regular semi-open in X .

1.23 Definition [5]: A fuzzy set α in fts X is called fuzzy weakly closed if $\text{p-cl}(\alpha) \leq \mu$ whenever $\alpha \leq \mu$ and μ is $\text{rg}\alpha$ -open set in X .

1.24 Defintion [5]:A fuzzy set α of a fts X is fuzzy weakly-open set, if it's complement α^c is a fuzzy weakly-closed in fts X .

1.25 Defintion[6]: Let X and Y be two fts. A map $f: (X,T) \rightarrow (Y,T)$ is called fuzzy weakly-open map if the inverse image of every fuzzy open set in X is fuzzy weakly-open in Y .

1.26 Defintion[6]:Let X and Y be two fuzzy topological spaces. A map $f: (X,T) \rightarrow (Y,T)$ is called fuzzy weakly-closed map if the image of every fuzzy closed set in X is a fuzzy weakly closed set in Y .

1.27 Definition[7]:Let X and Y be fts. A map $f: X \rightarrow Y$ is said to be fuzzy weakly-continuous if the inverse image of every fuzzy open set in Y is fuzzy weakly-open in X .

1.28 Definition[7]: Let X and Y be fts. A map $f: X \rightarrow Y$ is said to be a fuzzy weakly-irresolute map if the inverse image of every fuzzy weakly-open in Y is a fuzzy weakly-open set in X .

1.29 definition [2]: Let X and Y be fts. A map $f: X \rightarrow Y$ is said to be a fuzzy continuous mapping if $f^{-1}(\mu)$ is fuzzy open in X for each fuzzy open set μ in Y .

1.30 Definition [8]: Let X and Y be fts. A map $f: X \rightarrow Y$ is said to be a fuzzy -irresolute map if the inverse image of every fuzzy semi-open in Y is a fuzzy semi-open set in X .

1.31 Definition [9]: Let X and Y be fts. A bijection $f: X \rightarrow Y$ is said to be a fuzzy-homeomorphism iff f and f^{-1} are fuzzy continuous.

2. Fuzzy weakly-homeomorphism in fuzzy topological spaces.

Definition 2.1: Let X and Y be fts. A bijective map $f: (X,T_1) \rightarrow (Y,T_2)$ is called fuzzy weakly-homeomorphism if f and f^{-1} are fuzzy weakly-continuous map.

The family of all fuzzy weakly-homeomorphism from (X,T) on to itself is denoted by fuzzy weakly-Homeomorphism (X,T) .

Theorem 2.2: Every fuzzy homeomorphism is fuzzy weakly-homeomorphism.

Proof: Let a map $f: (X,T_1) \rightarrow (Y,T_2)$ be a fuzzy homeomorphism. Then f and f^{-1} are fuzzy continuous map. Since every fuzzy continuous map is fuzzy weakly-continuous map f and f^{-1} are fuzzy weakly-continuous map therefore f is fuzzy a weakly-homeomorphism.

The converse of the above theorem need not be true as seen from the following example.

Example 2.3: Let $X=Y= \{a, b, c,d\}$ and the functions $\alpha, \beta, \gamma, \delta: X \rightarrow [0, 1]$ be defined as

$$\alpha(x) = \begin{cases} 1 & \text{if } x = a \\ 0 & \text{otherwise} \end{cases}$$

$$\beta(x) = \begin{cases} 1 & \text{if } x = b \\ 0 & \text{otherwise} \end{cases}$$

$$\gamma(x) = \begin{cases} 1 & \text{if } x = a,b \\ 0 & \text{Otherwise} \end{cases}$$

$$\delta(x) = \begin{cases} 1 & \text{if } x = a,b,c \\ 0 & \text{otherwise.} \end{cases}$$

Consider $T_1=\{0,1,\alpha,\beta,\gamma,\delta\}$ and $T_2 = \{0,1,\alpha,\beta,\gamma,\delta\}$ then (X,T_1) and (Y,T_2) are fts. Define a map $f:(X,T_1) \rightarrow (Y,T_2)$ by $f(a)=c, f(b)=a, f(c)=b, f(d)=d$ then f is fuzzy weakly-homeomorphism but it is not fuzzy homeomorphism, as the image of closed set γ in X is not fuzzy closed set in (Y,T_2) .

Theorem 2.4: Let X and Y be fuzzy topological spaces and $f: (X,T_1) \rightarrow (Y,T_2)$ be a bijective map. Then the following statements are equivalent.

- (a) f^{-1} is fuzzy weakly continuous map.
- (b) f is fuzzy weakly-open map.
- (c) f is fuzzy weakly-closed map.

Proof: (a) \rightarrow (b) Let α be any fuzzy open set in X . Since f^{-1} is fuzzy weakly continuous map, $(f^{-1})^{-1}(\alpha) = f(\alpha)$ is fuzzy weakly-open in Y . Hence f is fuzzy weakly-open map.

(b) \rightarrow (c) Let α be any fuzzy closed set in X . Then $1-\alpha$ is fuzzy weakly-open in X . Since f is a fuzzy weakly-open, $f(1-\alpha)$ is fuzzy weakly-open in Y . but $f(1-\alpha) = 1 - f(\alpha)$, as f is a bijective map. Hence $f(\alpha)$ is fuzzy weakly set in Y . Therefore f is fuzzy weakly-closed map.

(c) \rightarrow (a) Let α be any fuzzy closed set in X . Then $f(\alpha)$ is a fuzzy weakly closed set in Y but $(f^{-1})^{-1}(f(\alpha)) = \alpha$. Therefore f^{-1} is fuzzy weakly-continuous map.

Theorem 2.5: Let X and Y be fuzzy topological spaces and $f: (X,T_1) \rightarrow (Y,T_2)$ be a bijective and fuzzy weakly-continuous map. Then the following statements are equivalent.

- (a) f is fuzzy weakly open map.
- (b) f is fuzzy weakly-homeomorphism.
- (c) f is fuzzy weakly-closed map.

Proof: (a) implies (b) by hypothesis and assumption f is a fuzzy weakly-homeomorphism

(b) implies (c) since f is fuzzy weakly-homeomorphism, it is fuzzy weakly-open, so by the above theorem 2.4 it is a fuzzy weakly-closed map.

(c) implies (b) Let σ be any fuzzy open set in X . so that $1-\sigma$ is a closed set and f being weakly-closed, $f(1-\sigma)$ is fuzzy weakly-closed in Y . but $f(1-\sigma) = 1-f(\sigma)$, thus $f(\sigma)$ is fuzzy weakly open set in Y . Therefore f is fuzzy weakly-open map.

Defintion 2.6 : A bijective map $f : (X, T_1) \rightarrow (Y, T_2)$ is called a fuzzy weakly-closed homeomorphism if f and f^{-1} are fuzzy weakly-irresolute map. We say that spaces (X, T_1) and (Y, T_2) are fuzzy weakly closed homeomorphism if there exist a fuzzy weakly-closed homomorphism from (X, T_1) onto (Y, T_2) . The family of all fuzzy weakly-homeomorphism from (X, T) onto itself is denoted by f weakly-h (X, T) .

Theorem 2.7: Every fuzzy weakly closed homeomorphism is fuzzy weakly-homeomorphism but not conversely.

Proof: The proof follows from the fact that every fuzzy weakly-irresolute map is fuzzy weakly-continuous map but not conversely.

Theorem 2.8: Let $(X, T_1), (Y, T_2)$ & (Z, T_3) be a fts and $f: (X, T_1) \rightarrow (Y, T_2), g: (Y, T_2) \rightarrow (Z, T_3)$ be fuzzy Weakly-homeomorphism. Then their composition $g \circ f : (X, T_1) \rightarrow (Z, T_3)$ is a fuzzy weakly-closed homeomorphism.

Proof: Let μ be a fuzzy weakly-open set in (Z, T_3) , Since g is a fuzzy weakly-irresolute map, $g^{-1}(\mu)$ is a fuzzy weakly-open set in (Y, T_2) . Since f is a fuzzy weakly-irresolute map, $f^{-1}(g^{-1}(\mu))$ is a fuzzy weakly-open set in (X, T_1) but $f^{-1}(g^{-1}(\mu)) = (g \circ f)^{-1}(\mu)$. Therefore $(g \circ f)$ is fuzzy weakly-irresolute map.

To prove: $(g \circ f)^{-1}$ is fuzzy weakly-irresolute map. Let α be a fuzzy weakly-open set in (X, T_1) . Since f^{-1} is fuzzy weakly-irresolute map, $(f^{-1})^{-1}(\alpha)$ is a fuzzy weakly-open set in (Y, T_2) also $(f^{-1})^{-1}(\alpha) = f(\alpha)$. Since g^{-1} is fuzzy weakly-irresolute map, $((g^{-1})^{-1})(f(\alpha))$ is a fuzzy weakly-open set in (Z, T_3) that is $((g^{-1})^{-1})(f(\alpha)) = g(f(\alpha)) = (g \circ f)(\alpha) = ((g \circ f)^{-1})^{-1}(\alpha)$. Therefore $(g \circ f)^{-1}$ is fuzzy weakly-irresolute map. Thus $g \circ f$ and $(g \circ f)^{-1}$ are fuzzy weakly-irresolute map. Hence $g \circ f$ is fuzzy weakly-closed homeomorphism.

Theorem 2.9: The set fuzzy weakly-closed homeomorphism (X, T) is a group under the composition map.

Proof: Define a binary operation $*$ f -weakly closed homeomorphism $(X, T) \times f$ -weakly closed homeomorphism $(X, T) \rightarrow f$ weakly-c-h (X, T) by $f * g = g \circ f$ for all $f, g \in f$ weakly-c-h (X, T) and o is the usual operation of composition of maps. Then by theorem 2.8, $g \circ f \in f$ weakly closed h (X, T) we know that, the composition of maps is associate and the identity map $I: (X, T) \rightarrow (X, T)$ belonging to f weakly closed-h (X, T) serves as the identity element. If $f \in f$ weakly closed-h (X, T) then $f^{-1} \in f$ weakly closed-h (X, T) s.t $f \circ f^{-1} = f^{-1} \circ f = I$ and so inverse exist for each element of f weakly-c-h (X, T) , Therefore $[f$ weakly-c-h $(X, T), o]$ is a group under the operation of composition of maps.

Theorem 2.10: Let $f: (X, T_1) \rightarrow (Y, T_2)$ be a fuzzy weakly-closed homeomorphism then f induces an isomorphism from the group fuzzy weakly closed h (X, T_1) on to the group f weakly-h (Y, T_2) .

Proof: Using the map f , we define a map $\mu_f: f$ weakly closed-h $(X, T_1) \rightarrow f$ -weakly closed-h (Y, T_2) by $\mu_f(h) = f \circ h \circ f^{-1}$ for every $h \in f$ weakly closed-h (X, T_1) . Then μ_f is a bijection.

Further for all $h_1, h_2 \in f$ weakly-c-h(X, T_1), $\mu_f(h_1 \circ h_2) = f \circ (h_1 \circ h_2) \circ f^{-1} = (f \circ h_1 \circ f^{-1}) \circ (f \circ h_2 \circ f^{-1}) = \mu_f(h_1) \circ \mu_f(h_2)$. Therefore μ_f is a homeomorphism and so it is an isomorphism induced by f .

References

- [1] L.A.Zadeh, Fuzzy sets, Information and control, 8 (1965) 338-353.
- [2] C.L.Chang, Fuzzy topological spaces, JI. Math. Anal. Appl., 24(1968), 182-190.
- [3] K.K.Azad, On fuzzy semi continuity, fuzzy almost continuity and fuzzy weakly continuity. JI. Math. Anal. Appl. 82 No. 1 (1981), 14-32.
- [4] S. S. Benchalli, R. S. Wali and Basavaraj M. Ittanagi on fuzzy rw-closed sets and fuzzy rw-open sets in fuzzy topological spaces Int. J. of Mathematical Sciences and Applications, Vol. 1, No. 2, May 2011.
- [5] R.S.Wali and Vivekananda Dembre, R.S.Wali and Vivekananda Dembre, Fuzzy weakly-closed sets and Fuzzy weakly-open sets in Topological Spaces.
- [6] R.S.Wali and Vivekananda Dembre, Fuzzy weakly-open maps and fuzzy weakly-closed maps in fuzzy topological spaces
- [7] R.S.Wali and Vivekananda Dembre, Fuzzy weakly-continuous maps and fuzzy weakly-irresolute in fuzzy topological spaces
- [8] Mukerjee, M.N. and Sinha, S.P., irresolute and almost open function between fts, fuzzy sets and systems, 29(1989), 141-148.
- [9] Ferraro, M and Foster, D.H., differentiation of fuzzy continuous mappings on fts, jour. Math. anal & appl 121(1987), 1-7
- [10] Thakur S.S. and Bajpai Pandey Jyoti "Intuitionistic Fuzzy α -closed sets", International Journal of Fuzzy system and Rough System 4(1), 67-73.
- [11] R.S.Wali and Vivekananda Dembre; on α -closed sets in topological spaces; Journal of computer and Mathematical sciences.