Application of Browder’s and Göhde’s Fixed Point Theorem to Solutions of Operator Equations in Banach Spaces

Neeta Singh
Department of Mathematics
University of Allahabad, Prayagraj-211002, India

Abstract — *the Browder's and Göhde's fixed point theorem for the existence of solutions of operator equations involving asymptotically nonexpansive mappings in uniformly convex Banach spaces.*

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I. PRELIMINARIES

We recall some definitions and the Browder's and Göhde's fixed point theorem.

Definition 1.1 [2] We apply A mapping T from a metric space (X, d) into another metric space (Y, ρ) is said to satisfy Lipschitz condition on X if there exists a constant $L > 0$ such that

$$\rho(Tx, Ty) \leq L\, d(x, y)$$

for all $x, y \in X$. If L is the least number for which Lipschitz condition holds, then L is called Lipschitz constant. If $L = 1$, the mapping is said to be nonexpansive.

Definition 1.2 [3] Let K be a nonempty subset of a Banach space X. A mapping $T: K \to K$ is said to be asymptotically nonexpansive if for each n $n \in \mathbb{N}$ there exists a positive constant $k_n \geq 1$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in K$.

Theorem 1.3 [1] Let X be a uniformly convex Banach space and C a nonempty, closed, convex and bounded subset of X. Then every nonexpansive mapping $T: C \to C$ has a fixed point in C.

We now state the Main theorem.

Theorem 1.4 Let X be a uniformly convex Banach space and K a nonempty subset of X. Let $T: K \to K$ be an asymptotically nonexpansive mapping and $f_n \in K$, then the operator equation

$$k_n x = T^n x + f_n,$$

where $n \in \mathbb{N}$ and k_n is the Lipschitz constant of the iterates T^n, has a solution if and only if, for any $x_1 \in K$, the sequence of iterates $\{x_n\}$ in K defined by

$$k_n x_{n+1} = T^n x_n + f_n ,$$

$n \in \mathbb{N}$ is bounded.

II. Proof of the Main Theorem

For every $n \in \mathbb{N}$, let T_{f_n} be defined to be a mapping from K into K by

$$T_{f_n}(u) = \frac{1}{k_n}\left[T^n u + f_n\right].$$

Then $u_n \in K$ is a solution of

$$x = \frac{1}{k_n}\left[T^n x + f_n\right]$$

if and only if u_n is a fixed point of T_{f_n}.

Since T is asymptotically nonexpansive it follows that T_{f_n} is nonexpansive for all $n \in \mathbb{N}$.

$$\left\|T_{f_n}(x) - T_{f_n}(y)\right\| = \frac{1}{k_n}\left\|T^n(x) - T^n(y)\right\| \le \|x - y\|.$$

Suppose T_{f_n} has a fixed point $u_n \in K$. Then

$$\|x_{n+1} - u_n\| = \left\|\frac{1}{k_n}\left[T^n x_n + f_n\right] - u_n\right\| = \left\|T_{f_n}(x_n) - T_{f_n}(u_n)\right\| \le \|x_n - u_n\|,$$

T_{f_n} being nonexpansive. Since $\{\|x_n - u_n\|\}$ is non-increasing, hence $\{x_n\}$ is bounded. Conversely, suppose that $\{x_n\}$ is bounded. Let $d = \operatorname{diam}(\{x_n\})$ and

$$B_d[x] = \{y \in K : \|x - y\| \le d\}$$

for each $x \in K$. Set

$$C_n = \bigcap_{i \ge n} B_d[x_i] \subset K.$$

Hence C_n is a nonempty, convex set for each n $n \in \mathbb{N}$.

Now we claim that $T_{f_n}(C_n) \subset C_{n+1}$. Let $y \in B_d[x_n]$ which implies $\|y - x_n\| \le d$. Since T_{f_n} is nonexpansive, we get

$$\left\|T_{f_n}(y) - T_{f_n}(x_n)\right\| \le d$$

$$\left\|\frac{1}{k_n}\left[T^n(y)+f_n\right]-\frac{1}{k_n}\left[T^n(x_n)+f_n\right]\right\|\le d$$

or

$$\left\|\frac{1}{k_n}\left[T^n(y)+f_n\right]-x_{n+1}\right\|\le d$$

or

$$\frac{1}{k_n}\left[T^n y+f_n\right]\in B_d\left[x_{n+1}\right]$$

giving

$$T_{f_n}(y)\in B_d\left[x_{n+1}\right]$$

proving that $T_{f_n}(C_n)\subset C_{n+1}$.

Let $C=\overline{\bigcup_{n\in\mathbb{N}} C_n}$. Since C_n increases with n, C is a closed, convex and bounded subset of K. We can easily see that T_{f_n} maps C into C.

$$T_{f_n}(C)=T_{f_n}\left(\overline{\bigcup_{n\in\mathbb{N}} C_n}\right)\subseteq\overline{T_{f_n}\left(\bigcup_{n\in\mathbb{N}} C_n\right)}=\overline{\bigcup_{n\in\mathbb{N}} T_{f_n}(C_n)}\subseteq\overline{\bigcup_{n\in\mathbb{N}} C_{n+1}}=C.$$

Applying the Browder and Göhde's theorem to T_{f_n} and C we get a fixed point of T_{f_n} in C. Since $C\subset K$, we obtain a fixed point of T_{f_n} in K.

References

[1] R. P. Agarwal, D. O'Regan and D. R. Sahu, "*Fixed point theory for Lipschitzian-type mappings with applications, Topological fixed point theory and its Applications*," Springer, 6, (2009).

[2] K. Goebel and W. A. Kirk, *Topics in metric fixed point theory*, Cambridge Studies in Advanced Mathematics 28, Cambridge University Press, 1990.

[3] M. A. Khamsi and W. A. Kirk, *An introduction to metric spaces and fixed point theory*, Pure and Applied Mathematics, John Wiley & Sons Inc., 2001.