CERTAIN SUBCLASSES OF MEROMORPHIC FUNCTIONS ASSOCIATED WITH GENERALIZED POLYLOGARITHM FUNCTIONS

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Abstract. In the current work, we define a subclass $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$ of meromorphic univalent functions with generalized polylogarithm functions. We obtain coefficient estimates, extreme points, growth and distortion bounds, radii of meromorphically starlikeness and meromorphically convexity, partial sums, and integral means inequality for the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$. Further, it is shown that the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$ is closed under convex linear combination.

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1. Introduction

Let Σ denote the class of functions of the form

$$f(z) = z^{-1} + \sum_{n=1}^{\infty} a_n z^n \tag{1.1}$$

which are analytic in the punctured open unit disk

$$\mathbb{U}^* := \{z : z \in \mathbb{C}, 0 < |z| < 1\} =: \mathbb{U} \setminus \{0\}.$$

Let $\Sigma_{\mathcal{S}}$, $\Sigma^*(\alpha)$ and $\Sigma_K(\alpha)$, $(0 \le \alpha < 1)$ denote the subclasses of Σ that are meromorphically univalent functions , meromorphically starlike functions of order α and meromophically convex functions of order α respectively. Analytically, $f \in \Sigma^*(\alpha)$ if and only if, f is of the form (1.1) and satisfies

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha, \quad z \in \mathbb{U}^*,$$

similarly, $f \in \Sigma_K(\alpha)$, if and only if, f is of the form (1.1) and satisfies

$$-\Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\alpha,\quad z\in\mathbb{U}^*,$$

and similar other classes of meromorphically univalent functions have been extensively studied by Altintas et al. [2], Aouf [3, 4], Ganigi and Uralegaddi [9], Mogra et al. [17], Uralegadi [25], Uralegaddi and Ganigi [26] and Uralegaddi and Somanatha [27] and others [1, 6, 7, 11, 14, 15, 16, 18, 19, 21, 22, 23, 24, 29].

Let Σ_P be the class of functions of the form

$$f(z)=z^{-1}+\sum_{n=1}^{\infty}a_nz^n,\ \ a_n\geq 0, \tag{1.2}$$

that are analytic and univalent in $\mathbb{U}^*$. For functions $f\in\Sigma$ given by (1.1) and $g\in\Sigma$ given by

$$g(z)=z^{-1}+\sum_{n=1}^{\infty}b_nz^n, \tag{1.3}$$

we define the Hadamard product (or convolution) of $f(z)$ and $g(z)$ by

$$(f*g)(z):=z^{-1}+\sum_{n=1}^{\infty}a_nb_nz^n=:(g*f)(z). \tag{1.4}$$

For $\varkappa\in\mathbb{N}$, the set of natural numbers with $\varkappa\geq 2$, an absolutely convergent series defined as

$$Li_{\varkappa}(z)=\sum_{n=1}\frac{z^n}{n^{\varkappa}}$$

is known as the polylogarithm. This class of functions was invented by Leibnitz [12]. For more works on polylogarithm and meromorphic function (see [1, 24, 29]).

We consider a linear operator

$$\Omega_{\varkappa}f(z)\ :\ \Sigma\to\Sigma$$

which is defined by the following Hadamard product (or convolution) :

$$\Omega_{\varkappa}f(z)=\phi_{\varkappa}(z)*f(z)=\frac{1}{z}+\sum_{n=0}^{\infty}\frac{1}{(n+2)^{\varkappa}}a_nz^n,$$

where

$$\phi_{\varkappa}(z)=z^{-2}Li_{\varkappa}(z).$$

Next, we define the linear operator

$$\sigma_{\varkappa}:\Sigma\to\Sigma$$

as follows:

$$\sigma_{\varkappa}f(z)=\left(\Omega_{\varkappa}f(z)-\frac{1}{2^{\varkappa}}a_0\right)=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{1}{(n+2)^{\varkappa}}a_nz^n.$$

For function f in the class Σ_P, we define a linear operator $\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z)$ as follows

$$\begin{aligned}
\mathscr{D}^{0}_{\mu,\,\varkappa}f(z) &= \sigma_{\varkappa}f(z)\\
\mathscr{D}^{1}_{\mu,\,\varkappa}f(z) &= (1-\mu)\sigma_{\varkappa}f(z)+\mu z(\sigma_{\varkappa}f(z))' = \mathscr{D}_{\mu,\,\varkappa}f(z)\\
\mathscr{D}^{2}_{\mu,\,\varkappa}f(z) &= \mathscr{D}_{\mu,\,\varkappa}\left(\mathscr{D}_{\mu,\,\varkappa}f(z)\right)\\
\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z) &= \mathscr{D}_{\mu,\,\varkappa}\left(\mathscr{D}^{\kappa-1}_{\mu,\,\varkappa}f(z)\right)\\
&= \frac{1}{z}+\sum_{n=1}^{\infty}\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}a_n z^n, \qquad \kappa\in\mathbb{N}.
\end{aligned}$$

Now, in the following definition, we define a subclass $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$ for functions in the class Σ_P.

Definition 1.1. For $0\le\alpha<1$, $0\le\mu,\ \lambda\le 1$, $\kappa,\varkappa\in\mathbb{N}$ and $\varkappa\ge 2$, let $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$ denote a subclass of Σ consisting of functions of the form (1.1) satisfying the condition that

$$\Re\left(\frac{z(\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z))'}{(\lambda-1)\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z)+\lambda z(\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z))'}\right)>\alpha, \quad z\in\mathbb{U}^*. \tag{1.5}$$

2. Coefficient Estimates

Our first theorem gives a necessary and sufficient condition for a function $f\in\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$.

Theorem 2.1. *Let $f\in\Sigma_P$ be given by (1.2). Then $f\in\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$ if and only if*

$$\sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(1+n)]\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ a_n\le(1-\alpha). \tag{2.1}$$

Proof. If $f\in\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$, then

$$\Re\left(\frac{z(\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z))'}{(\lambda-1)\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z)+\lambda z(\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z))'}\right)=\Re\left(\frac{-1+\sum\limits_{n=0}^{\infty}n\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ a_n z^{n+1}}{-1+\sum\limits_{n=0}^{\infty}(\lambda-1+n\lambda)\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ a_n z^{n+1}}\right).$$

By letting $z\to 1^{-1}$, we have

$$\left(\frac{-1+\sum\limits_{n=1}^{\infty}n\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ a_n z^{n+1}}{-1+\sum\limits_{n=1}^{\infty}(\lambda-1+n\lambda)\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ a_n z^{n+1}}\right)>\alpha.$$

This shows that (2.1) holds.

Conversely assume that (2.1) holds. It is sufficient to show that

$$\left|\frac{\omega-1}{\omega+1-2\alpha}\right|<1,$$

where

$$\omega=\frac{z(\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z))'}{(\lambda-1)\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z)+\lambda z(\mathscr{D}^{\kappa}_{\mu,\,\varkappa}f(z))'}.$$

Using (2.1) that

$$\left|\frac{z(\mathscr{D}^{\kappa}_{\mu,\varkappa}f(z))' - \left[(\lambda-1)\mathscr{D}^{\kappa}_{\mu,\varkappa}f(z) + \lambda z(\mathscr{D}^{\kappa}_{\mu,\varkappa}f(z))'\right]}{z(\mathscr{D}^{\kappa}_{\mu,\varkappa}f(z))' + (1-2\alpha)\left[(\lambda-1)\mathscr{D}^{\kappa}_{\mu,\varkappa}f(z) + \lambda z(\mathscr{D}^{\kappa}_{\mu,\varkappa}f(z))'\right]}\right|$$

$$= \left|\frac{\sum_{n=1}^{\infty}(1-\lambda)\,(n+1)\;\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\;a_n z^{n+1}}{-2(1-\alpha)+\sum_{n=1}^{\infty}\left[\left[(1+(1-2\alpha)\lambda)\right]n+(1-2\alpha)(\lambda-1)\right]\;\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\;a_n z^{n+1}}\right|$$

$$\leq \frac{\sum_{n=1}^{\infty}(1-\lambda)\,(n+1)\;\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\;a_n}{2(1-\alpha)-\sum_{n=1}^{\infty}\left[\left[(1+(1-2\alpha)\lambda)\right]n+(1-2\alpha)(\lambda-1)\right]\;\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\;a_n}$$

$$\leq 1.$$

Thus we have $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$. □

For the choice of $\lambda = 0$, we get the following.

Corollary 2.2. *Let $f(z) \in \Sigma_P$ be given by (1.2). Then $f \in \Sigma^*_P(\alpha)$ if and only if*

$$\sum_{n=1}^{\infty}(n+\alpha)\;\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\;a_n \leq 1-\alpha.$$

Our next result gives the coefficient estimates for functions in $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$.

Theorem 2.3. *If $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$, then*

$$a_n \leq \frac{1-\alpha}{\{n+\alpha-\alpha\lambda(1+n)\}\;\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}, \qquad n = 1,2,3,\ldots.$$

The result is sharp for the functions $F_n(z)$ given by

$$F_n(z) = \frac{1}{z} + \frac{1-\alpha}{\{n+\alpha-\alpha\lambda(1+n)\}\;\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}z^n, \qquad n = 1,2,3,\ldots.$$

Proof. If $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$, then we have, for each n,

$$\{n+\alpha-\alpha\lambda(1+n)\}\;\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\;a_n \leq \sum_{n=1}^{\infty}\{n+\alpha-\alpha\lambda(1+n)\}\;\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\;a_n \leq 1-\alpha.$$

Therefore we have

$$a_n \leq \frac{1-\alpha}{\{n+\alpha-\alpha\lambda(1+n)\}\;\frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}.$$

Since

$$F_n(z) = \frac{1}{z} + \frac{1-\alpha}{\{n+\alpha-\alpha\lambda(1+n)\}\ \dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}} z^n$$

satisfies the conditions of Theorem 2.1, $F_n(z) \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$ and the equality is attained for this function. □

For $\lambda = 0$, we have the following corollary.

Corollary 2.4. *If $f \in \Sigma_P^*(\alpha)$, then*

$$a_n \leq \frac{1-\alpha}{(n+\alpha)\ \dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}, \qquad n = 1, 2, 3, \ldots.$$

Theorem 2.5. *If $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$, then*

$$\frac{1}{r} - \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\ \dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} r \leq |f(z)| \leq \frac{1}{r} + \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\ \dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} r \quad (|z| = r).$$

The result is sharp for

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{\{1+\alpha-2\alpha\lambda\}\ \dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} z. \tag{2.2}$$

Proof. Since $f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n$, we have

$$|f(z)| \leq \frac{1}{r} + \sum_{n=1}^{\infty} a_n r^n \leq \frac{1}{r} + r \sum_{n=1}^{\infty} a_n.$$

Since,

$$\sum_{n=1}^{\infty} a_n \leq \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\ \dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}.$$

Using this, we have

$$|f(z)| \leq \frac{1}{r} + \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\ \dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} r.$$

Similarly

$$|f(z)| \geq \frac{1}{r} - \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\ \dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} r.$$

The result is sharp for $f(z) = \frac{1}{z} + \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\ \frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} z.$ □

Similarly we have the following:

Theorem 2.6. *If $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$, then If $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$, then*

$$\frac{1}{r^2} - \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\ \dfrac{[1+2\mu]^\kappa}{3^\varkappa}} \leq |f'(z)| \leq \frac{1}{r^2} + \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\ \dfrac{[1+2\mu]^\kappa}{3^\varkappa}} \quad (|z|=r).$$

The result is sharp for the function given by (2.2).

3. Closure Theorems

Let the functions $F_k(z)$ be given by

$$F_k(z) = \frac{1}{z} + \sum_{n=1}^{\infty} f_{n,k} z^n, \quad k=1,2,...,m. \tag{3.1}$$

We shall prove the following closure theorems for the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$.

Theorem 3.1. *Let the function $F_k(z)$ defined by (3.1) be in the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$ for every $k=1,2,...,m$. Then the function $f(z)$ defined by*

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \qquad (a_n \geq 0)$$

belongs to the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$, where $a_n = \frac{1}{m}\sum_{k=1}^{m} f_{n,k}$ $(n=1,2,..)$

Proof. Since $F_n(z) \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$, it follows from Theorem 2.1 that

$$\sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(1+n)]\ \frac{[1+\mu(n+1)]^\kappa}{(n+2)^\varkappa}\ f_{n,k} \leq 1-\alpha \tag{3.2}$$

for every $k=1,2,..,m$. Hence

$$\begin{aligned}
&\sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(1+n)]\ \frac{[1+\mu(n+1)]^\kappa}{(n+2)^\varkappa}\ a_n \\
&= \sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(1+n)]\ \frac{[1+\mu(n+1)]^\kappa}{(n+2)^\varkappa}\ \left(\frac{1}{m}\sum_{k=1}^{m} f_{n,k}\right) \\
&= \frac{1}{m}\sum_{k=1}^{m}\left(\sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(1+n)]\ \frac{[1+\mu(n+1)]^\kappa}{(n+2)^\varkappa}\ f_{n,k}\right) \\
&\leq 1-\alpha.
\end{aligned}$$

By Theorem 2.1, it follows that $f(z) \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$. □

Theorem 3.2. *The class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$ is closed under convex linear combination.*

Proof. Let the function $F_k(z)$ given by (3.1) be in the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$. Then it is enough to show that the function

$$H(z) = \lambda F_1(z) + (1-\lambda)F_2(z) \ \ (0 \leq \lambda \leq 1)$$

is also in the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$. Since for $0 \leq \lambda \leq 1$,

$$H(z) = \frac{1}{z} + \sum_{n=1}^{\infty}[\lambda f_{n,1} + (1-\lambda)f_{n,2}]z^n,$$

we observe that

$$\begin{aligned}
&\sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(1+n)]\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ [\lambda f_{n,1} + (1-\lambda)f_{n,2}] \\
&= \lambda\sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(1+n)]\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ f_{n,1} \\
&\quad +(1-\lambda)\sum_{n=1}^{\infty}[n+\alpha-\alpha\lambda(1+n)]\ \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}\ f_{n,2} \\
&\leq 1-\alpha.
\end{aligned}$$

By Theorem 2.1, we have $H(z) \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$. □

Theorem 3.3. *Let* $F_0(z) = \frac{1}{z}$ *and*

$$F_n(z) = \frac{1}{z} + \frac{1-\alpha}{[n+\alpha-\alpha\lambda(1+n)]\ \dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}z^n, \qquad n \in \mathbb{N}.$$

Then $f(z) \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$ *if and only if* $f(z)$ *can be expressed in the form*

$$f(z) = \sum_{n=0}^{\infty}\lambda_n F_n(z),$$

where $\lambda_n \geq 0$ *and* $\sum_{n=0}^{\infty}\lambda_n = 1$.

Proof. Let

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty}\lambda_n F_n(z) \\
&= \frac{1}{z} + \sum_{n=1}^{\infty}\frac{\lambda_n(1-\alpha)}{[n+\alpha-\alpha\lambda(1+n)]\ \dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}z^n.
\end{aligned}$$

Then

$$\begin{aligned}
&\sum_{n=1}^{\infty}\lambda_n\frac{1-\alpha}{[n+\alpha-\alpha\lambda(1+n)]\ \dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}\frac{[n+\alpha-\alpha\lambda(1+n)]\ \dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}{(1-\alpha)} \\
&= \sum_{n=1}^{\infty}\lambda_n = 1-\lambda_0 \leq 1.
\end{aligned}$$

By Theorem 2.1, we have $f(z) \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$.

Conversely, let $f(z) \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$. From Theorem 2.3, we have

$$a_n \leq \frac{1-\alpha}{[n+\alpha-\alpha\lambda(1+n)]\ \dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}} \quad \text{for} \quad n=1,\ 2,\ \cdots$$

we may take

$$\lambda_n = \frac{[n+\alpha-\alpha\lambda(1+n)]\ \dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}{1-\alpha} a_n \quad \text{for} \quad n=1,\ 2\ \cdots$$

and

$$\lambda_0 = 1 - \sum_{n=1}^{\infty} \lambda_n.$$

Then

$$f(z) = \sum_{n=0}^{\infty} \lambda_n F_n(z).$$

□

4. Neighborhoods for the class $\mathscr{G}_P^{(\gamma)}(\alpha,\ \lambda,\ \mu,\ \varkappa)$

In this section, we determine the neighborhood for the class $\mathscr{G}_P^{(\gamma)}(\alpha,\ \lambda,\ \mu,\ \varkappa)$, which we define as follows:

Definition 4.1. A function $f \in \Sigma_p$ is said to be in the class $\mathscr{G}_P^{(\gamma)}(\alpha,\ \lambda,\ \mu,\ \varkappa)$ if there exists a function $g \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$ such that

$$\left| \frac{f(z)}{g(z)} - 1 \right| < 1-\gamma, \qquad (z \in \Delta, 0 \leq \gamma < 1). \tag{4.1}$$

Following the earlier works on neighborhoods of analytic functions by Goodman [10] and Ruscheweyh [20], we define the δ-neighborhood of a function $f \in \Sigma_p$ by

$$N_\delta(f) = \left\{ g \in \Sigma_p\ :\ g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta \right\}. \tag{4.2}$$

Particulary for the identity function $e(z) = \frac{1}{z}$, we have

$$N_\delta(e) = \left\{ g \in \Sigma_p : g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n|b_n| \leq \delta \right\}. \tag{4.3}$$

Theorem 4.2. *If*

$$\delta = \frac{1-\alpha}{(1+(1-2\lambda)\alpha)\ \dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} \tag{4.4}$$

then $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa) \subset N_\delta(e)$.

Proof. For $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$, Theorem 2.1 immediately yields

$$(1+\alpha-2\alpha\lambda)\ \frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}\ \sum_{n=1}^{\infty} a_n \leq 1-\alpha$$

so that

$$\sum_{n=1}^{\infty} a_n \leq \frac{(1-\alpha)}{(1+\alpha-2\alpha\lambda)\ \frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} \tag{4.5}$$

On the other hand, from (2.1) and (4.5) we see that

$$\begin{aligned}
\frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}\ \sum_{n=2}^{\infty} n a_n &\leq (1-\alpha)-(1-2\lambda)\alpha\ \frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}\ \sum_{n=1}^{\infty} a_n \\
&\leq (1-\alpha)-(1-2\lambda)\alpha\ \frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}\ \frac{(1-\alpha)}{(1+\alpha-2\alpha\lambda)\ \frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} \\
&\leq \frac{1-\alpha}{(1+(1-2\lambda)\alpha)},
\end{aligned}$$

that is

$$\sum_{n=2}^{\infty} n a_n \leq \frac{1-\alpha}{((1+\alpha-2\alpha\lambda)\ \frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}} = \delta \tag{4.6}$$

which in view of the definition (4.3) proves Theorem 4.2. □

Theorem 4.3. *If* $g \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$ *and*

$$\gamma = 1 - \frac{\delta\ \ (1+\alpha-2\alpha\lambda)\ \frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}{(1+\alpha-2\alpha\lambda)\ \frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}\ -(1-\alpha)}, \tag{4.7}$$

then

$$N_{\delta}(g) \subset M_p^{(\gamma)}(\alpha,\lambda).$$

Proof. Let $f \in N_{\delta}(g)$. Then we find from (4.2) that

$$\sum_{n=1}^{\infty} n|a_n - b_n| \leq \delta \tag{4.8}$$

which implies the coefficient inequality

$$\sum_{n=1}^{\infty} |a_n - b_n| \leq \delta, \qquad (n \in \mathbb{N}). \tag{4.9}$$

Since $g \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$, we have [cf. equation (2.1)]

$$\sum_{n=1}^{\infty} b_n \leq \frac{1-\alpha}{(1+\alpha-2\alpha\lambda)\ \frac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}, \tag{4.10}$$

so that

$$\begin{aligned}\left|\frac{f(z)}{g(z)}-1\right| &< \frac{\sum_{n=1}^{\infty}|a_n-b_n|}{1-\sum_{n=1}^{\infty}b_n}\\ &= \frac{\delta\ (1+\alpha-2\alpha\lambda)\ \dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}}{(1+\alpha-2\alpha\lambda)\ \dfrac{[1+2\mu]^{\kappa}}{3^{\varkappa}}\ -(1-\alpha)}\\ &= 1-\gamma\end{aligned}$$

provided γ is given by (4.7). Hence, by definition $f\in\mathscr{G}_P^{(\gamma)}(\alpha,\ \lambda,\ \mu,\ \varkappa)$ for γ given by (4.7) which completes the proof. □

5. Radii of meromorphically starlikeness and meromorphically convexity

The radii of starlikeness and convexity for the class are given by the following theorems for the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$.

Theorem 5.1. *Let the function $f\in\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$. Then f is meromorphically starlike of order ρ, $(0\le\rho<1)$, in $|z|<r_1(\alpha,\lambda,\rho)$, where*

$$r_1(\alpha,\lambda,\rho)=\inf_n\left[\frac{(1-\rho)(1-\alpha)}{(n+2-\rho)[n+\alpha-\alpha\lambda(1+n)]\ \dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}\right]^{\frac{1}{n+1}},\qquad n\geqq 1, \tag{5.1}$$

Proof. It is sufficient to show that

$$\left|-\frac{zf'(z)}{f(z)}-1\right|\le 1-\rho \text{ or equivalently} \tag{5.2}$$

$$\left|\frac{zf'(z)}{f(z)}+1\right|=\left|\frac{\sum_{n=1}^{\infty}(n+1)a_nz^n}{\frac{1}{z}-\sum_{n=1}^{\infty}a_nz^n}\right|\le 1-\rho \text{ or}$$

$$\sum_{n=1}^{\infty}\left(\frac{n+2-\rho}{1-\rho}\right)a_n|z|^{n+1}\le 1,$$

for $0\le\rho<1$, and $|z|<r_1(\alpha,\lambda,\rho)$. By Theorem 2.1, (5.2) will be true if

$$\left(\frac{n+2-\rho}{1-\rho}\right)|z|^{n+1}\le\frac{1-\alpha}{[n+\alpha-\alpha\lambda(1+n)]\ \dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}$$

or, if

$$|z| \leq \left[\frac{(1-\rho)(1-\alpha)}{(n+2-\rho)[n+\alpha-\alpha\lambda(1+n)] \, \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}} \right]^{\frac{1}{n+1}}, \quad n \geq 1. \tag{5.3}$$

This completes the proof of Theorem 5.1. ☐

Theorem 5.2. *Let the function $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$. Then f is meromorphically convex of order ρ, $(0 \leq \rho < 1)$, in $|z| < r_2(\alpha, \lambda, \rho)$, where*

$$r_2(\alpha,\lambda,\rho) = \inf_n \left[\frac{(1-\rho)(1-\alpha)}{n\,(n+2-\rho)[n+\alpha-\alpha\lambda(1+n)] \, \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}} \right]^{\frac{1}{n+1}}, \qquad n \geqq 1, \tag{5.4}$$

Proof. It is sufficient to show that

$$\left| -1 - \frac{zf''(z)}{f'(z)} - 1 \right| \leq 1-\rho \text{ or equivalently} \tag{5.5}$$

$$\left| \frac{zf''(z)}{f'(z)} + 2 \right| = \left| \frac{\sum_{n=1}^{\infty} n(n+1)a_n z^{n-1}}{-\frac{1}{z^2} - \sum_{n=1}^{\infty} n a_n z^{n-1}} \right| \leq 1-\rho \text{ or}$$

$$\sum_{n=1}^{\infty} \left(\frac{n(n+2-\rho)}{1-\rho} \right) a_n |z|^{n+1} \leq 1,$$

for $0 \leq \rho < 1$, and $|z| < r_2(\alpha,\lambda,\rho)$. By Theorem 2.1, (5.5) will be true if

$$\left(\frac{n(n+2-\rho)}{1-\rho} \right) |z|^{n+1} \leq \frac{(1-\alpha)}{[n+\alpha-\alpha\lambda(1+n)] \, \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}}$$

or, if

$$|z| \leq \left[\frac{(1-\rho)(1-\alpha)}{n(n+2-\rho)[n+\alpha-\alpha\lambda(1+n)] \, \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}}} \right]^{\frac{1}{n+1}}, \quad n \geq 1. \tag{5.6}$$

This completes the proof of Theorem 5.2. ☐

6. Integral Operators

In this section, we consider integral transforms of functions in the class $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$.

Theorem 6.1. *Let the function $f(z)$ given by (1.2) be in $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$. Then the integral operator*

$$F(z) = \nu \int_0^1 u^\nu f(uz)du \qquad (0 < u \leq 1, 0 < \nu < \infty)$$

is in $\mathscr{G}_P(\delta,\ \lambda,\ \mu,\ \varkappa)$, where

$$\delta = \frac{(\nu+2)\{1+\alpha-2\alpha\lambda\} - \nu(1-\alpha)}{\nu(1-\alpha)\{1-2\lambda\} + (1+\alpha)\{1-2\lambda\}(\nu+2)}.$$

The result is sharp for the function $f(z) = \frac{1}{z} + \frac{1-\alpha}{\{1+\alpha-2\alpha\lambda\}}z$.

Proof. Let $f(z) \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$. Then

$$\begin{aligned} F(z) &= \nu \int_0^1 u^\nu f(uz)du \\ &= \nu \int_0^1 \left(\frac{u^{\nu-1}}{z} + \sum_{n=1}^{\infty} f_n u^{n+\nu} z^n \right) du \\ &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\nu}{\nu+n+1} f_n z^n. \end{aligned}$$

It is sufficient to show that

$$\sum_{n=1}^{\infty} \frac{\nu\{n+\delta-\delta\lambda(1+n)\}\ \frac{[1+\mu(n+1)]^\kappa}{(n+2)^\varkappa}}{(\nu+n+1)(1-\delta)} a_n \leq 1. \tag{6.1}$$

Since $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$, we have

$$\sum_{n=1}^{\infty} \frac{\{n+\alpha-\alpha\lambda(1+n)\}\ \frac{[1+\mu(n+1)]^\kappa}{(n+2)^\varkappa}}{(1-\alpha)} a_n \leq 1.$$

Note that (6.1) is satisfied if

$$\frac{\nu\{n+\delta-\delta\lambda(1+n)\}}{(\nu+n+1)(1-\delta)} \leq \frac{\{n+\alpha-\alpha\lambda(1+n)\}}{(1-\alpha)}.$$

Rewriting the inequality, we have

$$\nu\{n+\delta-\delta\lambda(1+n)\}(1-\alpha) \leq (\nu+n+1)(1-\delta)\{n+\alpha-\alpha\lambda(1+n)\}.$$

Solving for δ, we have

$$\delta \leq \frac{(\nu+n+1)\{n+\alpha-\alpha\lambda(1+n)\} - \nu n(1-\alpha)}{\nu(1-\alpha)\{1-\lambda(1+n)\} + \{(n+\alpha-\alpha\lambda(1+n)\}(\nu+n+1)} = F(n).$$

A simple computation will show that $F(n)$ is increasing and $F(n) \geq F(1)$. Using this, the results follows. □

Corollary 6.2. *Let the function $f(z)$ defined by (1) be in $\Sigma_p^*(\alpha)$. Then the integral operator*

$$F(z) = \nu \int_0^1 u^\nu f(uz)du \qquad (0 < u \le 1, 0 < \nu < \infty)$$

is in $\Sigma_p^*(\delta)$, where $\delta = \frac{1+\alpha+\nu\alpha}{1+\alpha+\nu}$. The result is sharp for the function

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{1+\alpha} z.$$

Also we have the following:

Theorem 6.3. *Let $f(z)$, given by (1.2), be in $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$,*

$$F(z) = \frac{1}{\nu}[(\nu+1)f(z) + zf'(z)] = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{\nu+n+1}{\nu} f_n z^n, \qquad \nu > 0. \tag{6.2}$$

Then $F(z)$ is in $\mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$ for $|z| \le r(\alpha, \lambda, \beta)$ where

$$r(\alpha,\lambda,\beta) = \inf_n \left(\frac{\nu(1-\beta)\{n+\alpha-\alpha\lambda(1+n)\}}{(1-\alpha)(\nu+n+1)\{n+\beta-\beta\lambda(1+n)\}} \right)^{1/(n+1)}, \quad n = 1, 2, 3, \ldots.$$

The result is sharp for the function $f_n(z) = \frac{1}{z} + \frac{1-\alpha}{\{n+\alpha-\alpha\lambda(1+n)\}} z^n, \quad n = 1, 2, 3, \ldots.$

Proof. Let $w = \frac{z(\mathscr{D}^{\kappa}_{\mu,\,\varkappa} f(z))'}{(\lambda-1)\mathscr{D}^{\kappa}_{\mu,\,\varkappa} f(z) + \lambda z(\mathscr{D}^{\kappa}_{\mu,\,\varkappa} f(z))'}$. Then it is sufficient to show that

$$\left| \frac{w-1}{w+1-2\beta} \right| < 1.$$

A computation shows that this is satisfied if

$$\sum_{n=1}^{\infty} \frac{\{n+\beta-\beta\lambda(1+n)\} \dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}} (\nu+n+1)}{(1-\beta)\nu} a_n |z|^{n+1} \le 1. \tag{6.3}$$

Since $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$, by Theorem 2.1, we have

$$\sum_{n=1}^{\infty} \{n+\alpha-\alpha\lambda(1+n)\} \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}} a_n \le 1-\alpha.$$

The equation (6.3) is satisfied if

$$\frac{\{n+\beta-\beta\lambda(1+n)\}(\nu+n+1)}{(1-\beta)\nu} \frac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}} a_n |z|^{n+1} \le \frac{\{n+\alpha-\alpha\lambda(1+n)\} \dfrac{[1+\mu(n+1)]^{\kappa}}{(n+2)^{\varkappa}} a_n}{1-\alpha}.$$

Solving for $|z|$, we get the result. □

For the choice of $\kappa = \lambda = 0$, we have the following result of Uralegaddi and Ganigi [26].

Corollary 6.4. *Let the function* $f(z)$ *defined by* (1.2) *be in* $\Sigma_p^*(\alpha)$ *and* $F(z)$ *given by (6.2). Then* $F(z)$ *is in* $\Sigma_p^*(\alpha)$ *for* $|z| \leq r(\alpha, \beta)$ *where*

$$r(\alpha, \beta) = \inf_n \left(\frac{\nu(1-\beta)(n+\alpha)}{(1-\alpha)(\nu+n+1)(n+\beta)} \right)^{1/(n+1)}, \quad n = 1, 2, 3, \ldots.$$

The result is sharp for the function $f_n(z) = \frac{1}{z} + \frac{1-\alpha}{n+\alpha} z^n, \quad n = 1, 2, 3, \ldots.$

7. Partial Sums

In this section, we will investigate the ratio of a function of the form (1.2) to its sequence of partial sums

$$f_1(z) = \frac{1}{z} \text{ and } f_k(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n \tag{7.1}$$

when the coefficients are sufficiently small to satisfy the condition analogous to

$$\sum_{n=1}^{\infty} [n + \alpha - \alpha\lambda(1+n)] \, \frac{[1+\mu(n+1)]^\kappa}{(n+2)^\varkappa} \, a_n \leq 1 - \alpha.$$

For the sake of brevity we rewrite it as

$$\sum_{n=1}^{\infty} d_n |a_n| \leq 1 - \alpha, \tag{7.2}$$

where

$$d_n := [n + \alpha - \alpha\lambda(1+n)] \, \frac{[1+\mu(n+1)]^\kappa}{(n+2)^\varkappa} \tag{7.3}$$

More precisely we will determine sharp lower bounds for $\Re\{f(z)/f_k(z)\}$ and $\Re\{f_k(z)/f(z)\}$. In this connection we make use of the well known result that

$$\Re \left\{ \frac{1+w(z)}{1-w(z)} \right\} > 0, \quad (z \in \mathbb{U}^*)$$

if and only if $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$ satisfies the inequality $|\omega(z)| \leq |z|$.

Unless and otherwise stated, we will assume that f is of the form (1.2) and its sequence of partial sums is denoted by

$$f_k(z) = \frac{1}{z} + \sum_{n=1}^{k} a_n z^n.$$

Theorem 7.1. *Let* $f(z) \in \mathscr{G}_P(\alpha, \ \lambda, \ \mu, \ \varkappa)$ *given by* (1.2) *satisfy the condition,* (2.1)

$$\Re \left\{ \frac{f(z)}{f_k(z)} \right\} \geq \frac{d_{k+1}(\lambda, \alpha) - 1 + \alpha}{d_{k+1}(\lambda, \alpha)}, \qquad (z \in \mathbb{U}^*) \tag{7.4}$$

where $d_{n+1} \geq 1 - \alpha$ and

$$d_k \geq \begin{cases} 1-\alpha, & \text{if } k = 1, 2, 3, \ldots, n \\ d_{n+1}, & \text{if } k = n+1, n+2, \ldots. \end{cases} \tag{7.5}$$

The result (7.4) is sharp with the function given by

$$f(z) = \frac{1}{z} + \frac{1-\alpha}{d_{k+1}(\lambda,\alpha)} z^{k+1}. \tag{7.6}$$

Proof. Define the function $w(z)$ by

$$\frac{1+w(z)}{1-w(z)} = \frac{d_{k+1}(\lambda,\alpha)}{1-\alpha}\left[\frac{f(z)}{f_k(z)} - \frac{d_{k+1}(\lambda,\alpha)-1+\alpha}{d_{k+1}(\lambda,\alpha)}\right] \tag{7.7}$$

$$= \frac{1+\sum\limits_{n=1}^{k} a_n z^{n+1} + \left(\frac{d_{k+1}(\lambda,\alpha)}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty} a_n z^{n+1}}{1+\sum\limits_{n=1}^{k} a_n z^{n+1}}. \tag{7.8}$$

It suffices to show that $|w(z)| \leq 1$. Now, from (7.8) we can write

$$w(z) = \frac{\left(\frac{d_{k+1}(\lambda,\alpha)}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty} a_n z^{n+1}}{2+2\sum\limits_{n=1}^{k} a_n z^{n+1} + \left(\frac{d_{k+1}(\lambda,\alpha)}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty} a_n z^{n+1}}.$$

Hence we obtain

$$|w(z)| \leq \frac{\left(\frac{d_{k+1}(\lambda,\alpha)}{1-\alpha}\right)\sum\limits_{k=n+1}^{\infty} |a_n|}{2-2\sum\limits_{n=1}^{k} |a_n| - \left(\frac{d_{k+1}(\lambda,\alpha)}{1-\alpha}\right)\sum\limits_{n=k+1}^{\infty} |a_n|}.$$

Now $|w(z)| \leq 1$ if

$$2\left(\frac{d_{k+1}(\lambda,\alpha)}{1-\alpha}\right)\sum_{n=k+1}^{\infty} |a_n| \leq 2 - 2\sum_{n=1}^{k} |a_n|$$

or equivalently,

$$\sum_{n=1}^{k} |a_n| + \frac{d_{k+1}(\lambda,\alpha)}{1-\alpha}\sum_{n=k+1}^{\infty} |a_n| \leq 1.$$

From the condition (1.2), it is sufficient to show that

$$\sum_{n=1}^{k} |a_n| + \frac{d_{k+1}(\lambda,\alpha)}{1-\alpha}\sum_{n=k+1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty}\frac{d_n(\lambda,\alpha)}{1-\alpha}|a_n|$$

which is equivalent to

$$\sum_{n=1}^{k}\left(\frac{d_n(\lambda,\alpha)-1+\alpha}{1-\alpha}\right)|a_n| + \sum_{n=k+1}^{\infty}\left(\frac{d_n(\lambda,\alpha)-d_{k+1}(\lambda,\alpha)}{1-\alpha}\right)|a_n| \geq 0. \tag{7.9}$$

To see that the function given by (7.6) gives the sharp result, we observe that for $z = re^{i\pi/k}$

$$\begin{aligned}\frac{f(z)}{f_k(z)} &= 1 + \frac{1-\alpha}{d_{k+1}(\lambda,\alpha)} z^n \to 1 - \frac{1-\alpha}{d_{k+1}(\lambda,\alpha)} \\ &= \frac{d_{k+1}(\lambda,\alpha) - 1 + \alpha}{d_{k+1}(\lambda,\alpha)} \quad \text{when } r \to 1^-\end{aligned}$$

which shows the bound (7.4) is the best possible for each $k \in \mathbb{N}$. □

We next determine bounds for $f_k(z)/f(z)$.

Theorem 7.2. *If $f \in \mathscr{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$ of the form (1.2) satisfies the condition (2.1), then*

$$\Re\left\{\frac{f_k(z)}{f(z)}\right\} \geq \frac{d_{k+1}(\lambda,\alpha)}{d_{k+1}(\lambda,\alpha) + 1 - \alpha}, \qquad (z \in \mathbb{U}^*), \tag{7.10}$$

where $d_{n+1} \geq 1-\alpha$ *and*

$$d_k \geq \begin{cases} 1-\alpha, & if\ k = 1,2,3,\ldots,n \\ d_{k+1}, & if\ k = n+1, n+2, \ldots. \end{cases} \tag{7.11}$$

The result (7.10) is sharp with the function given by (7.6).

Proof. We write

$$\frac{1+w(z)}{1-w(z)} = \frac{d_{k+1}(\lambda,\alpha)+1-\alpha}{1-\alpha}\left[\frac{f_k(z)}{f(z)} - \frac{d_{k+1}(\lambda,\alpha)}{d_{k+1}(\lambda,\alpha)+1-\alpha}\right]$$

$$= \frac{1 + \sum\limits_{n=1}^{k} a_n z^{n+1} - \left(\frac{d_{k+1}(\lambda,\alpha)}{1-\alpha}\right) \sum\limits_{n=k+1}^{\infty} a_n z^{n+1}}{1 + \sum\limits_{n=1}^{\infty} a_n z^{n+1}},$$

where

$$|w(z)| \leq \frac{\left(\frac{d_{k+1}(\lambda,\alpha)+1-\alpha}{1-\alpha}\right) \sum\limits_{n=k+1}^{\infty} |a_n|}{2 - 2\sum\limits_{n=1}^{k} |a_n| - \left(\frac{d_{k+1}(\lambda,\alpha)-1+\alpha}{1-\alpha}\right) \sum\limits_{n=k+1}^{\infty} |a_n|} \leq 1.$$

This last inequality is equivalent to

$$\sum_{n=1}^{k} |a_n| + \frac{d_{k+1}(\lambda,\alpha)}{1-\alpha} \sum_{n=k+1}^{\infty} |a_n| \leq 1.$$

Make use of (2.1) to get (7.9). Finally, equality holds in (7.10) for the extremal function $f(z)$ given by (7.6). □

Theorem 7.3. *Let $k \in N$ and let the sequence $\{d_n\}$, defined by (7.3), satisfying the inequalities (2.1). If a function f belong to the class $M_{l,m}^{\alpha_1}(\lambda,\alpha)$, then*

$$\Re\left\{\frac{f'(z)}{f_k(z)}\right\} \geq 1 - \frac{(1-\alpha)(k+1)}{d_{k+1}(\lambda,\alpha)}, \qquad (z \in \mathbb{U}^*), \tag{7.12}$$

and

$$\Re\left\{\frac{f_k'(z)}{f(z)}\right\} \geq \frac{d_{k+1}(\lambda,\alpha)}{(1-\alpha)(k+1)+d_{k+1}(\lambda,\alpha)}, \qquad (z\in\mathbb{U}^*). \tag{7.13}$$

The bounds are sharp, with the function given by (7.6).

Proof. The proof is analogous to that of Theorems 7.1 and 7.2, so, we omit the details. □

8. Integral Means Inequalities

For analytic functions f and g in $\mathbb{U}^*$, f is said to be subordinate to g if there exists an analytic function w such that

$$w(0)=0, \qquad |w(z)|<1 \quad \text{and} \qquad f(z)=g(w(z)), \qquad z\in\mathbb{U}^*.$$

This subordination will be denoted here by

$$f\prec g, \qquad z\in\mathbb{U}^*$$

or, conventionally, by

$$f(z)\prec g(z), \qquad z\in\mathbb{U}^*.$$

In particular, when g is univalent in $\mathbb{U}^*$,

$$f\prec g \qquad (z\in\mathbb{U}^*) \;\Leftrightarrow\; f(0)=g(0) \quad \text{and} \quad f(\mathbb{U}^*)\subset g(\mathbb{U}^*).$$

Lemma 8.1. [13] *If the functions f and g are analytic in $\mathbb{U}$ with $g\prec f$ then*

$$\int_0^{2\pi}\left|g(re^{i\theta})\right|^\eta d\theta \leq \int_0^{2\pi}\left|f(re^{i\theta})\right|^\eta d\theta, \quad \eta>0,\ \ z=re^{i\theta} \text{ and } 0<r<1. \tag{8.1}$$

Applying Theorem 2.1 with the extremal function and Lemma 8.1, we prove the following theorem.

Theorem 8.2. *Let $\eta>0$. If $f\in\mathcal{G}_P(\alpha,\ \lambda,\ \mu,\ \varkappa)$, and f_n is defined by*

$$f_n(z)=\frac{1}{z}+\frac{(1-\alpha)}{[n+\alpha-\alpha\lambda(1+n)]}\ \frac{(n+2)^c}{[1+\mu(n+1)]^\kappa}\ z^n, \qquad n=1,\ 2,\ \cdots.$$

If there exists an analytic function $w(z)$ such that

$$[w(z)]^{n+1}=\frac{[n+\alpha-\alpha\lambda(1+n)]}{(1-\alpha)}\ \frac{[1+\mu(n+1)]^\kappa}{(n+2)^c}\sum_{n=1}^{\infty}a_n z^{n+1},$$

then for $z=re^{i\theta}$ and $0<r<1$, we have

$$\int_0^{2\pi}\left|f(re^{i\theta})\right|^\eta d\theta \leq \int_0^{2\pi}\left|f_n(re^{i\theta})\right|^\eta d\theta, \qquad \eta>0. \tag{8.2}$$

Proof. Let f of the form 1.2 and $f_n(z) = \frac{1}{z} + \frac{(1-\alpha)}{[n+\alpha-\alpha\lambda(1+n)]}\ \frac{(n+2)^c}{[1+\mu(n+1)]^\kappa}\ z^n$, then we must show that

$$\int_0^{2\pi} \left|1+\sum_{n=1}^{\infty} a_n z^{n+1}\right|^\eta d\theta \le \int_0^{2\pi} \left|1+\frac{(1-\alpha)}{[n+\alpha-\alpha\lambda(1+n)]}\ \frac{(n+2)^c}{[1+\mu(n+1)]^\kappa}\ z^{n+1}\right|^\eta d\theta.$$

By Lemma 8.1, it suffices to show that

$$1+\sum_{n=2}^{\infty}|a_n|z^{n+1} \prec 1+\frac{(1-\alpha)}{[n+\alpha-\alpha\lambda(1+n)]}\ \frac{(n+2)^c}{[1+\mu(n+1)]^\kappa}\ z^{n+1}.$$

If we set w such that

$$[w(z)]^{n+1} = \frac{[n+\alpha-\alpha\lambda(1+n)]}{(1-\alpha)}\ \frac{[1+\mu(n+1)]^\kappa}{(n+2)^c}\sum_{n=1}^{\infty} a_n z^{n+1},$$

we get

$$1+\sum_{n=2}^{\infty}|a_n|z^{n+1} = 1+\frac{(1-\alpha)}{[n+\alpha-\alpha\lambda(1+n)]}\ \frac{(n+2)^c}{[1+\mu(n+1)]^\kappa}\ [w(z)]^{n+1}. \tag{8.3}$$

Clearly, $w(0)=0$, then from Theorem 2.1 we can write

$$\begin{aligned}|w(z)|^{n+1} &= \left|\frac{[n+\alpha-\alpha\lambda(1+n)]}{(1-\alpha)}\ \frac{[1+\mu(n+1)]^\kappa}{(n+2)^c}\sum_{n=1}^{\infty} a_n z^{n+1}\right| \\ &\le \sum_{n=2}^{\infty}\frac{[n+\alpha-\alpha\lambda(1+n)]}{(1-\alpha)}\ \frac{[1+\mu(n+1)]^\kappa}{(n+2)^c}|a_n||z|^{n+1} \\ &< |z| < 1.\end{aligned}$$

This completes the proof of the Thoerem 8.2. □

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