Teoplitz Determinant For A Subclass of Generalized Distribution Function Involving Jackson's Derivative Operator Through Chebyshev Polynomials

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Abstract: *In this paper, the author finds the coefficient bounds and Teoplitz determinants of Generalized distribution function involving Jackson's q-derivative operator using the subordination principle.*

Keywords: *Analytic function, univalent function, generalyzed distribution function, chebyshev polynomial, surbordination, Teoplitz determinant, q-derivative.*

I. INTRODUCTION, PRELIMINARIES AND LEMMA

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1}$$

which are analytic in the open unit disk $U = \{z : z \in C \text{ and } |z| < 1\}$

and normalized by the condition $f(0) = 0 = f'(0) - 1$. Also let $H \in A$ be the class of analytic univalent function in U .a function $f \in A$ is called a starlike function denoted with S^* if and only if

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \tag{2}$$

A function $f \in A$ which maps U onto a convex domain is called convex function denoted by K if and only if

$$\text{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in u \tag{3}$$

Generally, $S^*(0) = S^*$ and $k(0) = k$

A function $f \in H$ is called a close-to-convex in U if the range $f(U)$ is close to convex and this is the compliment of $f(U)$ which is written as the union of non-intersecting half lines.

Moreover, a function $f \in H$ is said to be close-to-convex with respect to a fixed Starlike functions g (g not necessarily normalized) denoted by C_g, if and only if

$$\left\{ \frac{zf'(z)}{g(z)} \right\} > 0, \quad z \in U. \tag{4}$$

The decompositional approach to matrix computation, one of the top ten algorithms of the 20th century had severally been employed by many researchers particularly the Hankel and Teoplitz determinants having numerous applications. For example, they provide a platform on which a variety of scientific and engineering problems can be solved and furthermore, they permit reasonably simple rounding-error analysis and afford high quality software implementations. Hankel determinants play a vital role in different branches and have many applications [13].

Moreover, it is interesting to note that a closer relation of the Hankel matrix (or determinant) is the Teoplitz matrix (or determinant). In linear algebra, a Teoplitz matrix or diagonal-constant matrix named after Otto Teoplitz is that in which each descending diagonal from left to right is constant. A Teoplitz matrix can be thought of as an "upside down" Hankel matrix. This is so because while Hankel matrix (or determinant) have constant entries along the reverse diagonal, a Teoplitz matrix (or determinant) on the other hand have constant entries along the diagonal.

The Hankel determinant of f for $q \geq 1$ and $n \geq 1$ was defined by Pommereke [13] and [14] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \tag{5}$$

while the symmetric Teoplitz determinant $T_q(n)$ for $q \geq 1$ and $n \geq 1$ was defined as

$$T_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_n & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_n \end{vmatrix} \tag{6}$$

and in particular

$$T_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_2 \end{vmatrix}, \qquad T_2(3) = \begin{vmatrix} a_3 & a_4 \\ a_4 & a_3 \end{vmatrix} \quad \text{and} \quad T_3(2) = \begin{vmatrix} a_2 & a_3 & a_4 \\ a_3 & a_2 & a_3 \\ a_4 & a_3 & a_2 \end{vmatrix} \tag{7}$$

For a good summary of the applications of Teoplitz determinant to a wide range of areas of pure and applied Mathematics; (see [18],[17] and [16] for details).

A matrix equation of the form $Ax = b$ is called a Teoplitz system if A is a Teoplitz matrix. If A is an $n \times n$ Teoplitz matrix, then the system has only $2n-1$ degrees of freedom rather than n^2. We might therefore expect that the solution of a Teoplitz system would be easier, and indeed that is the case. Moreover, Teoplitz matrices are subspaces of the vector spaces of $n \times n$ matrices under matrix addition and scalar multiplication and Teoplitz matrices commute asymptotically [19].

Since Noonan and Thomas in [7] began the study of Hankel determinant when they like defined the *qth* Hankel determinant of the function in (1) as in (5); various researchers had investigated the Hankel and Teoplitz determinants for different univalent and bi-univalent functions, excluding generalized distribution function which is a new research area in the field of geometric function theory. Beginning with Porwal in [15] who investigated the geometric properties of generalized distribution associated with univalent functions, Oladipo in [8] and [9] investigated bonds for probabilities of the generalized distribution polylogarithm and generalized distribution associated with univalent functions in conical domain respectively. In this work, we aim at filling the vacuum created by Porwal and Oladipo thus finding the Teoplitz determinant for the generalized distribution involving Jackson's q- derivative operator.

Let the series $\sum_{k=0}^{\infty} a_k, a_k \geq 0, n \in N$ be convergent and its sum is denoted by S such that

$$S = \sum_{k=0}^{\infty} a_k \tag{8}$$

We now introduce the generalized discrete probability distribution whose probability mass function is $p(k) = \frac{a_k}{s}, k = 0,1,2$ (9)

Obviously, $p(k)$ is a probability mass function because $p(k) \geq 0$ and $\sum_k p_k \geq 1$

We then introduce a power series whose coefficients are probabilities of the generalized distribution, that is

$$G_d(z) = z + \sum_{k=2}^{\infty} \frac{a_k - 1}{s} z^k \tag{10}$$

Applying the q-derivative $[k]_q$ operator as defined in (12) on (10) we

$$D_q G_d(z) = z + \sum_{k=2}^{\infty} [k]_q \frac{a_k - 1}{s} z^k \tag{11}$$

In the field of Geometric function theory, various subclasses of the normalized analytic function class A have been studied from different points of view. The q-calculus as well as the fractional q-calculus such as fractional q-integral and fractional q-derivative operators are used to investigate several subclasses of analytic functions [Details are found in [2], [1], [10] and [12]. The application of q-calculus which plays vital role in the theory of hyper-geometric series, quantum physics and operator theory was initiated by Jackson [5]. He was the first Mathematician who developed q-derivative and q-integral in a systematic way. Both operators play crucial role in the theory of relativity, usually encompasses two theories by Einstein, one in special relativity and the other in general relativity. While special relativity applies to the elementary particles and their interactions, general relativity on the other hand applies to the cosmological and astrophysical realm including astronomy. Of interest is the fact that special relativity theory had rapidly become a significant and necessary tool for theorists and experimentalists in the new fields of atomic physics, nuclear physics and quantum mechanics.

In the present paper, we consider the symmetric Teoplitz determinants and obtain the estimates of those determinants and whose elements are the coefficients of a_n of q-derivative operator $[k]_q = \frac{1-q^k}{1-q}$, for $k \in \mathbb{N}$ on $f \in T^t(\theta, \mu)$, a subclass of generalized distribution function by the method of surbodination principle on Chebyshev polynomials. Specifically, we obtain the coefficient bounds for the symmetric Teoplitz determinants $T_2(2)$, $T_2(3)$, $T_3(2)$ and $T_3(1)$.

Definition 1.1 Let $q \in (0,1)$ and define

$$[k]_q = \frac{1-q^k}{1-q} \ , \text{ for } k \in N \tag{12}$$

Definition 1.2 The Jackson's q-derivative of a function $f \in A,\ 0 < q < 1$ is defined as follows:

$$D_q f(z) = \begin{cases} f(z) - f(qz), & for\ \ z \neq 0 \\ f'(0), & for\ \ z = 0 \end{cases}$$

and

$$D_q^{\ 2} f(z) = \Delta_q\left(\Delta_q f(z)\right) \tag{13}$$

We note that $Lim_{q \to 1^-}\left(D_q f(z)\right) = f'(z)$ if f is differentiable at z.

From (12) and (1) we deduce that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1} \tag{14}$$

Definition 1.3 The symmetric q-derivative $\overline{\Delta}_q f$ of a function f given by (1) is defined as follows:

$$\overline{D}_q f(z) = \begin{cases} \dfrac{f(qz) - f\left(q^{-1}(z)\right)}{\left(q - q^{-1}\right)z} & for \;\; z \neq 0 \\ f'(0) & for \;\; z = 0 \end{cases} \tag{15}$$

From (14), we deduce that $\tilde{D}_q z^k = [\tilde{k}]_q \, z^{k-1}$ and a power series of

$\tilde{D}_q f$ is $\tilde{D}_q f(z) = 1 + \sum_{k=2}^{\infty} [\tilde{k}]_q a_k z^{k-1}$

when f has the form (1) and the symbol $(k)_q$ denotes the number $[\tilde{k}]_q = \dfrac{q^k - q^{-k}}{q - q^{-1}}$

Chebyshev polynomials have become increasingly important in numerical analysis from both the theoretical and practical points of view. They are sequences of orthogonal polynomials which are practically related to De-Moivres formular and which are defined recursively. There exists four kinds though, the first and second kinds $T_n(x)$ and $U_n(x)$ are well known having more results, uses and applications [3] and [6]. In this work we shall limit ourselves to the second kind given as

$$U_k(t) = \frac{\sin(k+1)\alpha}{\sin\alpha}, t \in (-1, 1) \tag{16}$$

where k denotes the degree of the polynomial and $t = \cos\alpha$

The Chebyshev polynomials of the second kind $U_k(t); t \in (-1, 1)$ have the generating function of the form

$$H(z,t) = \frac{1}{1 - 2tz + z^2} = 1 + \sum_{k=1}^{\infty} \frac{\sin(k+1)\alpha}{\sin\alpha} z^k \left(z \in D, |t| < 1\right)$$

Note that $t = \cos\alpha, \alpha \in \left(\frac{-\pi}{3}, \frac{\pi}{3}\right)$, then

$$H(z,t) = \frac{1}{1 - 2\cos\alpha z + z^2} = 1 + \sum_{k=1}^{\infty} \frac{\sin(k+1)\alpha}{\sin\alpha} z^k \tag{17}$$

Thus,

$$H(z,t) = 1 + 2\cos\alpha z + \left(3\cos^2\alpha - \sin^2\alpha\right)z^2 + \ldots.. \tag{18}$$

Following the relationship in Fadipe *et al* [4], we have

$$H(z,t) = 1 + u_1(t)z + u_2(t)z^2 + \ldots.. \quad (19)$$

where $U_{k-1} = \frac{\sin(k\cos^{-1} t)}{\sqrt{1-t^2}}, k \in N$

are the Chebyshev polynomials of the second kind. It is known that

$$U_k(t) = 2tU_{k-1}(t) - U_{k-2}(t)$$

So that

$$U_1(t) = 2t,, \; U_2(t) = 4t^2 - 1 \text{ and } U_3(t) = 8t^3 - 4t \quad (20)$$

Definition 1.4 If $f(z)$ and $g(z)$ are analytic in U, we say that $f(z)$ is subordinate to $g(z)$ written symbolically as $f \prec g$ or $f(z) \prec g(z)$, $z \in U$ if there exists a Schwarz function $w(z)$ 3333333which by definition is analytic in U such that

$$f(z) = g(w(z))$$

Definition 1.5 The Convolution (or Hadamard product) of two series

$f(z) = \sum_{k=0}^{\infty} a_k z^k$ and $g(z) = \sum_{k=0}^{\infty} b_k z^k$ is defined by

$$f * g(z) = \sum_{k=0}^{\infty} a_k b_k z^k$$

A Set of Lemmas:

For this study, the following existing lemmas are established:
Let P be the class of functions $p(z)$ with positive real part consisting of all analytic function $p : U \to C$ satisfying the following conditions

$$p(0) = 1 \text{ and } R(p(z)) > 0$$

Lemma 1.1 "Reference [11] shows that **i**f the function $p \in P$ is defined by

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots$$

then $|c_n| \leq 2, \; n \in \mathbb{N} = \{1,2,3 \ldots\}$ "

Lemma 1.2 [11]: If the function $p \in P$ is defined by

$$c(z) = 1 + c_1, z + c_2, z^2 + c_3, z^3 + \ldots$$

then

$$2c_2 = c_1^2 + x(4 - c_1^2)$$

and

$$4c_3 = c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z$$

for some values of z, $|z| \le 1$

II. MAIN RESULTS

Definition 2.1 A function $f \in A$ is said to be in class $G_d(\theta, \mu, H(z,t))$ if

$$\mathrm{Re}\left\{(1-\theta)\frac{G_d(z)}{z} + \theta G_d{}'(z) + \mu z G_d{}''(z)\right\} \prec H(z,t) \tag{21}$$

where $0 \le \theta \le 1$, $\mu \ge 1$ and $H(z,t)$ is the Chebyshev polynomial

Remark 2.2. if $\theta = 0$, we have

$$\mathrm{Re}\left\{\frac{G_d(z)}{z} + \mu z G_d{}''(z)\right\} \prec H(z,t) \tag{22}$$

Remark 2.3. if $\theta = 1$, we have

$$\mathrm{Re}\left\{G_d{}'(z) + \mu z G_d{}''(z)\right\} \prec H(z,t) \tag{23}$$

Theorem 1: Let $f \in G_d(\theta, \mu, H(z,t))$, the generalized distribution which satisfies the subordination principle. Then

$$\left|\frac{a_1}{S}\right| \le \frac{2t}{(1+\theta+2\mu)[2]_q}, \tag{24}$$

$$\left|\frac{a_2}{S}\right| \le \frac{2t^2-1}{(1+2\theta+6\mu)[3]_q} \tag{25}$$

$$\left|\frac{a_3}{S}\right| \le \frac{4t(2t^2-1)}{(1+3\theta+12\mu)[4]_q} \tag{26}$$

Proof:

Let $f \in G_d(\theta, \mu, H(z,t))$. Then there exists a Chebyshev polynomial $H(u(z),t)$ such that

$$\mathrm{Re}\left\{(1-\theta)\frac{G_d(z)}{z} + \theta G_d{}'(z) + \mu z G_d{}''(z)\right\} \prec H(u(z),t) \tag{27}$$

where $H(z,t) = 1 + \tfrac{1}{2}U_1(t)c_1 z + \left(\frac{U_1(t)c_2}{2} - \frac{U_1(t)c_1{}^2}{4} + \frac{U_2(t)c_1{}^2}{4}\right)z^2$

$$+\left(\frac{U_1(t)c_3}{2} - \frac{U_1(t)c_1c_2}{2} + \frac{U_1(t)c_1{}^3}{8} + \frac{U_2(t)c_1c_2}{2} - \frac{U_2(t)c_1{}^3}{4} + \frac{U_3(t)c_1{}^3}{8}\right)z^3 + \ldots \tag{28}$$

Next, we define the function $p \in P$ by

$$p(z) = \frac{1+u(z)}{1-u(z)} = 1 + p_1 z + p_2 z^2 + ... \tag{29}$$

In the following, we can derive $u(z) = \dfrac{p(z)-1}{p(z)+1}$ (30)

From (27) we have

$$(1-\theta)\left(1+\sum_{k=2}^{\infty}[k]_q \frac{a_{k-1}}{S} z^{k-1}\right) + \theta\left(1+\sum_{k=2}^{\infty}k[k]_q \frac{a_{k-1}}{S} z^{k-1}\right) + \sum_{k=2}^{\infty}\mu k(k-1)[k]_q \frac{a_{k-1}}{S} z^{k-1}$$

$$= \quad 1 + \left(\tfrac{1}{2}U_1(t)\right)c_1 z + \left(\frac{U_1(t)c_2}{2} - \frac{U_1(t)c_1^{\,2}}{4} + \frac{U_2(t)c_1^{\,2}}{4}\right)z^2$$

$$+\left(\frac{U_1(t)c_3}{2} - \frac{U_1(t)c_1 c_2}{2} + \frac{U_1(t)c_1^{\,3}}{8} + \frac{U_2(t)c_1 c_2}{2} - \frac{U_2(t)c_1^{\,3}}{4} + \frac{U_3(t)c_1^{\,3}}{8}\right)z^3 + ... \tag{31}$$

Expanding and equating the coefficients of z, z^2 and z^3 in both sides of (31) gives:

$$(1+\theta+2\mu)[2]_q \frac{a_1}{S} = \tfrac{1}{2}U_1(t)c_1 \tag{32}$$

$$(1+2\theta+6\mu)[3]_q \frac{a_2}{S} = \frac{U_1(t)c_2}{2} - \frac{U_1(t)c_1^{\,2}}{4} + \frac{U_2(t)c_1^{\,2}}{4} \tag{33}$$

$$(1+3\theta+12\mu)[4]_q \frac{a_3}{S} = \frac{U_1(t)c_3}{2} - \frac{U_1(t)c_1 c_2}{2} + \frac{U_1(t)c_1^{\,3}}{8}$$

$$+\frac{U_2(t)c_1 c_2}{2} - \frac{U_2(t)c_1^{\,3}}{4} + \frac{U_3(t)c_1^{\,3}}{8} \tag{34}$$

which yields:

$$\frac{a_1}{S} = \frac{U_1(t)c_1}{2(1+\theta+2\mu)[2]_q} \tag{35}$$

$$\frac{a_2}{S} = \frac{U_1(t)c_2}{2(1+2\theta+6\mu)[3]_q} - \frac{U_1(t)c_1^{\,2}}{4(1+2\theta+6\mu)[3]_q} + \frac{U_2(t)c_1^{\,2}}{4(1+2\theta+6\mu)[3]_q} \tag{36}$$

$$\frac{a_3}{S} = \frac{U_1(t)c_3}{2(1+3\theta+12\mu)[4]_q} - \frac{U_1(t)c_1c_2}{2(1+3\theta+12\mu)[4]_q} + \frac{U_1(t)c_1^{\,3}}{8(1+3\theta+12\mu)[4]_q} \tag{37}$$

Applying lemmas 1.1 and 1.2 and (20) on (35), (36) and (37) together with triangle inequality we have:

$$\left|\frac{a_1}{S}\right| \le \frac{2t}{(1+\theta+2\mu)[2]_q},$$

$$\left|\frac{a_2}{S}\right| \le \frac{2t^2-1}{(1+2\theta+6\mu)[3]_q}$$

$$\left|\frac{a_3}{S}\right| \le \frac{4t(2t^2-1)}{(1+3\theta+12\mu)[4]_q}$$

which ends the proof.

Remarks: With special choices of parameters involved in theorem 1, various interesting results could be derived.

Theorem 2: Let $f \in T^t(\theta,\mu)$, the Teoplitz determinant for the generalized distribution which satisfies the subordination principle. Then

$$T_2(2)=\left|\frac{a_2^2}{s^2}-\frac{a_1^2}{s^2}\right| \le \left(\frac{4t^2-1}{V}\right)^2-\left(\frac{2t}{M}\right)^2 \tag{38}$$

where $M=(1+\theta+2\mu)[2]_q$ and $V=(1+2\theta+6\mu)[3]_q$

Proof:

Substituting the values of $\frac{a_1}{s}$, $\frac{a_2}{s}$ as obtained in (35), and (36) in theorem I with Lemmas 1.1 and 1.2, we have

$$\left|\frac{a_2^2}{s^2}-\frac{a_1^2}{s^2}\right| = \frac{u_1{}^2(t)(4-c_1{}^2)^2\,x^2}{16(1+2\theta+6\mu)^2[3]_q{}^2}+\frac{u_2{}^2(t)\,c_1{}^4}{16(1+2\theta+6\mu)^2[3]_q{}^2}$$

$$+\frac{u_1(t)u_2(t)(4-c_1{}^2)c_1{}^2\,x}{8(1+2\theta+6\mu)^2[3]_q{}^2}-\frac{u_1{}^2(t)\,c_1{}^2}{4(1+\theta+2\mu)^2[2]_q{}^2} \tag{39}$$

Let $R=4-c_1{}^2$ and W = $(1-|x|^2)z$

Then $\left|\frac{a_2^2}{s^2}-\frac{a_1^2}{s^2}\right| = \frac{u_1{}^2(t)R^2\,x^2}{16(1+2\theta+6\mu)^2[3]_q{}^2}+\frac{u_2{}^2(t)\,c_1{}^4}{16(1+2\theta+6\mu)^2[3]_q{}^2}$

$$+\frac{u_1(t)u_2(t)R\,c_1{}^2x}{8(1+2\theta+6\mu)^2[3]_q{}^2}-\frac{u_2{}^2(t)\,c_1{}^2}{4(1+\theta+2\mu)^2[2]_q{}^2}$$

By Lemma 3.1, we have $|c_n| \leq 2$, □ ∈ □ $\{1,2,3,...\}$. For convenience of notation we take $c_n = c$ and we may assume without loss of generality that $c \in [0,2]$. Applying the triangle inequality with $R = 4 - c_1^{\;2}$. Then

$$\left|\frac{a_2^2}{s^2} - \frac{a_1^2}{s^2}\right| \leq \frac{u_1^{\;2}(t) R^2 x^2}{16(1+2\theta+6\mu)^2 [3]_q^{\;2}} + \frac{u_2^{\;2}(t)\, c^4}{16(1+2\theta+6\mu)^2 [3]_q^{\;2}}$$

$$+ \frac{u_1(t)\, u_2(t) R c^2 x}{8(1+2\theta+6\mu)^2 [3]_q^{\;2}} - \frac{u_2^{\;2}(t)\, c^2}{4(1+\theta+2\mu)^2 [2]_q^{\;2}} = \phi(|x|) \tag{40}$$

Trivially, we can show that this expression has a maximum value

$$\left|\frac{a_2^2}{s^2} - \frac{a_1^2}{s^2}\right| \leq \left[\frac{u_2(t)}{(1+2\theta+6\mu)[3]_q}\right]^2 - \left[\frac{u_1(t)}{(1+\theta+2\mu)[2]_q}\right]^2 \tag{41}$$

on [0,2] when c=2.
Applying (20) on (41) gives the results

Thus $= \left|\frac{a_2^2}{s^2} - \frac{a_1^2}{s^2}\right| \leq \left[\frac{4t^2 - 1}{V}\right]^2 - \left[\frac{2t}{M}\right]^2$

where $M = (1+\theta+2\mu)[2]_q$ and $V = (1+2\theta+6\mu)[3]_q$
which completes the proof.

Corollary 2.1: Let $f \in T^t(o,\mu)$ in theorem 2. Then

$$\left|\frac{a_2^2}{s^2} - \frac{a_1^2}{s^2}\right| \leq \left[\frac{4t^2 - 1}{(1+6\mu)[3]_q}\right]^2 - \left[\frac{2t}{(1+2\mu)[2]_q}\right]^2$$

Corollary 2.2: Let $f \in T^t(1,\mu)$. Then

$$\left|\frac{a_2^2}{s^2} - \frac{a_1^2}{s^2}\right| \leq \left[\frac{4t^2 - 1}{3(1+2\mu)[3]_q}\right]^2 - \left[\frac{2t}{2(1+\mu)[2]_q}\right]^2$$

Corollary 2.3: Let $f \in T^t(\theta,1)$. Then

$$\left|\frac{a_2^2}{s^2} - \frac{a_1^2}{s^2}\right| \leq \left[\frac{4t^2 - 1}{(7+2\theta)[3]_q}\right]^2 - \left[\frac{2t}{(3+\theta)[2]_q}\right]^2$$

Corollary 2. 4: Let $f \in T^{\frac{1}{2}}(\theta,\mu)$. Then

$$\left|\frac{a_2^2}{s^2} - \frac{a_1^2}{s^2}\right| \leq \left[\frac{1}{M}\right]^2$$

Theorem 3: Let $f \in T^{t}(\theta,\mu)$ satisfying the subordination principle. Then

$$T_2(3)=\left|\frac{a_3^2}{s^2}-\frac{a_2^2}{s^2}\right| \le \left[\frac{8t^3-4t}{Y}\right]^2-\left[\frac{4t^2-1}{V}\right]^2 \tag{42}$$

where $V=(1+2\theta+6\mu)[3]_q$ and $Y=(1+3\theta+12\mu)[4]_q$.

Proof:

Substituting $\frac{a_3}{s}$, $\frac{a_2}{s}$ from (36) and (37) in theorem 1 and Lemmas 1.1 and 1.2, we have

$$\left|\frac{a_3^2}{s^2}-\frac{a_2^2}{s^2}\right| = \frac{u_2^{\,2}(t)\left(4-c_1^{\,2}\right)^2 c_1^{\,2}x^2}{16(1+3\theta+12\mu)^2[4]_q^{\,2}}+\frac{u_1^{\,2}(t)\left(4-c_1^{\,2}\right)^2 c_1^{\,2}x^4}{64(1+3\theta+12\mu)^2[4]_q^{\,2}}+\frac{u_3^{\,2}(t)\,c_1^{\,6}}{16(1+3\theta+12\mu)^2[4]_q^{\,2}}$$

$$+\frac{u_1^{\,2}(t)\left(4-c_1^{\,2}\right)\left(1-|x|^2\right)^2 z^2}{16(1+3\theta+12\mu)^2[4]_q^{\,2}}-\frac{u_1(t)u_2(t)\left(4-c_1^{\,2}\right)c_1^{\,2}x^3}{16(1+3\theta+12\mu)[4]_q^{\,2}}-\frac{u_2(t)u_3(t)\left(4-c_1^{\,2}\right)c_1^{\,4}x}{16(1+3\theta+12\mu)^2[4]_q^{\,2}}$$

$$+\frac{u_1(t)u_2(t)\left(4-c_1^{\,2}\right)\left(1-|x|^2\right)c_1xz}{8(1+3\theta+12\mu)^2[4]_q^{\,2}}-\frac{u_1(t)u_3(t)\left(4-c_1^{\,2}\right)c_1^{\,4}x^2}{32(1+3\theta+12\mu)^2[4]_q^{\,2}}-\frac{u_1^{\,2}(t)\left(4-c_1^{\,2}\right)^2\left(1-|x|^2\right)c_1x^2z}{16(1+3\theta+12\mu)^2[4]_q^{\,2}}$$

$$+\frac{u_1(t)u_3(t)\left(4-c_1^{\,2}\right)\left(1-|x|^2\right)c_1^{\,3}z}{16(1+3\theta+12\mu)^2[4]_q^{\,2}}-\frac{u_2^{\,2}(t)\left(4-c_1^{\,2}\right)^2x^2}{16(1+2\theta+6\mu)^2[3]_q^{\,2}}-\frac{u_2^{\,2}(t)\,c_1^{\,4}}{16(1+2\theta+6\mu)^2[3]_q^{\,2}}$$

$$-\frac{u_1(t)u_2(t)\left(4-c_1^{\,2}\right)c_1^{\,2}x}{8(1+2\theta+6\mu)^2[3]_q^{\,2}} \tag{43}$$

Let $\left(4-c_1^{\,2}\right)=$ R and $\left(1-|x|^2\right)z$ =W. Then

$$\left|\frac{a_3^2}{s^2}-\frac{a_2^2}{s^2}\right| = \frac{u_2^{\,2}(t)R^2c_1^{\,2}x^2}{16(1+3\theta+12\mu)^2[4]_q^{\,2}}+\frac{u_1^{\,2}(t)\,Rc_1^{\,2}x^4}{64(1+3\theta+12\mu)^2[4]_q^{\,2}}$$

$$+\frac{u_3^{\,2}(t)\,c_1^{\,6}}{64(1+3\theta+12\mu)^2[4]_q^{\,2}}-\frac{u_1^{\,2}(t)RW^2}{16(1+3\theta+12\mu)^2[4]_q^{\,2}}$$

$$-\frac{u_1(t)\,u_2(t)\,Rc_1^{\,2}x^3}{16(1+3\theta+12\mu)^2[4]_q^{\,2}}-\frac{u_2(t)\,u_3(t)\,Rc_1^{\,4}x}{64(1+3\theta+12\mu)^2[4]_q^{\,2}}$$

$$+\frac{u_1(t)\,u_2(t)\,RWc_1x}{8(1+3\theta+12\mu)^2[4]_q^{\,2}}-\frac{u_1(t)\,u_3(t)\,Rc_1^{\,4}x^2}{32(1+3\theta+12\mu)^2[4]_q^{\,2}}-\frac{u_1^{\,2}(t)\,R^2W\,c_1x^2}{16(1+3\theta+12\mu)^2[4]_q^{\,2}}$$

$$+\frac{u_1(t)\,u_3(t)\,RWc_1^{\,3}}{16(1+3\theta+12\mu)^2[4]_q^{\,2}}-\frac{u_1^{\,2}(t)R^2x^2}{16(1+2\theta+6\mu)^2[3]_q^{\,2}}$$

$$-\frac{u_2^{\,2}(t)\;c_1^{\,4}}{16(1+2\theta+6\mu)^2[3]_q^{\,2}}-\frac{u_1(t)\,u_2(t)\,Rc_1^{\,2}x}{8(1+2\theta+6\mu)^2[3]_q^{\,2}} \tag{44}$$

By Lemma 3.1, we have $|c_n|\le 2$, □ ∈ □ $\{1,2,3,\ldots\}$. For convenience of notation we take $c_n = c$ and we may assume without loss of generality that $c\in[0,2]$. Applying the triangle inequality with $R=4-c_1^{\,2}$. Then

$$\left|\frac{a_3^2}{s^2}-\frac{a_2^2}{s^2}\right|=\frac{u_3^{\,2}(t)c^6}{64(1+3\theta+12\mu)^2[4]_q^{\,2}}-\frac{u_2^{\,2}(t)\;c^4}{16(1+2\theta+6\mu)^2[3]_q^{\,2}}$$

$$+\frac{u_2^{\,2}(t)R^2\,c^2x^2}{16(1+3\theta+12\mu)^2[4]_q^{\,2}}+\frac{u_1^{\,2}(t)Rc^2x^4}{64(1+3\theta+12\mu)^2[4]_q^{\,2}}-\frac{u_1^{\,2}(t)\;RW^2}{16(1+3\theta+12\mu)^2[4]_q^{\,2}}$$

$$-\frac{u_1(t)u_2(t)Rc^2x^3}{16(1+3\theta+12\mu)^2[4]_q^{\,2}}-\frac{u_2(t)u_3(t)Rc^4x}{16(1+3\theta+12\mu)^2[4]_q^{\,2}}-\frac{u_1(t)u_2(t)RWcx}{8(1+3\theta+12\mu)^2[4]_q^{\,2}}$$

$$-\frac{u_1(t)u_3(t)Rc^4x^2}{32(1+3\theta+12\mu)^2[4]_q^{\,2}}-\frac{u_1^{\,2}(t)\;R^2Wcx^2}{16(1+3\theta+12\mu)^2[4]_q^{\,2}}+\frac{u_1(t)u_3(t)RWc^3}{16(1+3\theta+12\mu)^2[4]_q^{\,2}}$$

$$-\frac{u_1^{\,2}(t)\;R^2x^2}{16(1+2\theta+6\mu)^2[3]_q^{\,2}}+\frac{u_1(t)\;u_2(t)Rc^2x}{8(1+2\theta+6\mu)^2[3]_q^{\,2}}=\phi(|x|) \tag{45}$$

Trivially, we can show that this expression has a maximum value

$$\left|\frac{a_3^2}{s^2}-\frac{a_2^2}{s^2}\right|\le\left[\frac{u_3(t)}{(1+3\theta+12\mu)[4]_q}\right]^2-\left[\frac{u_2(t)}{(1+2\theta+6\mu)[3]_q}\right]^2 \text{ on } [0,\,2] \text{ when } c=2.$$

and applying (20) we obtained the results

$$\left|\frac{a_3^2}{s^2}-\frac{a_2^2}{s^2}\right|\le\left[\frac{4t(2t^2-1)}{Y}\right]^2-\left[\frac{(4t^2-1)}{V}\right]^2$$

where $V=(1+2\theta+6\mu)[3]_q$ and $Y=(1+3\theta+12\mu)[4]_q$.

and this ends the proof.

Corollary 3.1: Let $f\in T^t(o,\mu)$. Then

$$\left|\frac{a_3^2}{s^2}-\frac{a_2^2}{s^2}\right|\leq\left[\frac{4t(2t^2-1)}{(1+12\mu)[4]_q}\right]^2-\left[\frac{(4t^2-1)}{(1+6\mu)[3]_q}\right]^2$$

Corollary 3.2: Let $f\in T^t(1,\mu)$. Then

$$\left|\frac{a_3^2}{s^2}-\frac{a_2^2}{s^2}\right|\leq\left[\frac{4t(2t^2-1)}{4(1+3\mu)[4]_q}\right]^2-\left[\frac{(4t^2-1)}{3(1+2\mu)[3]_q}\right]^2$$

Corollary 3.3: Let $f\in T^t(\theta,1)$. Then

$$\left|\frac{a_3^2}{s^2}-\frac{a_2^2}{s^2}\right|\leq\left[\frac{4t(2t^2-1)}{(13+3\theta)[4]_q}\right]^2-\left[\frac{(4t^2-1)}{(7+2\theta)[3]_q}\right]^2$$

Corollary 3. 4: Let $f\in T^{\frac{1}{2}}(\theta,\mu)$. Then

$$\left|\frac{a_3^2}{s^2}-\frac{a_2^2}{s^2}\right|\leq\left[\frac{1}{Y}\right]^2$$

Theorem 4: Let $f\in T^t(\theta,\mu)$ satisfying the subordination principle. Then

$$T_3(2)=\left|\left(\frac{a_1}{s}-\frac{a_3}{s}\right)\left[\frac{a_1^2}{s^2}-2\frac{a_2^2}{s^2}+\frac{a_1}{s}\frac{a_3}{s}\right]\right|\leq\frac{2t}{M}\left[\frac{2t}{V}\right]^2-\frac{4t}{M}\left[\frac{(4t^2-1)}{V}\right]^2$$
$$+\frac{(8t^3-4t)}{Y}\left[\frac{2t}{M}\right]^2-\frac{(8t^3-4t)}{Y}\left[\frac{2t}{V}\right]^2+\frac{(8t^3-4t)}{Y}\left[\frac{2(4t^2-1)}{V}\right]^2$$
$$-\frac{2t}{M}\left[\frac{(8t^3-4t)}{Y}\right]^2 \qquad (46)$$

where $M=(1+\theta+2\mu)[2]_q$, $V=(1+2\theta+6\mu)[3]_q$ and $Y=(1+3\theta+12\mu)[4]_q$

Proof:

Substituting for the values of $\frac{a_1}{s}$, $\frac{a_2}{s}$, and $\frac{a_3}{s}$ as in (35), (36) and (37) in theorem 1, together with Lemmas 1.1 and 1.2, we have

$$T_3(2)=\left|\left(\frac{a_1}{s}-\frac{a_3}{s}\right)\left[\frac{a_1^2}{s^2}-2\frac{a_2^2}{s^2}+\frac{a_1}{s}\frac{a_3}{s}\right]\right|$$

$$=\left[\frac{u_1(t)c_1}{2M}+\frac{u_1(t)(4-c_1^2)c_1x^2}{8Y}-\frac{u_1(t)(4-c_1^2)(1-|x|^2)z}{8Y}-\frac{u_2(t)(4-c_1^2)c_1x}{4Y}-\frac{u_3(t)\,c_1^2}{8Y}\right]$$

$$\left[\frac{u_1^2(t)c_1^2}{4V^2}-\frac{u_1^2(t)(4-c_1^2)^2x^2}{8V^2}-\frac{u_2^2(t)c_1^4}{8V^2}-\frac{u_1(t)u_2(t)(4-c_1^2)c_1^2x}{4V^2}-\frac{u_1^2(t)\,(4-c_1^2)c_1^2x^2}{16MY}\right.$$

$$\left.+\frac{u_1^2(t)(4-c_1^2)(1-|x|^2)z}{16MY}+\frac{u_1(t)u_2(t)(4-c_1^2)c_1^2x}{8MY}+\frac{u_1(t)u_3(t)\,c_1^4}{16MY}\right] \quad (47)$$

Let $R=(4-c_1^2)$ and $W=(1-|x|^2)z$ and expand, we have

$$\begin{aligned}T_3((2)=&\frac{u_1^3(t)c_1^3}{8MV^2}-\frac{u_1^3(t)R^2c_1x^2}{16MV^2}-\frac{u_1(t)v_2^2(t)c_1^5}{16MV^2}-\frac{u_1^2(t)u_2(t)Rc_1^3x}{8MV^2}-\frac{u_1^3(t)\,Rc_1^3x^2}{32M^2Y}\\
&+\frac{u_1^3(t)Rwc_1}{32M^2Y}+\frac{u_1^2(t)u_2(t)Rc_1^3x}{16M^2Y}+\frac{u_1^2(t)u_3(t)c_1^5}{32M^2Y}+\frac{u_1^3(t)Rc_1^3x^2}{32V^2Y}-\frac{u_1^3(t)R^3c_1x^4}{64V^2Y}\\
&-\frac{u_1(t)u_2^2(t)Rc_1^5x^2}{64V^2Y}-\frac{u_1^2(t)u_2(t)R^2\,c_1^3x^3}{32V^2Y}-\frac{u_1^3(t)\,R^2\,c_1^3x^4}{128MY^2}+\frac{u_1^3(t)\,R^2\,c_1wx^2}{128MY^2}\\
&+\frac{u_1^2(t)u_2(t)R^2c_1^3x^3}{64MY^2}+\frac{u_1^2(t)u_3(t)Rc_1^5x^2}{128MY^2}-\frac{u_1^3(t)\,Rwc_1^2}{32V^2Y}+\frac{u_1^3(t)\,R^3\,wx^2}{64V^2Y}\\
&+\frac{u_1(t)u_2^2(t)Rwc_1^4}{64V^2Y}+\frac{u_1^2(t)u_2(t)R^2\,wc_1^2x}{32V^2Y}+\frac{u_1^3(t)\,R^2wc_1^2x^2}{128MY^2}-\frac{u_1^3(t)\,R^2\,w^2}{128MY^2}\\
&-\frac{u_1^2(t)u_2(t)R^2wc_1^2x}{64MY^2}-\frac{u_1^2(t)u_3(t)Rwc_1^4}{128MY^2}-\frac{u_1^2(t)u_2\,(t)\,Rc_1^3x}{16V^2Y}+\frac{u_1^2(t)u_2(t)R^3\,c_1x^3}{32V^2Y}\\
&+\frac{u_2^3(t)Rc_1^5}{32V^2Y}+\frac{u_1(t)u_2^2(t)R^2c_1^3x^2}{16V^2Y}+\frac{u_1^2(t)u_2(t)R^2\,c_1^3x^3}{64MY^2}-\frac{u_1^2(t)u_2(t)\,R^2wc_1x}{64MY^2}\\
&-\frac{u_1(t)u_2^2(t)R^2c_1^3x^2}{32MY^2}-\frac{u_1(t)u_2(t)u_3(t)Rc_1^5x}{64MY^2}-\frac{u_1^2(t)u_3(t)c_1^5}{32V^2Y}+\frac{u_1^2(t)u_3(t)\,R^2c_1^3x^2}{64V^2Y}\\
&+\frac{u_2^2(t)u_3(t)c_1^7}{64V^2Y}+\frac{u_1(t)u_2(t)u_3(t)\,c_1^5Rx}{32V^2Y}+\frac{u_1^2(t)u_3(t)\,c_1^5x^2R}{128MY^2}-\frac{u_1^2(t)u_3(t)\,Rwc_1^3}{128MY^2}\\
&-\frac{u_1(t)u_2(t)u_3(t)\,Rc_1^5x}{64MY^2}-\frac{u_1(t)u_3^2(t)\,c_1^7}{128MY^2}\end{aligned} \quad (48)$$

By Lemma 3.1, we have $|c_n|\le 2$, □ ∈ □ $\{1,2,3,\ldots\}$. For convenience of notation we take $c_n=c$ and we may assume without loss of generality that $c\in[0,2]$. Applying the triangle inequality with $R=4-c_1^2$. Then

$$T_3((2)=\left|\lambda_1 c^3-(\lambda_2-\lambda_3+\lambda_4)c^5+(\lambda_5-\lambda_6)c^7\right|-\frac{u_1^{\,3}(t)R^2cx^2}{16MV^2}-\frac{u_1^{\,2}(t)u_2(t)Rc^3x}{8MV^2}$$

$$-\frac{u_1^{\,3}(t)Rc^3x^2}{32M^2Y}+\frac{u_1^{\,3}(t)Rwc}{32M^2Y}+\frac{u_1^{\,2}(t)u_2(t)Rc^3x}{16M^2Y}+\frac{u_1^{\,3}(t)Rc^3x^2}{32V^2Y}-\frac{u_1^{\,3}(t)R^3cx^4}{64V^2Y}$$

$$-\frac{u_1(t)u_2^{\,2}(t)Rc^5x^2}{64V^2Y}-\frac{u_1^{\,2}(t)u_2(t)R^2c^3x^3}{32V^2Y}-\frac{u_1^{\,3}(t)R^2c^3x^4}{128MY^2}+\frac{u_1^{\,3}(t)R^2cwx^2}{128MY^2}$$

$$+\frac{u_1^{\,2}(t)u_2(t)R^2c^3x^3}{64MY^2}+\frac{u_1^{\,2}(t)u_3(t)Rc^5x^2}{128MY^2}-\frac{u_1^{\,3}(t)\,Rwc^2}{32V^2Y}+\frac{u_1^{\,3}(t)\,R^3wx^2}{64V^2Y}$$

$$+\frac{u_1(t)u_2^{\,2}(t)Rwc^4}{64V^2Y}+\frac{u_1^{\,2}(t)u_2(t)R^2\,wc^2x}{32V^2Y}+\frac{u_1^{\,3}(t)R^2wc^2x^2}{128MY^2}-\frac{u_1^{\,3}(t)R^2\,w^2}{128MY^2}$$

$$-\frac{u_1^{\,2}(t)u_2(t)R^2wc^2x}{64MY^2}-\frac{u_1^{\,2}(t)u_3(t)Rwc^4}{128MY^2}-\frac{u_1^{\,2}(t)u_2(t)Rc^3x}{16V^2Y}+\frac{u_1^{\,2}(t)u_2(t)R^3cx^3}{32V^2Y}$$

$$+\frac{u_2^{\,3}(t)Rc^5}{32V^2Y}+\frac{u_1(t)u_2^{\,2}(t)R^2c^3x^2}{16V^2Y}+\frac{u_1^{\,2}(t)u_2(t)R^2\,c^3x^3}{64MY^2}-\frac{u_1^{\,2}(t)u_2(t)R^2wcx}{64MY^2}$$

$$-\frac{u_1(t)u_2^{\,2}(t)R^2c^3x^2}{32MY^2}-\frac{u_1(t)u_2(t)u_3(t)Rc^5x}{64MY^2}+\frac{u_1^{\,2}(t)u_3(t)R^2c^3x^2}{64V^2Y}$$

$$+\frac{u_1(t)u_2(t)u_3(t)c^5Rx}{32V^2Y}+\frac{u_1^{\,2}(t)u_3(t)c^5x^2R}{128MY^2}-\frac{u_1^{\,2}(t)u_3(t)Rwc^3}{128MY^2}-\frac{u_1(t)u_2(t)u_3(t)Rc^5x}{64MY^2}$$

$$=\ \phi(|x|) \tag{49}$$

Trivially, we can show that this expression has a maximum value

$$T_3(2)\le\frac{u_1^{\,3}(t)}{MV^2}-\frac{2u_1(t)u_2^{\,2}(t)}{MV^2}+\frac{2u_1^{\,2}(t)u_3(t)}{M^2Y}$$

$$-\frac{u_1^{\,2}(t)u_3(t)}{V^2Y}+\frac{4u_2^{\,2}(t)u_3(t)}{V^2Y}-\frac{u_1(t)\,u_3^{\,2}(t)}{MY^2} \tag{50}$$

on [0,2] when c=2 and substituting for (20) in (50) results into

$$T_3(2)=\left|\left(\frac{a_1}{s}-\frac{a_3}{s}\right)\left[\frac{a_1^2}{s^2}-2\frac{a_2^2}{s^2}+\frac{a_1}{s}\frac{a_3}{s}\right]\right|\le\frac{2t}{M}\left[\frac{2t}{V}\right]^2-\frac{4t}{M}\left[\frac{(4t^2-1)}{V}\right]^2$$

$$+\frac{(8t^3-4t)}{Y}\left[\frac{2t}{M}\right]^2-\frac{(8t^3-4t)}{Y}\left[\frac{2t}{V}\right]^2$$

$$+\frac{(8t^3-4t)}{Y}\left[\frac{2(4t^2-1)}{V}\right]^2-\frac{2t}{M}\left[\frac{(8t^3-4t)}{Y}\right]^2$$

where $M=(1+\theta+2\mu)[2]_q$, $V=(1+2\theta+6\mu)[3]_q$ and $Y=(1+3\theta+12\mu)[4]_q$

Thus the proof ends.

Corollary 4.1: Let $f \in T^t(o,\mu)$ in theorem 4. Then

$$T_3(2) \le \frac{2t}{(1+2\mu)[2]_q}\left[\frac{2t}{(1+6\mu)[3]_q}\right]^2 - \frac{4t}{(1+2\mu)[2]_q}\left[\frac{(4t^2-1)}{(1+6\mu)[3]_q}\right]^2$$

$$+\frac{(8t^3-4t)}{(1+12\mu)[4]_q}\left[\frac{2t}{(1+2\mu)[2]_q}\right]^2 - \frac{(8t^3-4t)}{(1+12\mu)[4]_q}\left[\frac{2t}{(1+6\mu)[3]_q}\right]^2$$

$$+\frac{(8t^3-4t)}{(1+12\mu)[4]_q}\left[\frac{2(4t^2-1)}{(1+6\mu)[3]_q}\right]^2 - \frac{2t}{(1+2\mu)[2]_q}\left[\frac{(8t^3-4t)}{(1+12\mu)[4]_q}\right]^2$$

Corollary 4.2: Let $f \in T^t(1,\mu)$ in theorem 4. Then

$$T_3(2) \le \frac{t}{(1+\mu)[2]_q}\left[\frac{2t}{3(1+2\mu)[3]_q}\right]^2 - \frac{2t}{(1+\mu)[2]_q}\left[\frac{(4t^2-1)}{3(1+2\mu)[3]_q}\right]^2$$

$$+\frac{(2t^3-t)}{(1+3\mu)[4]_q}\left[\frac{2t}{2(1+\mu)[2]_q}\right]^2 - \frac{(2t^3-4t)}{(1+3\mu)[4]_q}\left[\frac{2t}{3(1+2\mu)[3]_q}\right]^2$$

$$+\frac{(2t^3-t)}{(1+3\mu)[4]_q}\left[\frac{2(4t^2-1)}{3(1+2\mu)[3]_q}\right]^2 - \frac{t}{(1+\mu)[2]_q}\left[\frac{(8t^3-4t)}{4(1+3\mu)[4]_q}\right]^2$$

Corollary 4.3: Let $f \in T^t(\theta,1)$ in theorem 4. Then

$$T_3(2) \le \frac{2t}{(3+\theta)[2]_q}\left[\frac{2t}{(7+2\theta)[3]_q}\right]^2 - \frac{4t}{(3+\theta)[2]_q}\left[\frac{(4t^2-1)}{(7+2\theta)[3]_q}\right]^2$$

$$+\frac{(8t^3-4t)}{(13+3\theta)[4]_q}\left[\frac{2t}{(3+\theta)[2]_q}\right]^2 - \frac{(8t^3-4t)}{(13+3\theta)[4]_q}\left[\frac{2t}{(7+2\theta)[3]_q}\right]^2$$

$$-\frac{2t}{(3+\theta)[2]_q}\left[\frac{(8t^3-4t)}{(13+3\theta)[4]_q}\right]^2$$

Corollary 4.4: Let $f \in T^{\frac{1}{2}}(\theta,\mu)$ in theorem 4. Then

$$T_3(2) = \le \frac{1}{M}\left[\frac{1}{V}\right]^2 + \frac{1}{Y}\left[\frac{1}{V}\right]^2 - \frac{1}{V}\left[\frac{1}{Y}\right]^2$$

Theorem 5. Let $f \in T^t(\theta,\mu)$ satisfying the subordination principle. Then

$$T_3(1) = \left[1 + 2\frac{a_1^2}{s^2} - \left(\frac{a_2}{s} - 1\right)^2 + \frac{a_3^2}{s^2}\right] \le 1 + \frac{8(4t^2-1)}{V}\left[\frac{t}{M}\right]^2$$

$$-8\left[\frac{4t}{M}\right]^2 - \left[\frac{(4t^2-1)}{V}\right]^2 \quad (51)$$

where $M = (1+\theta+2\mu)[2]_q$, $V = (1+2\theta+6\mu)[3]_q$ and $Y = (1+3\theta+12\mu)[4]_q$

Proof:

The proof follows from the earlier theorems and the Lemmas.

Remarks: With special choices of parameters θ, μ *and* t involved in theorem 5, various interesting results could be derived as corollaries.

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