Bayesian Approach to Series System Reliability Estimation

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Abstract — *The problem of finding the posterior confidence limits for the reliability of a series system is considered in this paper and the confidence limits for the reliability of a series system with k- independent exponential components are obtained. The method is illustrated with an example involving a system with three components. Some important special cases are also studied with illustration.*

Keywords — *Bayes Estimate, Series System, Exponential Distribution*

I. INTRODUCTION

We consider a series system consisting of k independent components, i.e., the system fails if at least one component fails. The systems which are too expensive and huge to put on test for life data are of interest here. Nevertheless components of the system can be subjected to life test and failure data can be collected from such life testing experiments. These life data are used to construct two sided Bayes confidence limits on the series system reliability.

The problem of finding confidence limits for the reliability of a series system was attempted by several authors. See, for example, [3], [7], [6], [10], [11], [9] among others. An excellent review under Bayesian frame-work can be found in [8]. Some important problems in reliability are recently addressed by many researchers including [12].

The author (see [4]) studied a multi-component system; they assessed the reliability via Pass/fail data and illustrated the methodology with a series system three components, with simultaneously estimating system, sub-system and component reliabilities. In a recent paper (see [1]) reported simulation studies on (i) parallel system with 4 components (ii) 2-out-of-3 system and (III) a Bridge structure with three parameter Weibull distribution.

In this paper, we find the Bayesian confidence limits of a series system reliability consisting of k independent exponential components. The paper is organized as follows. The Section 2 contains the main result. In Section 3, a simulation based illustrative example is given involving a series system of three components. We have considered some special cases in the Section 4.

II. MAIN RESULT

Consider a series system composed of k independent exponential components with survival function:

$$\bar{F}_i(\mathrm{t}) = e^{-\lambda_i t} \qquad (2.1)$$

for i=1, …, n. Thus the system reliability is given by:

$$\mathrm{R(t)} = e^{-\lambda t} \qquad (2.2)$$

where

$$\lambda = \sum_{i=1}^{k} \lambda_i \qquad (2.3)$$

We consider a Jeffer's prior given by

$$g(\lambda_1,\ldots,\lambda_k) \propto \frac{1}{\prod_1^k \lambda_i} \tag{2.4}$$

The objective of the present study is to obtain a two-sided confidence limit for R(t).

Let n_i denote the number of component i put on test and $x_j^{(i)}$ be the observed life-length of the j-th prototype of component i, for j=1,…, n_i. Letting

$$T_i = \sum_{j=1}^{n_i} x_j^{(i)} \tag{2.5}$$

the posterior density using the prior in (2.4) is given by:

$$\pi(\lambda_1, \ldots, \lambda_k \mid x^{(1)}, \ldots, x^{(k)}) = c \left(\prod_{i=1}^k \lambda_i^{n_i-1}\right) e^{-\sum_{i=1}^k \lambda_i T_i} \tag{2.6}$$

where

$$c^{-1} = \prod_{i=1}^k \frac{\Gamma(n_i)}{T_i^{n_i}} \tag{2.7}$$

and

$$x^{(i)} = (x_1^{(i)}, \ldots, x_{n_i}^{(i)})' \tag{2.8}$$

In order to find the posterior distribution of λ given $x^{(1)}, \ldots, x^{(k)}$, we apply the following transformation

$$\lambda_i = \lambda\, u_i \tag{2.9}$$

for i = 1, … , k such that $0 \le u_i \le 1$ and $\sum_{i=1}^k u_i = 1$.

Hence, using (2.6) and (2.9), we get

$$\Pi(\lambda \mid x^{(1)}, \ldots, x^{(k)}) = c\, \lambda^{n_1 + \ldots + n_{k-1}} \iiint \prod_{i=1}^{k-1} u_i^{n_i-1} (1 - \prod_{i=1}^{k-1} u_i)^{n_k-1} \times$$

$$\exp\{\lambda[-\sum_1^{k-1} u_i T_i - (1 - \sum_1^{k-1} u_i) T_k]\} \prod_1^k du_i \tag{2.10}$$

Let

$$\frac{\alpha}{2} = \int_0^{\Lambda_1} \Pi(\lambda \mid x^{(1)}, \ldots, x^{(k)}) d\lambda = \int_{\Lambda_2}^{\infty} \Pi(\lambda \mid x^{(1)}, \ldots, x^{(k)}) d\lambda \tag{2.11}$$

so that

$$1 - \alpha = P(\Lambda_1 \le \lambda \le \Lambda_2) \tag{2.12}$$

Hence, $(e^{-\Lambda_2 t}, e^{-\Lambda_1 t})$ is the 100(1- α)% confidence interval for R(t).

The $Bayes$ estimate of R(t)is given by:

$$\widehat{\mathrm{R}}_{\mathrm{B}}(t) = \int_0^\infty \dots \int_0^\infty e^{-\left(\sum_{i=1}^{k} \lambda_i\right)t}\, \Pi(\lambda \mid x^{(1)}, \dots, x^{(k)}) \prod_{i=1}^{k} d\,\lambda_i$$

$$= \prod_{i=1}^{k} \left(1 + \frac{t}{T_i}\right)^{-n_i} \qquad (2.13)$$

IIII. NUMERICAL ILLUSTRATION

Consider a series system composed of three exponential components with means 100, 500 and 350 respectively. Independent samples of size 15 are simulated on each of the three components.

Let α = 0.05. Then solving the three dimensional integral in (2.11) we obtain the following results:

$$\Lambda_1 = 3.29 \text{ and } \Lambda_2 = 3.45.$$

Some values of the confidence limits and the $Bayes$ estimate are given in the Table I.

Table I:

Calculations of lower and upper limits and the $Bayes$ estimate of system reliability

t	Lower limit	Bayes estimate	Upper limit
1	0.03174	0.03560	0.03725
1.01	0.03066	0.03443	0.03604
1.03	0.02862	0.03221	0.03375
1.05	0.02671	0.03014	0.03160
1.07	0.02493	0.02819	0.02959
1.09	0.02327	0.02637	0.02870
1.10	0.02248	0.02551	0.02780
1.11	0.02172	0.02467	0.02694
1.12	0.02098	0.02386	0.02510
1.13	0.02027	0.02308	0.02428
1.14	0.01958	0.02232	0.02350
1.15	0.01892	0.02159	0.02274

III. SOME IMPORTANT SPECIAL CASES

There are many practical situations where one assumes that components of a system are identically distributed viz., in the case of calculating electric power system network reliability.

Consider a system composed of independent exponential components with component reliabilities given by:

$$p_i = e^{-\lambda t}, i = 1, \dots, n \qquad (4.1)$$

Then the system reliability can be obtained from the expression in (4.2):

$$R(t) = \sum_{j=1}^{2^m-1} \mathbf{1}(j) e^{-\lambda t \sum_{i=1}^{n} D(i,j)} \tag{4.2}$$

where m is the number of minimal path sets for the coherent structure with n components and D =((D(i,j))) is the corresponding Design Matrix. For details see [2]..

We discuss one special case of a series system below:

It follows from (4.2) that the system reliability of the series system is:

$$R(t) = e^{-n\lambda t}, \tag{4.3}$$

since $D = \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \vdots \\ 1 \end{pmatrix}$ *and* $\mathbf{1}$(j)=1, m= 1

Taking a Gamma prior and following [5], the Bayes estimate of R(t) is given by:

$$\frac{2+p}{1+\sum_1^p t_i} \tag{4.4}$$

whete

t_i : Observed failure times of the series system under consideration

As an illustration, we took n=2 and λ =$\frac{1}{2}$, we simulated 100 system failure data to get the *Bayes* estimate as

$$\frac{2+100}{1+103.3905} = 0.9771$$

IV. CONCLUSIONS

A coherent system can be realistically approximated by a series system through a ‘fault tree” representation. A fault tree basically displays the relationship among the various levels of failure events of the subsystems a coherent system is composed of (see [4] for further details). In principle, for a given prior, one can obtain the Bayes estimate of any coherent system using the relation (4.2). In this article, we have demonstrated the procedure for a series system.

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