Dynamic Behavior for a System of Four Coupled Nonlinear Oscillators with Delays

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Abstract: In this paper, a system of two coupled damped Duffing resonators driven by two van der Pol oscillators with delays is studied. Some sufficient conditions to ensure the periodic and partial periodic oscillations for the system are established. Computer simulation is given to demonstrate our result.

Keywords: nonlinear system, time delay, periodic and partial periodic oscillation

1 Introduction

Recently, the study of nonlinear dynamics of micro-electro-mechanical systems (MEMS) and nano-electro-mechanical systems (NEMS) has grown rapidly over the last decades. Analysis of simple cases as the building blocks in MEMS or NEMS can gain insight into larger complicated systems. In 2009, Karabalin *et al*. have discussed a system of two coupled nonlinear nano-electro-mechanical resonators using a structure of doubly clamped beams with a shared mechanical ledge. The authors modeled the behavior of the two strongly interacting nonlinear resonators by a coupled equations of motion for the beam as follows [1]:

$$\begin{cases} x_1''(t) + \gamma_1 x_1'(t) + \omega_1^2 x_1(t) + \alpha_1 x_1^3(t) + D(x_1(t) - x_2(t)) = g_{D1}(t), \\ x_2''(t) + \gamma_2 x_2'(t) + \omega_2^2 x_2(t) + \alpha_2 x_2^3(t) + D(x_2(t) - x_1(t)) = g_{D2}(t). \end{cases} \tag{1}$$

By using the standard methods of secular perturbation theory, the complex nonlinear behavior of the system has been demonstrated. The nonlinear behavior of coupled equations can be understood, controlled, and exploited. In order to understand the emergent behavior of complex dynamical systems and develop novel NEMS devices, Leung *et al*. have discussed the following damped Duffing resonator driven by a van der Pol oscillator [2]:

$$\begin{cases} u_1''(t) - \varepsilon_1 u_1'(t) + \Omega_1^2 u_1(t) + k_1 u_1^3(t) - k_c(u_2(t) - u_1(t)) = 0, \\ u_2''(t) - \varepsilon_2 (u_2^2(t) - 1) u_2'(t) + \Omega_2^2 u_2(t) - k_c(u_1(t) - u_2(t)) = 0. \end{cases} \tag{2}$$

By solving nonlinear algebraic equations, highly accurate bifurcation frequencies for various parameters are provided. Rand and Wong have considered a system of four coupled phase-only oscillators. The qualitative dynamics is depended upon a parameter representing coupling strength. This work has been used to MEMS artificial intelligence decision-making devices [3]. It is known that time delay is ubiquitous in many physical systems, due to finite switching speeds of amplifiers in electronic circuits, finite signal propagation times in networks and circuits, and so on. Recently, many researchers have studied the dynamical behavior of various isolated and coupled time delay systems [4-10]. Zhang and Gu have considered the existence of the Hopf bifurcation for a coupled van der Pol system as follows [11]:

$$\begin{cases} x_1''(t)+\varepsilon(x_1^2(t)-1)x_1'(t)+(1+\alpha)x_1(t)=\alpha y_1'(t-\tau_2), \\ y_1''(t)+\varepsilon(y_1^2(t)-1)y_1'(t)+(1+\alpha)y_1(t)=\alpha x_1'(t-\tau_1). \end{cases} \tag{3}$$

The stability and direction of the Hopf bifurcation were also determined by using the normal form theory and the center manifold theorem. Zhang *et al*. have investigated three coupled van der Pol oscillators with delay as follows [12]:

$$\begin{cases} x_1''+x_1-\varepsilon_1(1-x_1^2)x_1'=k[x_2(t-\tau)-x_1(t-\tau)]+k[x_3(t-\tau)-x_1(t-\tau)], \\ x_2''+x_2-\varepsilon_1(1-x_2^2)x_2'=k[x_3(t-\tau)-x_2(t-\tau)]+k[x_1(t-\tau)-x_2(t-\tau)], \\ x_3''+x_3-\varepsilon_1(1-x_3^2)x_3'=k[x_1(t-\tau)-x_3(t-\tau)]+k[x_2(t-\tau)-x_3(t-\tau)]. \end{cases} \tag{4}$$

By using a symmetric Hopf bifurcation theory, the Hopf bifurcations at zero point appear as the delay increases and the existence of multiple periodic solutions are also established. For various Duffing oscillators, van der Pol oscillators or van der Pol-Duffing oscillators, many results have been appeared in the literature [13-20]. Recently, Feng and Akujuobi have discussed the following two damped Duffing resonators driven by a van der Pol oscillator [21]:

$$\begin{cases} x_1''+\varepsilon_1x_1'+\Omega_1^2x_1+k_1x_1^3=p_1[x_2(t-\tilde{\tau}_2)-x_1(t-\tilde{\tau}_1)]+q_1[x_3(t-\tilde{\tau}_3)-x_1(t-\tilde{\tau}_1)], \\ x_2''+\varepsilon_2x_2'+\Omega_2^2x_2+k_2x_2^3=p_2[x_3(t-\tilde{\tau}_3)-x_2(t-\tilde{\tau}_2)]+q_2[x_1(t-\tilde{\tau}_1)-x_2(t-\tilde{\tau}_2)], \\ x_3''+\varepsilon_3(x_3^2-1)x_3'+\Omega_3^2x_3=p_3[x_1(t-\tilde{\tau}_1)-x_3(t-\tilde{\tau}_3)]+q_3[x_2(t-\tilde{\tau}_2)-x_3(t-\tilde{\tau}_3)]. \end{cases} \tag{5}$$

In system (5), the first two Duffing oscillators are coupled and driven by a van der Pol oscillator, in which the system appeared a partial oscillation under some restrictive conditions. Motivated by the above models, in this paper we extend model (5) to the following system of two damped

Duffing resonator driven by two van der Pol oscillators.

$$\begin{cases} x_1'' + \varepsilon_1 x_1' + \Omega_1^2 x_1 + k_1 x_1^3 & = & p_1[x_2(t-\tilde{\tau}_2) - x_1(t-\tilde{\tau}_1)] + q_1[x_3(t-\tilde{\tau}_3) \\ & & -x_1(t-\tilde{\tau}_1)] + r_1[x_4(t-\tilde{\tau}_4) - x_1(t-\tilde{\tau}_1)], \\ x_2'' + \varepsilon_2 x_2' + \Omega_2^2 x_2 + k_2 x_2^3 & = & p_2[x_3(t-\tilde{\tau}_3) - x_2(t-\tilde{\tau}_2)] + q_2[x_1(t-\tilde{\tau}_1) \\ & & -x_2(t-\tilde{\tau}_2)] + r_2[x_4(t-\tilde{\tau}_4) - x_2(t-\tilde{\tau}_2)], \\ x_3'' + \varepsilon_3(x_3^2-1)x_3' + \Omega_3^2 x_3 & = & p_3[x_1(t-\tilde{\tau}_1) - x_3(t-\tilde{\tau}_3)] + q_3[x_2(t-\tilde{\tau}_2) \\ & & -x_3(t-\tilde{\tau}_3)] + r_3[x_4(t-\tilde{\tau}_4) - x_3(t-\tilde{\tau}_3)], \\ x_4'' + \varepsilon_4(x_4^2-1)x_4' + \Omega_4^2 x_4 & = & p_4[x_1(t-\tilde{\tau}_1) - x_4(t-\tilde{\tau}_4)] + q_3[x_2(t-\tilde{\tau}_2) \\ & & -x_4(t-\tilde{\tau}_4)] + r_4[x_3(t-\tilde{\tau}_4) - x_4(t-\tilde{\tau}_4)]. \end{cases} \tag{6}$$

where $x_i = x_i(t)$ represents coordinate, $\varepsilon_i, \Omega_i (i = 1, 2, 3, 4), k_j (j = 1, 2)$ are the damping coefficient, linear frequency and nonlinear stiffness of the Duffing resonator respectively. $p_i, q_i, r_i (i = 1, 2, 3, 4)$ are the coupling linear stiffness between the four resonators. By means of mathematical analysis method, some sufficient conditions to ensure the periodic and partial periodic oscillations of system (6) were obtained. Numerical simulation is provided to support our result. It should be emphasized that if the constants $\varepsilon_i, \Omega_i, p_i, q_i, r_i \tilde{\tau}_i (i = 1, 2, 3, 4), k_j (j = 1, 2)$ are different values, then the method of Hopf bifurcation is very hard to deal with system (6). This is due to the complexity of finding the bifurcating parameter.

2 Preliminaries

Let $\tau_1 = \tilde{\tau}_1, \tau_3 = \tilde{\tau}_2, \tau_5 = \tilde{\tau}_3, \tau_7 = \tilde{\tau}_4$. It is convenient to write (6) as an equivalent eight-dimensional first order system:

$$\begin{cases} u_1' & = & u_2, \\ u_2' & = & -\varepsilon_1 u_2 - \Omega_1^2 u_1 - k_1 u_1^3 + p_1[u_3(t-\tau_3) - u_1(t-\tau_1)] \\ & & +q_1[u_5(t-\tau_5) - u_1(t-\tau_1)] + r_1[u_7(t-\tau_7) - u_1(t-\tau_1)], \\ u_3' & = & u_4, \\ u_4' & = & -\varepsilon_2 u_4 - \Omega_2^2 u_3 - k_2 u_3^3 + p_2[u_5(t-\tau_5) - u_3(t-\tau_3)] \\ & & +q_2[u_1(t-\tau_1) - u_3(t-\tau_3)] + r_2[u_7(t-\tau_7) - u_3(t-\tau_3)], \\ u_5' & = & u_6, \\ u_6' & = & -\varepsilon_3(u_5^2-1)u_6 - \Omega_3^2 u_5 + p_3[u_1(t-\tau_1) - u_5(t-\tau_5)] \\ & & +q_3[u_3(t-\tau_3) - u_5(t-\tau_5)] + r_3[u_7(t-\tau_7) - u_5(t-\tau_5)], \\ u_7' & = & u_8, \\ u_8' & = & -\varepsilon_4(u_7^2-1)u_8 - \Omega_4^2 u_7 + p_4[u_1(t-\tau_1) - u_7(t-\tau_7)] \\ & & +q_4[u_3(t-\tau_3) - u_7(t-\tau_7)] + r_4[u_5(t-\tau_5) - u_7(t-\tau_7)]. \end{cases} \tag{7}$$

where $u_i = u_i(t)(i = 1, 2, \cdots, 8)$. The matrix form of system (7) is as follows:

$$U'(t) = AU(t) + BU(t-\tau) + f(U(t)) \tag{8}$$

where $U(t) = [u_1(t), u_2(t), u_3(t), u_4(t), u_5(t), u_6(t), , u_7(t), u_8(t)]^T$, $U(t-\tau) = [u_1(t-\tau_1), 0, u_3(t-\tau_3), 0, u_5(t-\tau_5), 0, u_7(t-\tau_7), 0]^T$,

$$A = (a_{ij})_{8\times 8} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\Omega_1^2 & -\varepsilon_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\Omega_2^2 & -\varepsilon_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\Omega_3^2 & \varepsilon_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0-\Omega_4^2 & \varepsilon_8 \end{pmatrix},$$

$$B = (b_{ij})_{8\times 8} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ l_1 & 0 & p_1 & 0 & q_1 & 0 & r_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q_2 & 0 & l_2 & 0 & p_2 & 0 & r_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_3 & 0 & q_3 & 0 & l_3 & 0 & r_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ p_4 & 0 & q_4 & 0 & r_4 & 0 & l_4 & 0 \end{pmatrix}, \quad f(U(t)) = \begin{pmatrix} 0 \\ -k_1 u_1^3 \\ 0 \\ -k_2 u_3^3 \\ 0 \\ -\varepsilon_3 u_5^2 u_6 \\ 0 \\ -\varepsilon_4 u_7^2 u_8 \end{pmatrix}.$$

where $l_i = -p_i - q_i - r_i (i = 1, 2, 3, 4)$. Obviously, the linearized system of (8) is the following:

$$U'(t) = AU(t) + BU(t-\tau) \tag{9}$$

Definition 1 A solution of system (7) is called oscillatory if the solution is neither eventually positive nor eventually negative.

Definition 2 An oscillatory solution of system (7) is called partial oscillation if there is at least one component of the solution is non-oscillatory.

Lemma 1 Assume that system (7) has a unique equilibrium point and all solutions are bounded. If the unique equilibrium point of system (7) is unstable, then system (7) generates a limit cycle. In other words, there exists a periodic oscillatory solution of system (7).

Proof See [22] and the appendix of [23].

Lemma 2 For selected parameter values $\Omega_i, p_i, q_i, r_i (i = 1, 2, 3, 4)$, if M is a nonsingular matrix, then system (7) has a unique equilibrium point. where

$$M = (m_{ij})_{4\times 4} = \begin{pmatrix} m_{11} & -p_1 & -q_1 & -r_1 \\ -q_2 & m_{22} & -p_2 & -r_2 \\ -p_3 & -q_3 & m_{33} & -r_3 \\ -p_4 & -q_4 & -r_4 & m_{44} \end{pmatrix}.$$

with $m_{ii} = -\Omega_i^2 + p_i + q_i + r_i (i = 1, 2, 3, 4)$.

Proof An equilibrium point $u^* = (u_1^*, u_2^*, u_3^*, u_4^*, u_5^*, u_6^*)^T$ of system (7) is a constant solution of the following algebraic equation

$$\begin{cases} u_2^* = 0, \\ -\varepsilon_1 u_2^* - \Omega_1^2 u_1^* - k_1(u_1^*)^3 - p_1[u_3^* - u_1^*] - q_1[u_5^* - u_1^*] - r_1[u_7^* - u_1^*] = 0, \\ u_4^* = 0, \\ -\varepsilon_2 u_4^* - \Omega_2^2 u_3^* - k_2(u_3^*)^3 - p_2[u_5^* - u_3^*] - q_2[u_1^* - u_3^*] - r_2[u_7^* - u_3^*] = 0, \\ u_6^* = 0, \\ -\varepsilon_3[(u_5^*)^2 - 1]u_6^* - \Omega_3^2 u_5^* - p_3[u_1^* - u_5^*] - q_3[u_3^* - u_5^*] - r_3[u_7^* - u_5^*] = 0, \\ u_8^* = 0, \\ -\varepsilon_4[(u_7^*)^2 - 1]u_8^* - \Omega_4^2 u_7^* - p_4[u_1^* - u_7^*] - q_4[u_3^* - u_7^*] - r_4[u_5^* - u_7^*] = 0. \end{cases} \tag{10}$$

Since $u_2^* = 0, u_4^* = 0, u_6^* = 0, u_8^* = 0$, from (10) we have

$$\begin{cases} (-\Omega_1^2 + p_1 + q_1 + r_1)u_1^* - p_1 u_3^* - q_1 u_5^* - r_1 u_7^* = k_1(u_1^*)^3, \\ -q_2 u_1^* + (-\Omega_2^2 + p_2 + q_2 + r_2)u_3^* - p_2 u_5^* - r_2 u_7^* = k_2(u_3^*)^3, \\ -p_3 u_1^* - q_3 u_3^* + (-\Omega_3^2 + p_3 + q_3 + r_3)u_5^* - r_3 u_7^* = 0, \\ -p_4 u_1^* - q_4 u_3^* - r_4 u_5^* + (-\Omega_4^2 + p_4 + q_4 + r_4)u_7^* = 0. \end{cases} \tag{11}$$

We first consider the homogeneous system associated with system (11) as follows:

$$\begin{cases} (-\Omega_1^2 + p_1 + q_1 + r_1)u_1^* - p_1 u_3^* - q_1 u_5^* - r_1 u_7^* = 0, \\ -q_2 u_1^* + (-\Omega_2^2 + p_2 + q_2 + r_2)u_3^* - p_2 u_5^* - r_2 u_7^* = 0, \\ -p_3 u_1^* - q_3 u_3^* + (-\Omega_3^2 + p_3 + q_3 + r_3)u_5^* - r_3 u_7^* = 0, \\ -p_4 u_1^* - q_4 u_3^* - r_4 u_5^* + (-\Omega_4^2 + p_4 + q_4 + r_4)u_7^* = 0. \end{cases} \tag{12}$$

Since M is a non-singular matrix, the determinant of the coefficient matrix of system (12) does not equal to zero. According to the linear algebraic basic theorem, system (12) implies that $u_1^* = 0, u_3^* = 0, u_5^* = 0$ and $u_7^* = 0$. In other words, system (12) has a unique trivial solution. We see that the third equation and the fourth equation of system (11) are the same as the third equation and the fourth equation of system (12). Noting that $g(u_1^*) = k_1(u_1^*)^3$ and $h(u_3^*) = k_2(u_3^*)^3$ both are monotone functions, and only $g(0) = h(0) = 0$. This implies that $u^* = (0, 0, 0, 0, 0, 0, 0, 0)^T$ is the unique equilibrium point of system (7). The proof is completed.

For a matrix $D = (d_{ij})_{8\times 8}$, we adopt the following norms of the matrix $\|D\|$ and the measure of matrix $\mu(D)$ [24]: $\|D\| = \max_{1\leq j\leq 8}\sum_{i=1}^{8}|d_{ij}|$, and $\mu(D) = \max_{1\leq j\leq 8}(d_{jj} + \sum_{i=1,i\neq j}^{8}|d_{ij}|)$.

Lemma 3 Assume that $0 < \varepsilon_i (i = 3, 4), k_j > 0(j = 1, 2)$, then all solutions of system (7) are bounded.

Proof Note that time delay affects the stability of the solutions, it does not affect the boundedness of the solutions. To avoid unnecessary complexity, consider a special case of system (7) as $\tau_1 = \tau_3 = \tau_5 = \tau_7 = 0$. Consider a Lyapunov function

$$V(t, u(\cdot)) = \sum_{i=1}^{8}\frac{1}{2}u_i^2(t) \tag{13}$$

Calculating the upper right derivative D^+V of V along the solution of (7), we derive that

$$\begin{aligned}
& D^+V(t, u(\cdot))|_{(7)} = \sum_{i=1}^{8} u_i(t)u_i'(t) \\
= \quad & u_1u_2 - \varepsilon_1 u_2^2 - \Omega_1^2 u_1u_2 - k_1u_1^3u_2 + p_1(u_3u_2 - u_1u_2) + q_1(u_5u_2 - u_1u_2) \\
& + r_1(u_7u_2 - u_1u_2) + u_3u_4 - \varepsilon_2 u_4^2 - \Omega_2^2 u_3u_4 - k_2u_3^3u_4 + \cdots \\
& - \varepsilon_3 u_5^2u_6^2 + \cdots - \varepsilon_4 u_7^2u_8^2 + \cdots + r_4(u_5u_8 - u_7u_8)
\end{aligned} \tag{14}$$

Obviously, when $u_i \to \infty (1 \leq i \leq 8)$, $u_1^3u_2, u_3^3u_4u_5^2u_6^2, u_7^2u_8^2$ are higher order infinity than u_i^2, u_iu_j, respectively. Since $0 < \varepsilon_i (i = 3, 4), k_j > 0(j = 1, 2)$, therefore, there exists suitably large $L > 0$ such that $V'(t)|_{(7)} < 0$ as $u_i > L$. This means that all solutions of system (7) are bounded.

3 Periodic and partial periodic oscillations

Note that k_1, k_2 and $\varepsilon_3, \varepsilon_4$ are constants, u_1^3, u_3^3, u_5^2 and u_7^2 are high order infinitesimal as u_1, u_3, u_5, and u_7 tend toward to zero respectively. So, the unique equilibrium point which is exactly the zero point of system (7) and system (9), have the same instability. The oscillatory behavior of the solution of system (9) implied that the solution of system (7) is also oscillatory. Assume that $\varepsilon_i > 0 (i = 1, 2, 3, 4)$ and all solutions of system (7) are bounded. We first point out that the component $u_i (i = 5, 6, 7, 8)$ of the trivial solution of system (7) is unstable. Consider the subsystem constructing by the fifth, sixth, seventh and eighth equations of system (7) as follows ($u_1 = u_3 = 0$):

$$\begin{cases} u_5' &= u_6, \\ u_6' &= -\varepsilon_3(u_5^2-1)u_6 - \Omega_3^2 u_5 - p_3 u_5(t-\tau_5) - q_3 u_5(t-\tau_5) \\ & \quad +r_3[u_7(t-\tau_7) - u_5(t-\tau_5)], \\ u_7' &= u_8, \\ u_8' &= -\varepsilon_4(u_7^2-1)u_8 - \Omega_4^2 u_7 - p_4 u_7(t-\tau_7) - q_3 u_7(t-\tau_7) \\ & \quad +r_3[u_5(t-\tau_5) - u_7(t-\tau_7)]. \end{cases} \tag{15}$$

For simplify, let $u_5 = u_7 = 0$ in system (15), then we get

$$\begin{cases} u_5' = u_6, & u_6' = \varepsilon_3 u_6, \\ u_7' = u_8, & u_8' = \varepsilon_4 u_8. \end{cases} \tag{16}$$

We see that $u_5 = \frac{1}{\varepsilon_3}e^{\varepsilon_3 t}, u_6 = e^{\varepsilon_3 t}, u_7 = \frac{1}{\varepsilon_4}e^{\varepsilon_4 t}, u_8 = e^{\varepsilon_4 t}$ are solutions of system (16). Since $\varepsilon_3 > 0, \varepsilon_4 > 0$, implying that u_i(t) $(i = 5, 6, 7, 8)$ of the trivial solution in system (16) are unstable. Thus, the components $u_i (i = 5, 6, 7, 8)$ of the trivial solution in system (15) are also unstable. Therefore, model (7), if the components u_1, u_2, u_3, and u_4 of the trivial solution are globally asymptotically stable, then the system generates a partial periodic oscillation. Now we investigate the subsystem constructed by the first four equations of system (9) $(u_5(t) = 0)$ since u_1^3, u_3^3 are high order infinitesimal as u_1, u_3 tend toward to zero respectively:

$$\begin{cases} u_1' = u_2, \\ u_2' = -\varepsilon_1 u_2 - \Omega_1^2 u_1 + p_1[u_3(t-\tau_3) - u_1(t-\tau_1)] - (q_1+r_1)u_1(t-\tau_1), \\ u_3' = u_4, \\ u_4' = -\varepsilon_2 u_4 - \Omega_2^2 u_3 - (p_2+r_2)u_3(t-\tau_3) + q_2[u_1(t-\tau_1) - u_3(t-\tau_3)] \end{cases} \tag{17}$$

For convenience, we make the change of variables as $y_1(t) = u_1(t - \frac{\tau_1-\tau_3}{2}), y_2(t) = u_2(t - \frac{\tau_1-\tau_3}{2}), y_3(t) = u_3(t), y_4(t) = u_4(t)$ if $\tau_1 > \tau_3$, or $y_1(t) = u_1(t - \frac{\tau_3-\tau_1}{2}), y_2(t) = u_2(t - \frac{\tau_3-\tau_1}{2}), y_3(t) = u_3(t), y_4(t) = u_4(t)$ if $\tau_1 < \tau_3$ [11]. We can then rewrite system (17) as the following equivalent system

$$\begin{cases} y_1' = y_2, \\ y_2' = -\varepsilon_1 y_2 - \Omega_1^2 y_1 + p_1[y_3(t-\bar\tau) - y_1(t-\bar\tau)] - (q_1+r_1)y_1(t-\bar\tau), \\ y_3' = y_4, \\ y_4' = -\varepsilon_2 y_4 - \Omega_2^2 y_3 - (p_2+r_2)y_3(t-\bar\tau) + q_2[y_1(t-\bar\tau) - y_3(t-\bar\tau)] \end{cases} \tag{18}$$

where $\bar\tau = \frac{\tau_1+\tau_3}{2}$. The matrix form of (18) is as follows:

$$Y'(t) = A_1 Y(t) + B_1 Y(t-\bar\tau) \tag{19}$$

where $Y(t) = (y_1(t), \cdots, y_4(t))^T$, $Y(t-\bar{\tau}) = (y_1(t-\bar{\tau}), 0, y_3(t-\bar{\tau}), 0)^T$.

Theorem 1 Suppose that there exists a unique equilibrium point and all solutions of system (7) are bounded. Let $\varepsilon_i > 0 (i = 1, 2), \gamma_1, \gamma_2, \gamma_3, \gamma_4$ be the eigenvalues of matrix $A_1 + B_1$, which has a negative real part, namely, $Re\gamma_i < 0 (i = 1, 2, 3, 4)$. Setting $\gamma = \min\{|Re\gamma_1|, |Re\gamma_2|, |Re\gamma_3|, |Re\gamma_4|\}$. Assume that there exists a suitably small positive constant $\alpha (\alpha < \gamma)$ such that

$$\|B_1\| \cdot \frac{\|A_1\| + \|B_1\| e^{\alpha\bar{\tau}}}{\alpha(\gamma - \alpha)} (e^{\alpha\bar{\tau}} - 1) \leq 1. \tag{20}$$

Then system (7) has a partial periodic oscillation.

Proof Since $\varepsilon_i > 0 (i = 1, 2)$, and all solutions of system (7) are bounded, according to the above analysis, the component u_5, u_6, u_7 and u_8 are unstable. Therefore, we only need to show that the components u_i or $y_i (i = 1, \cdots, 4)$ are stable. Then system (7) generates a partial periodic oscillation. Consider system (19) for $t \geq \bar{\tau}$ we have

$$\begin{aligned} Y'(t) &= (A_1 + B_1) Y(t) - B_1 \int_{t-\bar{\tau}}^{t} Y'(s) ds \\ &= (A_1 + B_1) Y(t) - B_1 \int_{t-\bar{\tau}}^{t} [A_1 Y(s) + B_1 Y(s - \bar{\tau})] ds \end{aligned} \tag{21}$$

By variation of parameter, we have

$$\begin{aligned} Y(t) &= e^{(A_1+B_1)(t-\bar{\tau})} Y(\bar{\tau})) \\ &\quad - \int_{\bar{\tau}}^{t} e^{(A_1+B_1)(t-s)} ds \int_{s-\bar{\tau}}^{s} B_1 [A_1 Y(\sigma) + B_1 Y(\sigma - \bar{\tau})] d\sigma \end{aligned} \tag{22}$$

and hence

$$\begin{aligned} \|Y(t)\| &\leq \|Y\|_{\bar{\tau}} e^{-\gamma(t-\bar{\tau})} \\ &\quad + \|B_1\| \int_{\bar{\tau}}^{t} e^{-\gamma(t-s)} ds \int_{s-\bar{\tau}}^{s} (\|A_1\| \|Y(\sigma)\| + \|B_1\| \|Y(\sigma - \bar{\tau})\|) d\sigma \end{aligned} \tag{23}$$

where $\|Y\|_{\bar{\tau}} = \sup_{t \in [-\bar{\tau}, \bar{\tau}]} \|Y(t)\|$. From condition (20), we will show that

$$Y(t) \leq \|Y\|_{\bar{\tau}} e^{-\alpha(t-\bar{\tau})}, \quad t > \bar{\tau} \tag{24}$$

Indeed,

$$\begin{aligned} & \|Y\|_{\bar{\tau}} e^{-\gamma(t-\bar{\tau})} + \|B_1\| \int_{\bar{\tau}}^{t} e^{-\gamma(t-s)} ds \int_{s-\bar{\tau}}^{s} (\|A_1\| \|Y(\sigma)\| + \|B_1\| \|Y(\sigma - \bar{\tau})\|) d\sigma \\ \leq\ & \|Y\|_{\bar{\tau}} e^{-\gamma(t-\bar{\tau})} + \|Y\|_{\bar{\tau}} \|B_1\| \int_{\bar{\tau}}^{t} e^{-\gamma(t-s)} ds \int_{s-\bar{\tau}}^{s} (\|A_1\| e^{-\alpha(\sigma-\bar{\tau})} + \|B_1\| e^{-\alpha(\sigma-2\bar{\tau})}) d\sigma \\ =\ & \|Y\|_{\bar{\tau}} e^{-\gamma(t-\bar{\tau})} + \|Y\|_{\bar{\tau}} \|B_1\| \cdot \frac{\|A_1\| e^{\alpha\bar{\tau}} + \|B_1\| e^{2\alpha\bar{\tau}}}{-\alpha} \int_{\bar{\tau}}^{t} e^{-\gamma(t-s)} (e^{-\alpha s} - e^{-\alpha s + \alpha\bar{\tau}}) ds \\ =\ & \|Y\|_{\bar{\tau}} e^{-\gamma(t-\bar{\tau})} + \|Y\|_{\bar{\tau}} \|B_1\| \cdot \frac{\|A_1\| e^{\alpha\bar{\tau}} + \|B_1\| e^{2\alpha\bar{\tau}}}{-\alpha(\gamma - \alpha)} (1 - e^{\alpha\bar{\tau}}) e^{-\gamma t} (e^{(\gamma-\alpha)t} - e^{(\gamma-\alpha)\bar{\tau}}) \\ =\ & \|Y\|_{\bar{\tau}} e^{-\gamma(t-\bar{\tau})} + \|Y\|_{\bar{\tau}} \|B_1\| \cdot \frac{\|A_1\| + \|B_1\| e^{\alpha\bar{\tau}}}{\alpha(\gamma - \alpha)} (e^{\alpha\bar{\tau}} - 1)(e^{-\alpha(t-\bar{\tau})} - e^{-\gamma(t-\bar{\tau})}) \end{aligned} \tag{25}$$

Noting that when α is suitably small we have $e^{\alpha\bar{\tau}} \sim 1$. Thus one can select α suitably small such that $\|B_1\| \cdot \frac{\|A_1\|+\|B_1\|e^{\alpha\bar{\tau}}}{\alpha(\gamma-\alpha)}(e^{\alpha\bar{\tau}}-1) \leq 1$. Therefore, we have

$$\begin{aligned}\|Y(t)\| &\leq \|Y\|_{\bar{\tau}}e^{-\gamma(t-\bar{\tau})} + \|B_1\|\int_{\bar{\tau}}^{t} e^{-\gamma(t-s)}ds\int_{s-\bar{\tau}}^{s}(\|A_1\|\|Y(\sigma)\| + \|B_1\|\|Y(\sigma-\bar{\tau})\|)d\sigma \\ &\leq \|Y\|_{\bar{\tau}}e^{-\gamma(t-\bar{\tau})} + \|Y\|_{\bar{\tau}}(e^{-\alpha(t-\bar{\tau})} - e^{-\gamma(t-\bar{\tau})}) \\ &= \|Y\|_{\bar{\tau}}e^{-\alpha(t-\bar{\tau})} \qquad (26)\end{aligned}$$

Since $e^{-\alpha(t-\bar{\tau})} \to 0$ as $t \to \infty$. Inequality (26) implies the global asymptotic stability of the trivial solution of system (19). This suggests that the equilibrium point of system (19) is globally asymptotically stable. So system (7) has a partial periodic oscillation.

Theorem 2 Suppose that system (7) has a unique equilibrium point and all solutions of system (7) are bounded. If $A_1 + B_1 > 0$, then system (7) generates a periodic oscillation.

Proof According to the Lemma 1 we only need to show that the equilibrium point of subsystem (19) is unstable since the components u_5, u_6, u_7, and u_8 of the equilibrium point of system (9) are unstable. The characteristic equation associated with system (19) is given by:

$$\lambda = A_1 + B_1e^{-\lambda\bar{\tau}} \qquad (27)$$

Noting that (27) is a transcendental equation and λ may be a complex number. We prove that there exists a positive eigenvalue of (19) under the condition $A_1 + B_1 > 0$. If we set $f(\lambda) = \lambda - A_1 - B_1e^{-\lambda\bar{\tau}}$, then $f(\lambda)$ is a continuous function of λ. Since $A_1 + B_1 > 0$, then $f(0) = -A_1 - B_1 = -(A_1 + B_1) < 0$. When λ is sufficiently large, say $\lambda = \lambda^* > 0$, $e^{-\lambda^*\bar{\tau}}$ is sufficiently small, and $f(\lambda^*) = \lambda^* - A_1 - B_1e^{-\lambda^*\bar{\tau}} > 0$, thus there exists a $\lambda = \tilde{\lambda}$, $\tilde{\lambda} \in (0, \lambda^*)$ such that $f(\tilde{\lambda}) = 0$ by the Intermediate Value Theorem. This means that there is a positive eigenvalue of the characteristic equation (19) for any time delay $\bar{\tau}$, implying that the equilibrium point of system (7) is unstable for arbitrary time delays τ_1 and τ_3, and system (7) generates a periodic oscillation. The proof is completed.

Theorem 3 Suppose that system (7) has a unique equilibrium point and all solutions of system (7) are bounded. If matrix A_1 has a positive eigenvalue, then system (7) generates a periodic oscillation.

Proof We shall show that the equilibrium point of subsystem (19) is unstable since the components u_5, u_6, u_7, and u_8 of the equilibrium point of system (9) are unstable. The same as Theorem 2, The characteristic equation associated with system (19) is given by (27). Let $\theta_1, \theta_2, \theta_3$, and θ_4 are eigenvalues of matrix A_1, while $\omega_1, \omega_2, \omega_3$ and ω_4 are eigenvalues of matrix B_1. Then we have immediately that

$$\Pi_{i=1}^{4}[\lambda - \theta_i - \omega_ie^{-\lambda\bar{\tau}}] = 0 \qquad (28)$$

Since there is a positive eigenvalue of θ_i, without loss of generality, assume that $\theta_1 > 0$. Noting that there are two row entries are zeros of matrix B_1, implying that there are at least one eigenvalue is zero of matrix B_1. Suppose that $\omega_1 = 0$ and therefore we get

$$\lambda - \theta_1 - \omega_1 e^{-\lambda\bar{\tau}} = \lambda - \theta_1 = 0 \tag{29}$$

Thus, $\lambda = \theta_1$. In other words, θ_1 is a positive eigenvalue of system (19). This means that the equilibrium point of system (7) is unstable, and system (7) generates a periodic oscillation. The proof is completed.

4 Computer simulation result

This simulation is based on system (7), we select $\varepsilon_1 = 0.00125, \varepsilon_2 = 0.00145, \varepsilon_3 = 0.00185, \varepsilon_4 = 0.00165; \Omega_1^2 = 0.035, \Omega_2^2 = 0.025, \Omega_3^2 = 0.015, \Omega_4^2 = 0.024; k_1 = 195, k_2 = 185; p_1 = 0.12, p_2 = 0.18, p_3 = 0.24, p_4 = 0.15, q_1 = 0.0042, q_2 = 0.0025, q_3 = 0.0032, q_4 = 0.0015; r_1 = 0.042, r_2 = 0.038, r_3 = 0.054, r_4 = 0.055$. Thus $\|A_1\| = 1.00145, \|B_1\| = 0.02305$. The eigenvalues of matrix $A_1 + B_1$ are $-0.0175 \pm 0.4239i$, $-0.1250 \pm 0.4630i$, and $\gamma = 0.0175$. It is easy to check that the conditions of Lemma 2 and Lemma 3 hold. When α is selected by 0.01, time delays are selected as $\tau_1 = 0.0125, \tau_2 = 0.0145, \tau_3 = 0.0185, \tau_4 = 0.0165$, then $\bar{\tau} = 0.0155$ and $\|B_1\| \cdot \frac{\|A_1\|+\|B_1\|e^{\alpha\bar{\tau}}}{\alpha(\gamma-\alpha)}(e^{\alpha\bar{\tau}}-1) = \frac{0.2305(1.00145+0.2305(\exp(0.00155)))}{0.01(0.0175-0.01)} = 0.1218 < 1$. The conditions of Theorem 1 are satisfied, system (7) has a partial oscillation (see Fig 1 and Fig 2). When we select $\varepsilon_1 = 0.000135, \varepsilon_2 = 0.000125, \varepsilon_3 = 0.000245, \varepsilon_4 = 0.000225; \Omega_1^2 = 0.0132, \Omega_2^2 = 0.0125, \Omega_3^2 = 0.0124, \Omega_4^2 = 0.0115; k_1 = 100, k_2 = 120; p_1 = 0.078, p_2 = -0.18, p_3 = -0.024, p_4 = -0.15, q_1 = -0.42, q_2 = -0.25, q_3 = -0.32, q_4 = -0.15; r_1 = 0.42, r_2 = 0.38, r_3 = -0.54, r_4 = 0.55$. Time delays are selected as $\tau_1 = 0.0065, \tau_2 = 0.0075, \tau_3 = 0.0045\tau_4 = 0.0025$, then the eigenvalues of matrix A_1 are $0.2235, 0.8660, -0.2237, -0.8661$. The conditions of Theorem 3 are satisfied. We see that system (7) has an oscillatory solution (see Fig 3). In order to see the effects of time delays, we increase time delays to $\tau_1 = 0.65, \tau_2 = 0.75, \tau_3 = 0.45, \tau_4 = 0.25$, we see that the oscillations are maintained, but the oscillatory frequency and amplitude have been changed (see Fig 4).

5 Conclusion

This paper discussed a system of two coupled damped Duffing oscillators driven by two van der Pol oscillators with delays. Some sufficient conditions to ensure the periodic and partial periodic oscillations for the system are established. Our simple restrict conditions are very easy to check. The computer simulation indicates that time delay affects the oscillatory frequency and amplitude. The partial periodic oscillation is induced by unbalance damped oscillators.

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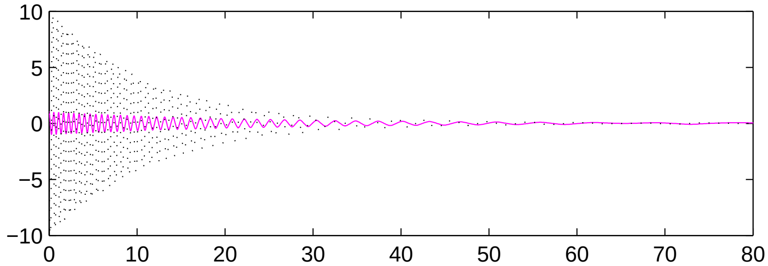
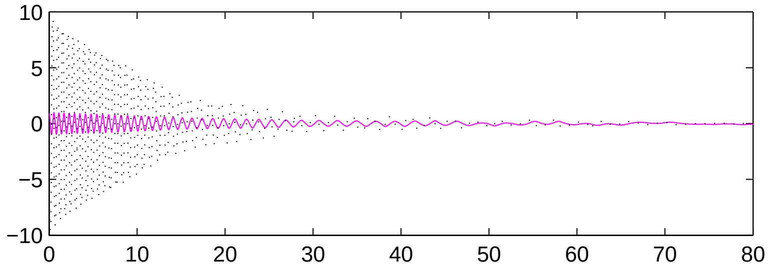
Fig 1 . Partial oscillation of the solutions, delays: 0.0125, 0.0145, 0.0185, 0.0165.

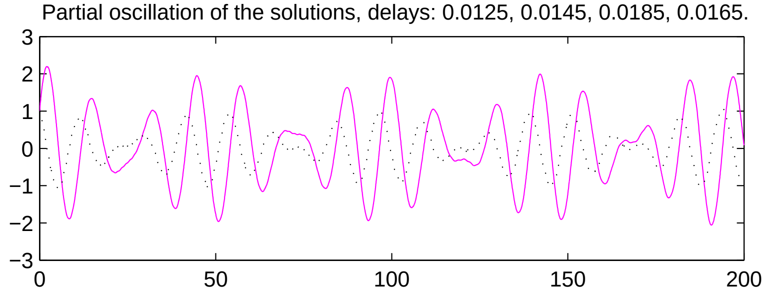


(a) Solid line: $u_1(t)$, dotted line: $u_2(t)$.

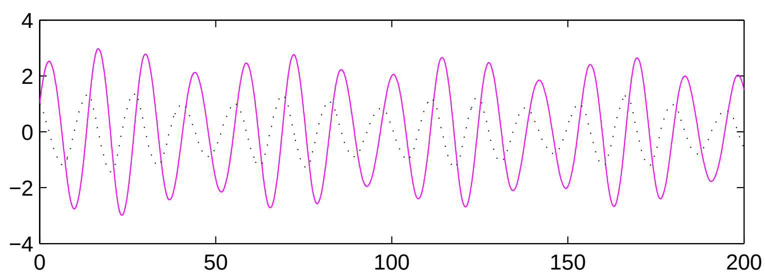


(b) Solid line: $u_3(t)$, dotted line: $u_4(t)$.

Partial oscillation of the solutions, delays: 0.0125, 0.0145, 0.0185, 0.0165.

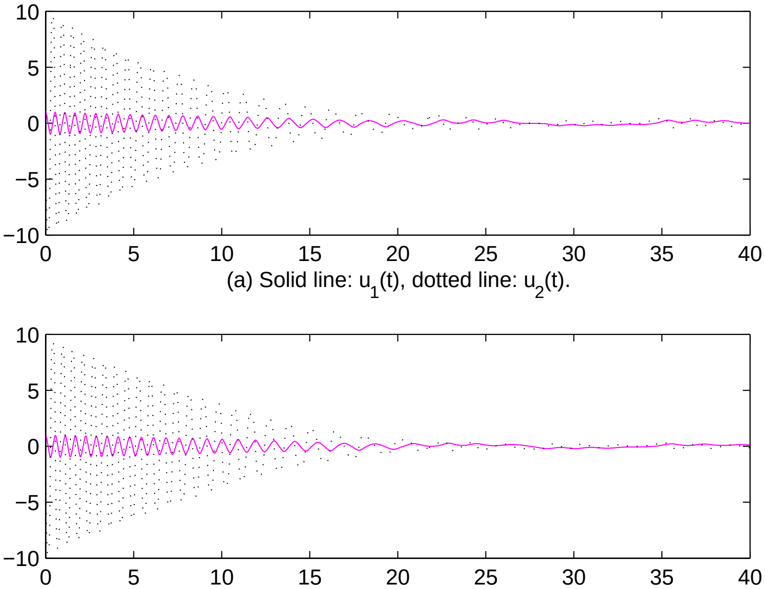


(c) Solid line: $u_5(t)$, dotted line: $u_6(t)$.



(d) Solid line:$u_7(t)$, dotted line: $u_8(t)$.

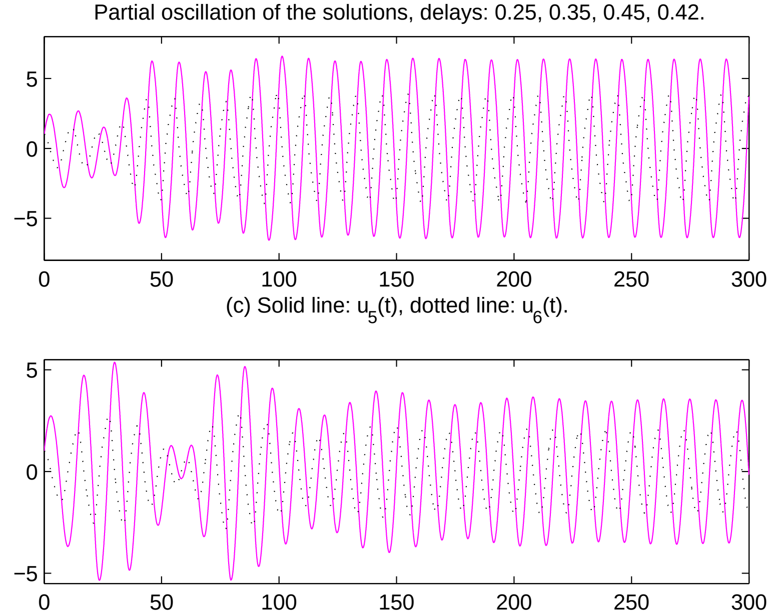
Fig 2. Partial oscillation of the solutions, delays: 0.25, 0.35, 0.45, 0.42.



(a) Solid line: $u_1(t)$, dotted line: $u_2(t)$.

(b) Solid line: $u_3(t)$, dotted line: $u_4(t)$.

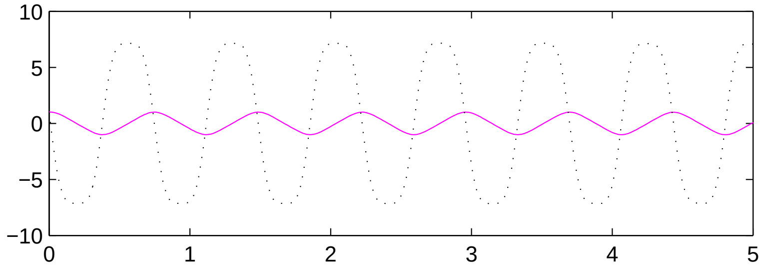
Partial oscillation of the solutions, delays: 0.25, 0.35, 0.45, 0.42.



(c) Solid line: $u_5(t)$, dotted line: $u_6(t)$.

(d) Solid line: $u_7(t)$, dotted line: $u_8(t)$.

Fig 3. Oscillation of the solutions, delays: 0.0065, 0.0075, 0.0045, 0.0025.



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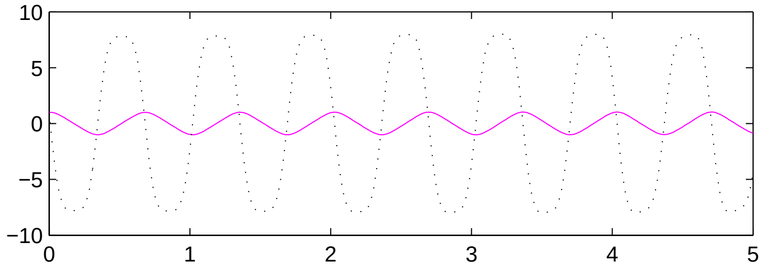
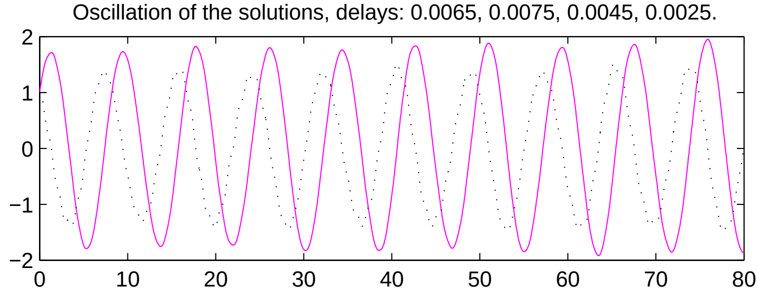
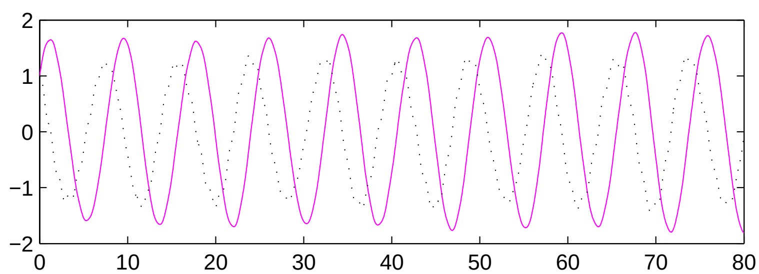


(b) Solid line: $u_3(t)$, dotted line: $u_4(t)$.

Oscillation of the solutions, delays: 0.0065, 0.0075, 0.0045, 0.0025.

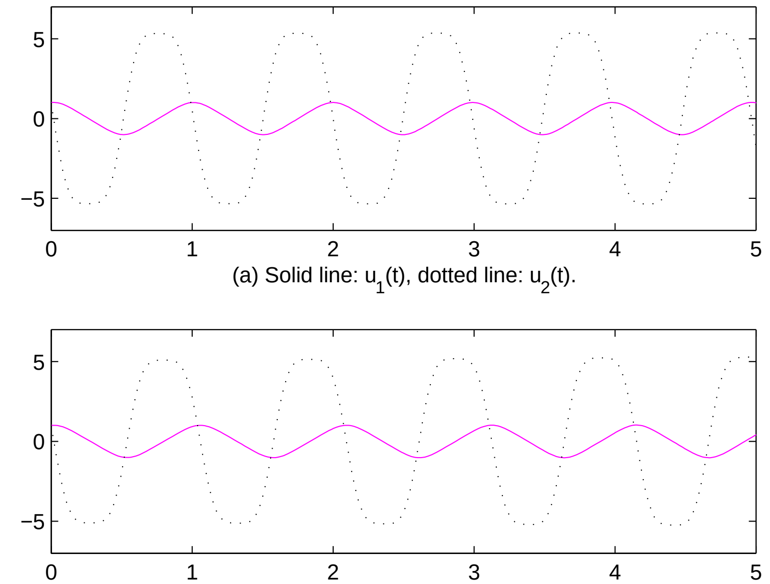


(c) Solid line: $u_5(t)$, dotted line: $u_6(t)$.



(d) Solid line: $u_7(t)$, dotted line: $u_8(t)$.

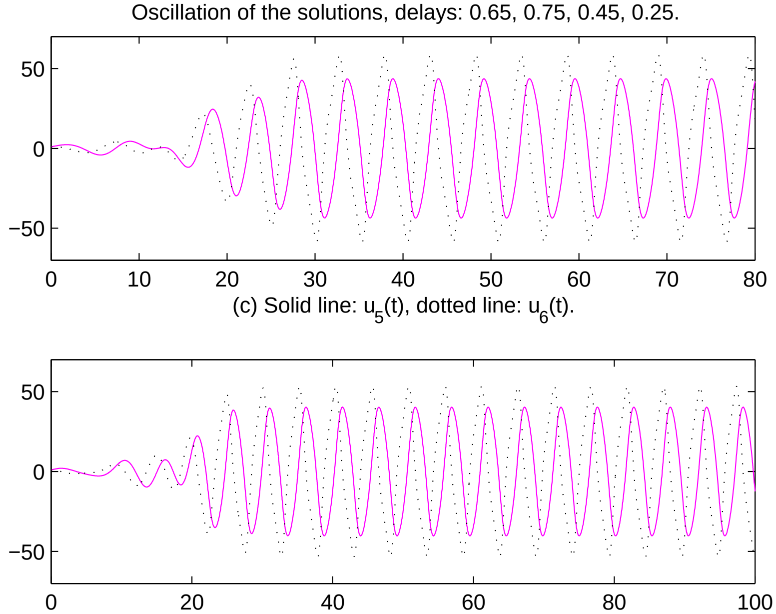
Fig 4. Oscillation of the solutions, delays: 0.65, 0.75, 0.45, 0.25.



(a) Solid line: $u_1(t)$, dotted line: $u_2(t)$.

(b) Solid line: $u_3(t)$, dotted line: $u_4(t)$.

Oscillation of the solutions, delays: 0.65, 0.75, 0.45, 0.25.



(c) Solid line: $u_5(t)$, dotted line: $u_6(t)$.

(d) Solid line: $u_7(t)$, dotted line: $u_8(t)$.