The Spectrum Of Wheel Graph Using Eigenvalues Circulant Matrix

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Abstract — *The purpose of this article is determining the spectrum of wheel graph. Some steps including drawing a wheel graph W_n, determining adjacency matrix of W_n and the eigenvalues of the circulant matrix of adjacency matrix are used. Furthermore, the spectrum of wheel graph is obtained based on the eigenvalues and its multiplicity. Spectrum of wheel graph W_n for W_4, W_6 and W_8 are presented.*

Keywords — ***wheel graph, spectrum graph, eigenvalues, circulant matrix***

I. INTRODUCTION

The theory of spectrum graph is a study of graph theory that studies the properties of graphs in relation to polynomial characteristics, eigenvalues and eigenvectors of adjacency matrices. Suppose graph G with ordo $n(n \geq 1)$ and size *m* and set point $V(G) = \{v_1, v_2, ..., v_n\}$. An adjacency matrix of graph *G*, denoted by $A(G)$ is a matrix $n \times n$ with elements of the rows *i* and column *j* valued 1 if v_i is directly connected to v_j, and valued 0 as v_i is not directly connected to v_j. In other words, the adjacency matrx can be written $A(G) = [a_{ij}], 1 \leq i, j \leq n$ [1].

The spectrum of graph is an ordered pair of adjacency matrices eigenvalues and their multiplicity. Suppose $\lambda_0, \lambda_1, ..., \lambda_s$ with $\lambda_0 > \lambda_1 > ... > \lambda_{s-1}$, and suppose $m(\lambda_0), m(\lambda_1), ..., m(\lambda_{s-1})$ is the multiplicity of each eigenvalue λ_i [2]. The ordo matrix $(2 \times n)$ that contains $\lambda_0, \lambda_1, ..., \lambda_s$ in the first row and $m(\lambda_0), m(\lambda_1), ..., m(\lambda_{s-1})$ in the second row is called spectrum of graph *G*, denoted by $Spec(G)$ [2]. From the literature review, spectrum of graph have been studied, among others, connectivity spectrum and Laplace spectrum on graphs G_l obtained from complete graphs K_l by adding isomorphic trees rooted for each point in K_l [3], spectrum connected to complete graphs (K_n), star graphs (S_n), complete biparty graph $(K_{m,n})$ [4], spectrum corona of graph *G* and K_1, cartesian multiplication graph *G* with K_2 and multiplication lexicographic graph *G* with K_2 [5].

II. LITERATURE REVIEW

Adjacency Matrix

Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Suppose that the vertices of *G* are listed arbitrarily as $v_1, v_2, ..., v_n$. The adjacency matrix *A* of *G*, with respect to this listing of the vertices, is the $n \times n$ zero-one matrix with 1 as its $(i, j)th$ entry when v_i and v_j are adjacent, and 0 as its $(i, j)th$ entry when they are not adjacent. In other words, if its adjacency matrix is $A = [a_{ij}]$, then [6]:

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge of } G \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Eigenvalues of Circulant Matrix

Suppose $[0, a_2, ..., a_n]$ is the first line of adjacency matrix. The eigenvalues of circulant matrix in graph *G* is:

$$\lambda_r = \sum_{j=2}^{n} a_j \omega^{(j-1)r} \tag{2}$$

for each $r = 0,1,2,...,n-1$ with $\omega = e^{\frac{2\pi i}{n}} = \cos\frac{2\pi i}{n} + i\sin\frac{2\pi i}{n}$ [2], [14].

If A is a matrix $n \times n$, then non-zero vector x on R^n is said that eigenvector of A if Ax is a scalar of multiple x, $Ax = \lambda x$ for any scalar λ. Scalar λ is called the eigenvalue of A and x is called the eigenvector of A associated with λ [8]. To obtain the eigenvalues of the matrix $n \times n$, rewrite $Ax = \lambda x$ as $Ax = I\lambda x$, equivalently $(I\lambda - A)x = 0$ [7], [8]. In order to be an eigenvalue must be non-zero solution from $(I\lambda - A)x = 0$ and have non-zero solution if $\det(I\lambda - A)x = 0$, in order to obtain the characteristic equation from matrix A, the scalars that satisfy this equation are eigenvalues [7].

Spectrum of Graph

If λ_0 is an eigenvalues of an $n \times n$ matrix A, then the dimension of the eigenspace corresponding to λ_0 is called the geometric multiplicity of λ_0, and the number of times that $\lambda - \lambda_0$ appears as a factor in the characteristic polynomial of A is called the algebraic multiplicity of λ_0 [8], [10] and [11].

Suppose $\lambda_0, \lambda_1, ..., \lambda_s$ with $\lambda_0 > \lambda_1 > ... > \lambda_{s-1}$ and suppose $m(\lambda_0), m(\lambda_1), ..., m(\lambda_{s-1})$ is the multiplicites of each eigenvalue λ_i. The ordo matrix $(2 \times n)$ that contains $\lambda_0, \lambda_1, ..., \lambda_s$ in first line and the second line is called spectrum graph G denoted by $Spec(G)$. The spectrum of graph G can be written [2], [15]:

$$Spec(G) = \begin{bmatrix} \lambda_0 & \lambda_1 & \cdots & \lambda_{s-1} \\ m(\lambda_0) & m(\lambda_1) & \cdots & m(\lambda_{s-1}) \end{bmatrix} \tag{3}$$

Wheel Graph W_n

Wheel graph W_n is a graph that contains one cycle that each point on cycle is directly connected to the center point [15], [16]. Wheel graph W_n is obtained by the summing operation of the cycle graph C_n with a complete graph K_1, so that [9]:

$$W_n = C_n + K_1, n > 2 \tag{4}$$

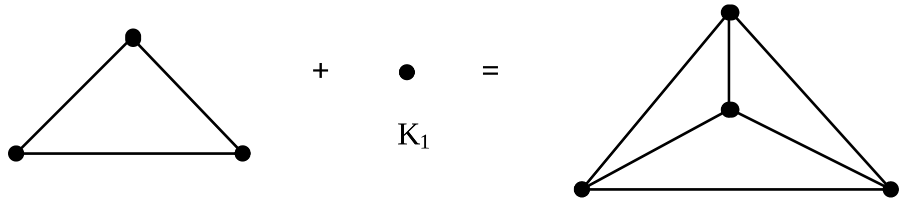


Fig. 1 Wheel graph W_4

III. METHDOS

The steps in determining spectrum matrix of wheel graph W_n using eigenvalues circulant matrix:

1) Drawing a wheel graph W_n

2) Looking for an adjacency matrix of wheel graph W_n

3) Looking for eigenvalues of wheel graph W_n with a circulant matrix

4) Determining a spectrum matrix of wheel graph W_n with a general equation of spectrum matrix.

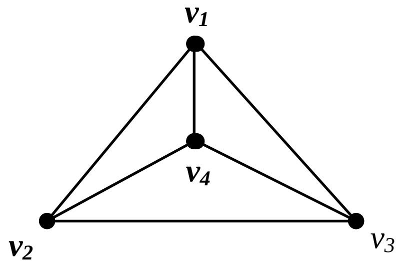
5) Obtain a spectrum matrix of wheel graph W_n

IV. RESULT AND DISCUSSION

Spectrum of Wheel Graph W_4

The steps are follows:

a. Draw a wheel graph W_4



b. Determine the adjacency matrix of wheel graph W_4

$$W_4 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

c. Determine eigenvalues of a circulant matrix

Eigenvalues of wheel graph W_4 with $r = 1,2,3,4$ and $n = 4$ can be obtained using equation (2).

For $r = 1$

$$\lambda_1 = \sum_{j=2}^{4} a_j \omega^{(j-1)r} = a_2\omega + a_3\omega^2 + a_4\omega^3 = (1)\omega + (1)\omega^2 + (1)\omega^3 = \omega^1 + \omega^2 + \omega^3$$

$$\omega^1 = e^{\frac{2\pi i}{4}} = \cos\frac{2\pi}{4} + i\sin\frac{2\pi}{4} = i,\ \omega^2 = e^{\frac{4\pi i}{4}} = \cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4} = -1, \omega^3 = e^{\frac{6\pi i}{4}} = \cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4} = -i$$

$$\lambda_1 = -1$$

For $r = 2$

$$\lambda_2 = \sum_{j=2}^{4} a_j \omega^{(j-1)2} = a_2\omega^2 + a_3\omega^4 + a_4\omega^6 = (1)\omega^2 + (1)\omega^4 + (1)\omega^6 = \omega^2 + \omega^4 + \omega^6$$

$$\omega^2 = e^{\frac{4\pi i}{4}} = \cos\frac{4\pi}{4} + i\sin\frac{4\pi}{4} = i,\ \omega^4 = e^{\frac{8\pi i}{4}} = \cos\frac{8\pi}{4} + i\sin\frac{8\pi}{4} = 1,\ \omega^6 = e^{\frac{12\pi i}{4}} = \cos\frac{12\pi}{4} + i\sin\frac{12\pi}{4} = -1$$

$$\lambda_2 = i$$

For $r = 3$

$$\lambda_3 = \sum_{j=2}^{4} a_j \omega^{(j-1)3} = a_2\omega^3 + a_3\omega^6 + a_4\omega^9 = (1)\omega^3 + (1)\omega^6 + (1)\omega^9 = \omega^3 + \omega^6 + \omega^9$$

$$\omega^3 = e^{\frac{6\pi i}{4}} = \cos\frac{6\pi}{4} + i\sin\frac{6\pi}{4} = -i,\ \omega^6 = e^{\frac{12\pi i}{4}} = \cos\frac{12\pi}{4} + i\sin\frac{12}{4} = -1,\ \omega^9 = e^{\frac{18\pi i}{4}} = \cos\frac{18\pi}{4} + i\sin\frac{18}{4} = i$$

$$\lambda_3 = -1$$

For $r = 4$

$$\lambda_4 = \sum_{j=2}^{4} a_j \omega^{(j-1)4} = a_2\omega^4 + a_3\omega^8 + a_4\omega^{12} = (1)\omega^4 + (1)\omega^8 + (1)\omega^{12} = \omega^4 + \omega^8 + \omega^{12}$$

$$\omega^4 = e^{\frac{8\pi i}{4}} = \cos\frac{8\pi}{4} + i\sin\frac{8\pi}{4} = 1,\ \omega^8 = e^{\frac{16\pi i}{4}} = \cos\frac{16\pi}{4} + i\sin\frac{16}{4} = 1,\ \omega^{12} = e^{\frac{24\pi i}{4}} = \cos\frac{24\pi}{4} + i\sin\frac{24}{4} = 1$$

$$\lambda_4 = 3$$

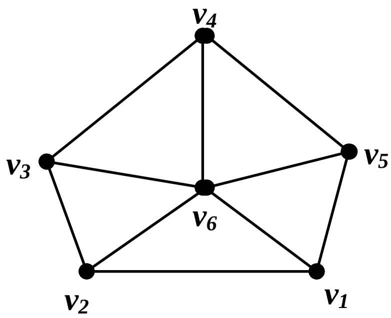
The spectrum of wheel graph W_4:

$$Spec(W_4) = \begin{bmatrix} \lambda_1, \lambda_3 & \lambda_2 & \lambda_4 \\ m(\lambda_1, \lambda_3) & m(\lambda_2) & m(\lambda_4) \end{bmatrix} = \begin{bmatrix} -1 & i & 3 \\ 2 & 1 & 1 \end{bmatrix}$$

Spectrum of Wheel Graph W_6

The steps are follows:

a. Draw a wheel graph W_6



b. Determine the adjacency matrix of wheel graph W_6

$$W_6 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 \end{bmatrix}$$

c. Determine eigenvalues of a circulant matrix

Eigenvalues of wheel graph W_6 with $r = 1,2,3,4,5,6$ and $n = 6$ can be obtained using equation (2).

For $r = 1$

$$\lambda_1 = \sum_{j=2}^{6} a_j \omega^{(j-1)r} = a_2\omega + a_3\omega^2 + a_4\omega^3 + a_5\omega^4 + a_6\omega^5 = (1)\omega + (0)\omega^2 + (0)\omega^3 + (1)\omega^4 + (1)\omega^5$$

$$\lambda_1 = \omega^1 + \omega^4 + \omega^5$$

$$\omega^1 = \cos\frac{2\pi}{6} + i\sin\frac{2\pi}{6} = \frac{1}{2} + \frac{1}{2}\sqrt{3}i,\ \ \omega^4 = \cos\frac{8\pi}{6} + i\sin\frac{8\pi}{6} = -\frac{1}{2} - \frac{1}{2}\sqrt{3}i,\ \ \omega^5 = \cos\frac{10\pi}{6} + i\sin\frac{10\pi}{6} = \frac{1}{2} - \frac{1}{2}\sqrt{3}i$$

$$\lambda_1 = \frac{1}{2} - \frac{1}{2}\sqrt{3}i$$

For $r = 2$

$$\lambda_2 = \sum_{j=2}^{6} a_j \omega^{(j-1)2} = a_2\omega^2 + a_3\omega^4 + a_4\omega^6 + a_5\omega^8 + a_6\omega^{10} = (1)\omega^2 + (0)\omega^4 + (0)\omega^6 + (1)\omega^8 + (1)\omega^{10}$$

$$\lambda_2 = \omega^2 + \omega^8 + \omega^{10}$$

$$\omega^2 = \cos\frac{4\pi}{6} + i\sin\frac{4\pi}{6} = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i,\ \ \omega^8 = \cos\frac{16\pi}{6} + i\sin\frac{16\pi}{6} = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i,\ \ \omega^{10} = \cos\frac{20\pi}{6} + i\sin\frac{20\pi}{6} = -\frac{1}{2} - \frac{1}{2}\sqrt{3}i$$

$$\lambda_2 = -\frac{3}{2} + \frac{1}{2}\sqrt{3}i$$

For $r = 3$

$$\lambda_3 = \sum_{j=2}^{6} a_j \omega^{(j-1)3} = a_2\omega^3 + a_3\omega^6 + a_4\omega^9 + a_5\omega^{12} + a_6\omega^{15} = (1)\omega^3 + (0)\omega^6 + (0)\omega^9 + (1)\omega^{12} + (1)\omega^{15}$$

$$\lambda_3 = \omega^3 + \omega^9 + \omega^{15}$$

$$\omega^3 = \cos\frac{6\pi}{6} + i\sin\frac{6\pi}{6} = -1,\ \ \omega^{12} = \cos\frac{24\pi}{6} + i\sin\frac{24\pi}{6} = 1,\ \ \omega^{15} = \cos\frac{30\pi}{6} + i\sin\frac{24\pi}{6} = -1$$

$$\lambda_3 = 1$$

For $r = 4$

$$\lambda_4 = \sum_{j=2}^{6} a_j \omega^{(j-1)4} = a_2\omega^4 + a_3\omega^8 + a_4\omega^{12} + a_5\omega^{16} + a_6\omega^{20} = (1)\omega^4 + (0)\omega^8 + (0)\omega^{12} + (1)\omega^{16} + (1)\omega^{20}$$

$$\lambda_4 = \omega^4 + \omega^{16} + \omega^{20}$$

$\omega^4 = \cos\frac{8\pi}{6} + i\sin\frac{8\pi}{6} = -\frac{1}{2} - \frac{1}{2}\sqrt{3}i,\ \omega^{16} = \cos\frac{32\pi}{6} + i\sin\frac{32\pi}{6} = -\frac{1}{2} - \frac{1}{2}\sqrt{3}i,\ \omega^{20} = \cos\frac{40\pi}{6} + i\sin\frac{40\pi}{6} = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$

$\lambda_4 = -\frac{3}{2} - \frac{1}{2}\sqrt{3}i$

For $r = 5$

$\lambda_5 = \sum_{j=2}^{6} a_j \omega^{(j-1)5} = a_2\omega^5 + a_3\omega^{10} + a_4\omega^{15} + a_5\omega^{20} + a_6\omega^{25} = (1)\omega^5 + (0)\omega^{10} + (0)\omega^{15} + (1)\omega^{20} + (1)\omega^{25}$

$\lambda_5 = \omega^5 + \omega^{20} + \omega^{25}$

$\omega^5 = \cos\frac{10\pi}{6} + i\sin\frac{10\pi}{6} = \frac{1}{2} - \frac{1}{2}\sqrt{3}i,\ \omega^{20} = \cos\frac{40\pi}{6} + i\sin\frac{40\pi}{6} = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i,\ \omega^{25} = \cos\frac{50\pi}{6} + i\sin\frac{50\pi}{6} = \frac{1}{2} + \frac{1}{2}\sqrt{3}i$

$\lambda_5 = \frac{1}{2} + \frac{1}{2}\sqrt{3}i$

For $r = 6$

$\lambda_6 = \sum_{j=2}^{6} a_j \omega^{(j-1)6} = a_2\omega^6 + a_3\omega^{12} + a_4\omega^{18} + a_5\omega^{24} + a_6\omega^{30} = (1)\omega^6 + (0)\omega^{12} + (0)\omega^{18} + (1)\omega^{24} + (1)\omega^{30}$

$\lambda_6 = \omega^6 + \omega^{24} + \omega^{30}$

$\omega^6 = \cos\frac{12\pi}{6} + i\sin\frac{12\pi}{6} = 1,\ \omega^{24} = \cos\frac{48\pi}{6} + i\sin\frac{48\pi}{6} = 1,\ \omega^{30} = \cos\frac{60\pi}{6} + i\sin\frac{60\pi}{6} = 1$

$\lambda_6 = 3$

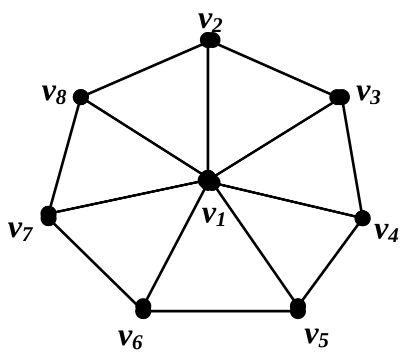
The spectrum of wheel graph W_6:

$$Spec(W_6) = \begin{bmatrix} \frac{1}{2} - \frac{1}{2}\sqrt{3}i & -\frac{3}{2} + \frac{1}{2}\sqrt{3}i & 1 & -\frac{3}{2} - \frac{1}{2}\sqrt{3}i & \frac{1}{2} + \frac{1}{2}\sqrt{3} & 3 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Spectrum of Wheel Graph W_8

The steps are follows:

a. Draw a wheel graph W_8



b. Determine the adjacency matrix of wheel graph W_8

$$W_8 = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

c. Determine eigenvalues with a circulant matrix

Eigenvalues of wheel graph W_8 with $r = 1,2,3,4,5,6,7,8$ and $n = 8$ can be obtained using equation (2).

$For\ \ r=1$

$\lambda_1=\sum_{j=2}^{8}a_j\omega^{(j-1)r}=a_2\omega^1+a_3\omega^2+a_4\omega^3+a_5\omega^4+a_6\omega^5+a_7\omega^6+a_8\omega^7$

$\lambda_1=(1)\omega+(1)\omega^2+(1)\omega^3+(1)\omega^4+(1)\omega^5+(1)\omega^6+(1)\omega^7$

$\lambda_1=\omega^1+\omega^2+\omega^3+\omega^4+\omega^5+\omega^6+\omega^7$

$\omega^1=\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}i,\ \omega^2=i,\ \omega^3=-\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}i,\ \omega^4=-1,\ \omega^5=-\frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{2}i,\ \omega^6=-i,\ \omega^7=\frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{2}i$

$\lambda_1=-1$

$For\ \ r=2$

$\lambda_2=\sum_{j=2}^{8}a_j\omega^{(j-1)r}=a_2\omega^2+a_3\omega^4+a_4\omega^6+a_5\omega^8+a_6\omega^{10}+a_7\omega^{12}+a_8\omega^{14}$

$\lambda_2=(1)\omega^2+(1)\omega^4+(1)\omega^6+(1)\omega^8+(1)\omega^{10}+(1)\omega^{12}+(1)\omega^{14}$

$\lambda_2=\omega^2+\omega^4+\omega^6+\omega^8+\omega^{10}+\omega^{12}+\omega^{14}$

$\omega^2=i,\ \omega^4=-1,\ \omega^6=-i,\ \omega^4=-1,\ \omega^6=-i,\ \omega^8=1,\ \omega^{10}=i,\ \omega^{12}=-1,\ \omega^{14}=-i$

$\lambda_2=-1$

$For\ \ r=3$

$\lambda_3=\sum_{j=2}^{8}a_j\omega^{(j-1)r}=a_2\omega^3+a_3\omega^6+a_4\omega^9+a_5\omega^{12}+a_6\omega^{15}+a_7\omega^{18}+a_8\omega^{21}$

$\lambda_3=(1)\omega^3+(1)\omega^6+(1)\omega^9+(1)\omega^{12}+(1)\omega^{15}+(1)\omega^{18}+(1)\omega^{21}$

$\lambda_3=\omega^3+\omega^6+\omega^9+\omega^{12}+\omega^{15}+\omega^{18}+\omega^{21}$

$\omega^3=-\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}i,\ \omega^6=-i,\ \omega^9=\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}i,\ \omega^{12}=-1,\ \omega^{15}=\frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{2}i,\ \omega^{18}=i,\ \omega^{21}=-\frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{2}i$

$\lambda_3=-1$

$For\ \ r=4$

$\lambda_4=\sum_{j=2}^{8}a_j\omega^{(j-1)r}=a_2\omega^4+a_3\omega^8+a_4\omega^{12}+a_5\omega^{16}+a_6\omega^{20}+a_7\omega^{24}+a_8\omega^{28}$

$\lambda_4=(1)\omega^4+(1)\omega^8+(1)\omega^{12}+(1)\omega^{16}+(1)\omega^{20}+(1)\omega^{24}+(1)\omega^{28}$

$\lambda_4=\omega^4+\omega^8+\omega^{12}+\omega^{16}+\omega^{20}+\omega^{24}+\omega^{28}$

$\omega^4=-1,\ \omega^8=1,\ \omega^{12}=-1,\ \omega^{16}=1,\ \omega^{20}=-1,\ \omega^{24}=1,\ \omega^{28}=-1$

$\lambda_4=-1$

$For\ \ r=5$

$\lambda_5=\sum_{j=2}^{8}a_j\omega^{(j-1)r}=a_2\omega^5+a_3\omega^{10}+a_4\omega^{15}+a_5\omega^{20}+a_6\omega^{25}+a_7\omega^{30}+a_8\omega^{35}$

$\lambda_5=(1)\omega^5+(1)\omega^{10}+(1)\omega^{15}+(1)\omega^{20}+(1)\omega^{25}+(1)\omega^{30}+(1)\omega^{35}$

$\lambda_5=\omega^5+\omega^{10}+\omega^{15}+\omega^{20}+\omega^{25}+\omega^{30}+\omega^{35}$

$\omega^5=-\frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{2}i,\ \omega^{10}=i,\ \omega^{15}=\frac{1}{2}\sqrt{2}-\frac{1}{2}\sqrt{2}i,\ \omega^{20}=-1,\ \omega^{25}=\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}i,\ \omega^{30}=-i,\ \omega^{35}=-\frac{1}{2}\sqrt{2}+\frac{1}{2}\sqrt{2}i$

$\lambda_5=-1$

$For\ \ r=6$

$\lambda_6=\sum_{j=2}^{8}a_j\omega^{(j-1)r}=a_2\omega^6+a_3\omega^{12}+a_4\omega^{18}+a_5\omega^{24}+a_6\omega^{30}+a_7\omega^{36}+a_8\omega^{42}$

$\lambda_6=(1)\omega^6+(1)\omega^{12}+(1)\omega^{18}+(1)\omega^{24}+(1)\omega^{30}+(1)\omega^{36}+(1)\omega^{42}$

$\lambda_6=\omega^6+\omega^{12}+\omega^{18}+\omega^{24}+\omega^{30}+\omega^{36}+\omega^{42}$

$\omega^6=-i,\ \omega^{12}=-1,\ \omega^{18}=i,\ \omega^{24}=1,\ \omega^{30}=-i,\ \omega^{36}=-1,\ \omega^{42}=i$

$\lambda_6=-1$

For $r = 7$

$$\lambda_7 = \sum_{j=2}^{8} a_j \omega^{(j-1)r} = a_2\omega^7 + a_3\omega^{14} + a_4\omega^{21} + a_5\omega^{28} + a_6\omega^{35} + a_7\omega^{42} + a_8\omega^{49}$$

$$\lambda_7 = (1)\omega^7 + (1)\omega^{14} + (1)\omega^{21} + (1)\omega^{28} + (1)\omega^{35} + (1)\omega^{42} + (1)\omega^{49}$$

$$\lambda_7 = \omega^7 + \omega^{14} + \omega^{21} + \omega^{28} + \omega^{35} + \omega^{42} + \omega^{49}$$

$$\omega^7 = \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}i,\ \omega^{14} = -i,\ \omega^{21} = -\frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}i,\ \omega^{28} = -1,\ \omega^{35} = -\frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i,\ \omega^{42} = i,\ \omega^{49} = \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2}i$$

$$\lambda_7 = -1$$

For $r = 8$

$$\lambda_8 = \sum_{j=2}^{8} a_j \omega^{(j-1)r} = a_2\omega^8 + a_3\omega^{16} + a_4\omega^{24} + a_5\omega^{32} + a_6\omega^{40} + a_7\omega^{48} + a_8\omega^{56}$$

$$\lambda_8 = (1)\omega^8 + (1)\omega^{16} + (1)\omega^{24} + (1)\omega^{32} + (1)\omega^{40} + (1)\omega^{48} + (1)\omega^{56}$$

$$\lambda_8 = \omega^8 + \omega^{16} + \omega^{24} + \omega^{32} + \omega^{40} + \omega^{48} + \omega^{56}$$

$$\omega^8 = 1,\ \omega^{16} = 1,\ \omega^{24} = 1,\ \omega^{32} = 1,\ \omega^{40} = 1,\ \omega^{48} = 1,\ \omega^{56} = 1$$

$$\lambda_8 = 7$$

The spectrum of wheel graph W_8 :

$$Spec(W_8) = \begin{bmatrix} \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 & \lambda_8 \\ m(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7) & m(\lambda_8) \end{bmatrix} = \begin{bmatrix} -1 & 7 \\ 7 & 1 \end{bmatrix}$$

V. CONCLUSIONS

Analitically it shown how to find the spectrum of wheel graph using eigenvalues of circulant matrix. It is more difficult to find the spectrum of the graph where the number of vertex of the wheel is large. Therefore it is necessary to build an algorithm that can be implemented to find the spectrum of the wheel graph..

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