Comparison Between Some Analysis Solutions for Solving Fredholm Integral Equation of Second Kind

ABDEL RADI ABDEL RAHMAN ABDEL GADIR [1]
Department of Mathematics, Faculty of Education
Omdurman Islamic University, Omdurman, Sudan
SAMIA ABDALLAH YAGOUB IBRAHIM [2]
Researcher of Ph.D. in Mathematics in Omdurman
Islamic University

Abstract:

Analysis solution is developed for solving Fredholm integral equation of the second kind. We reviewed different analysis solution for solving linear and nonlinear Fredholm integral equation of the second kind the aim is to classify the chosen method and evaluate their accuracy and efficiency we stress the importance of this comparison to drive the study on the analysis solution of Fredholm integral equation, we follow analysis method mathematical and we found the Adomian decomposition method gives the more accurate solutions.

Key Words: Fredholm integral equation second kind, Adomian decomposition method modified decomposition method, successive approximation method.

1. Introduction

In engineering and scientific applications of a natural phenomenon we usually find ourselves in front of one equation of three, differential equation, integral equation, or integro-differential equation. In fact the conversion of the scientific phenomenon to integral equation is the easy way to obtain analysis solutions.

Integral equations are encountered in a variety of applications: in potential theory, geophysics, electricity and magnetism, radiation, and control systems. However, the general form of Fredholm integral equation is defined as follows:

$$u(x)f(x) = f(x) + \lambda \int_a^b k(x,t)f(t)\,dt \tag{1}$$

where a and b are both constants. $f(x), u(x)$ and $k(x,t)$ are known functions while $f(x)$ is unknown function. λ (nonzero parameter) is called eigenvalue of integral equation. The function $k(x,t)$ is known as kernel of the integral equation.

Fredholm Integral Equation of the First Kind:

The integral equation is of the form (by setting $u(x) = 0$ in (1))

$$f(x) + \lambda \int_a^b k(x,t)f(t)\,dt \tag{2}$$

Equation (2) is known as Fredholm integral equation of first kind.

Fredholm Integral Equation of Second Kind:

The integral equation is of form (by setting $u(x) = 1$ in (1))

$$f(x) = f(x) + \int_a^b k(x,t)f(t)\,dt \tag{3}$$

Equation (3) is known as Fredholm integral equation of second kind.

System of Linear Fredholm Integral Equations:

The general form of system of linear Fredholm integral equations of second kind is defined as follows:

$$\sum_{j=1}^{n} u_{i,j} f_j(x) = f_i(x) + \sum_{j=1}^{n} \int_a^b k_{i,j}(x,t) f_j(t)\,dt, \;\; i = 1,2,\ldots,n \tag{4}$$

where $f_i(x)$ and $k_{i,j}(x,t)$ are known functions and $f_i(x)$ are unknown functions for $i,j = 1,2,\ldots,n$.

System of Nonlinear Fredholm Integral Equations:

System of nonlinear Fredholm integral equations of second kind is defined as follows:

$$\sum_{j=1}^{n} u_{i,j} f_j(x) = f_i(x) + \sum_{j=1}^{n} \int_a^b k_{i,j}(x,t) F_{i,j}(t, f_j(t))\,dt, \;\; i = 1,2,\ldots,n \tag{5}$$

where $f_i(x)$ and $k_{i,j}(x,t)$ are known functions and $f_j(x)$ are unknown functions for $i,j = 1,2,\ldots,n$.

2. Some Analysis Solution for solving Fredholm Integral Equations of the Second Kind

In this part we will present some analysis solutions for Fredholm integral equation of the second kind.

2.1 The Adomian Decomposition Method (ADM)

The (ADM) was introduced and developed by George Adomian [2], [13], [14], it consists of decomposing the unknown function $u(x)$ into a sum of an infinite number of components defined by decomposition series

$$u(x) = \sum_{n=0}^{\infty} u_n(x), \;\; n \geq 0 \tag{6}$$

The determination of these components can be achieved in an easy way through a recurrence relation that usually involves simple integrals that can be easily evaluated.
To establish the recurrence relation, we substitute equation (6) into the Fredholm integral equation of the second kind which takes the form

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)\,dt \tag{7}$$

that leads to

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \lambda \int_a^b k(x,t) \left(\sum_{n=0}^{\infty} u_n(t) \right) dt \tag{8}$$

where

$$u_0(x) = f(x), \quad u_{n+1}(x) = \lambda \textstyle\int_a^b k(x,t)u_n(t)\,dt, \quad n \geq 0 \tag{9}$$

As a result the solution of Fredholm integral equation is readily obtained in a series form by using the series assumption in (6), the more components we use, the higher accuracy we obtained, see references [11], [10] and [1].

Example 2.1. *Solve the following Fredholm integral equation*

$$u(x) = e^x - x + x \int_0^1 tu(t)\,dt \tag{10}$$

Solution:

The Adomian decomposition method assumes that the solution $u(x)$ has a series form given in (6). Substituting the decomposition series (6) into both sides of (10) gives

$$\sum_{n=0}^{\infty} u_n(x) = e^x - x + x \int_0^1 t \sum_{n=0}^{\infty} u_n(t)\, dt \tag{11}$$

or equivalently

$$u_0(x) + u_1(x) + u_2(x) + \cdots = e^x - x + x \int_0^1 t\, [u_0(t) + u_1(x) + u_2(x) + \cdots]\, dt \tag{12}$$

We identify the zeroth component by all terms that are not included under the integral sign. Therefore, we obtain the following recurrence relation

$$u_0(x)e^x - x, \quad u_{k+1}(x) = x \int_0^1 t u_k(t)\, dt, \quad k \geq 0 \tag{13}$$

Consequently, we obtain

$$\begin{aligned} u_0(x) &= e^x - x \\ u_1(x) &= x \int_0^1 t u_0(t)\, dt = x \int_0^1 t(e^t - t)\, dt = \frac{2}{3}x \\ u_2(x) &= x \int_0^1 t u_1(t)\, dt = x \int_0^1 \frac{2}{3}t^2\, dt = \frac{2}{9}x \\ u_3(x) &= x \int_0^1 t u_2(t)\, dt = x \int_0^1 \frac{2}{9}t^2\, dt = \frac{2}{27}x \\ u_4(x) &= x \int_0^1 t u_3(t)\, dt = x \int_0^1 \frac{2}{27}t^2\, dt = \frac{2}{81}x \end{aligned} \tag{14}$$

and so on. Using (6) gives the series solution

$$u(x) = e^x - x + \frac{2}{3}x \left(1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots\right). \tag{15}$$

Notice that the infinite geometric series at the right side has $a_1 = 1$, and the ratio $r = \frac{1}{3}$. The sum of the infinite series is therefore given by

$$S = \frac{1}{1 - \frac{1}{3}} = \frac{3}{2} \tag{16}$$

The series solution (15) converges to the closed form solution

$$u(x) = e^x \tag{17}$$

obtained upon using (16) into (15).

2.2 The Modified Decomposition Method

As shown before, the Adomian Decomposition method provides the solution in an infinite series of components. The components , $u_i, i \geq 0$ are easily computed if the term $f(x)$ in the Fredholm integral equation consists of a polynomial. However, if the function $f(x)$ consists of a combination of two or more polynomials, trigonometric Functions, hyperbolic functions or others, the evaluation of the components , $u_i, i \geq 0$ equires cumbersome work.
A reliable modification of the Adomian Decomposition method was developed by WazWaz. The modified decomposition method will facilitate the computational process and further accelerate the convergence of the series solution, it depends on splitting the function $f(x)$ into two parts, and therefore it cannot be used if the function $f(x)$ consists of only one term. To give a clear description of the technique, we recall that the standard Adomian decomposition method admits the use of the recurrence relation

$$u_0(x) = f(x), \quad u_{i+1}(x) = \lambda \int_a^b k(x,t)u_i(t)\,dt, \quad i \geq 0 \tag{18}$$

where the solution $u(x)$ is expressed by an infinite sum of components defined by

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{19}$$

The function $f(x)$ can be set as the sum of two partial functions, namely $f_1(x)$ and $f_2(x)$. In other words, we can set

$$f(x) = f_1(x) + f_2(x) \tag{20}$$

In view of (20), we introduce a qualitative change in the formation of the recurrence relation (18). To minimize the size of calculations, we identify the zeroth component $u_0(x)$ by only one part of $f(x)$, namely $f_1(x)$ or $f_2(x)$, the other part of $f(x)$ can be added to the component $u_1(x)$ that exists in the standard recurrence relation (18), that leads to equations

$$\begin{aligned} u_0(x) &= f_1(x) \\ u_1(x) &= f_2(x) + \lambda \int_a^b k(x,t)u_0(t)\,dt \\ u_{i+1}(x) &= \lambda \int_a^b k(x,t)u_i(t)\,dt, \ i \geq 1 \end{aligned} \tag{21}$$

Although this variation in the formation of $u_0(x)$ and $u_1(x)$ is slight, it plays a major role in accelerating the convergence of the solution and in minimizing the size of computational work. See references [11], [10] and [8].

Example 2.2. *Solve the Fredholm integral equation by using the modified decomposition method*

$$u(x) = x + \sin^{-1}\frac{x+1}{2} + \frac{2-\pi}{2}x^2 + \frac{1}{2}x^2\int_{-1}^{1} u(t)\,dt. \tag{22}$$

Solution:

We decompose $f(x)$ given by

$$f(x) = x + \sin^{-1}\frac{x+1}{2} + \frac{2-\pi}{2}x^2 \tag{23}$$

into two parts given by

$$f_1(x) = x + \sin^{-1}\tfrac{x+1}{2}, \quad f_2(x) = \tfrac{2-\pi}{2}x^2 \tag{24}$$

We next use the modified recurrence formula (21) to obtain

$$\begin{aligned} u_0(x) &= x + \sin^{-1}\frac{x+1}{2} \\ u_1(x) &= \frac{2-\pi}{2}x^2 + \frac{1}{2}x^2\int_{-1}^{1} u_0(t)\,dt = 0 \\ u_{i+1} &= -\int_{-1}^{1} u_i(t)\,dt = 0,\ i \geq 1. \end{aligned} \tag{25}$$

It is obvious that each component of $u_j, j \geq 1$ is zero. The exact solution is therefore given by

$$u(x) = x + \sin^{-1}\frac{x+1}{2} \tag{26}$$

2.3 The Noise Terms Phenomenon

It was shown that a proper selection of $f_1(x)$ and $f_2(x)$ is essential to use the modified decomposition method. However, the noise terms phenomenon, demonstrated a fast convergence of the solution. we present here the main steps for using this effect concept. The noise terms are the identical terms with opposite signs that may appear between components $u_0(x)$ and $u_1(x)$. Other noise terms may appear between other components. By canceling the noise terms between $u_0(x)$ and $u_1(x)$, even though $u_1(x)$ contains further terms, the remaining non-canceled terms of $u_0(x)$ may give the exact solution of the integral equation. The appearance of the noise terms between $u_0(x)$ and $u_1(x)$ is not always sufficient to obtain the exact solution by canceling these noise terms. Therefore, it is necessary to show that the non-canceled terms of $u_0(x)$ satisfy the given integral equation. It was formally proved in [12] that a necessary condition for the appearance of the noise terms is required. The conclusion made in [12] is that the zeroth component $u_0(x)$ must contain the exact solution $u(x)$ among other terms. The phenomenon of the useful noise terms will be explained by the following illustrative example:

Example 2.3. *Solve the Fredholm integral equation by using the noise terms phenomenon:*

$$u(x) = \sin x + \cos x - \frac{\pi}{2}x + \int_0^{\frac{\pi}{2}} xtu(t)\,dt \tag{27}$$

Solution:

The standard Adomian method gives the recurrence relation

$$\begin{aligned} u_0(x) &= \sin x + \cos x - \frac{\pi}{2}x \\ u_{i+1}(x) &= \int_0^{\frac{\pi}{2}} xtu_i(t)\,dt,\ \ i \geq 0 \end{aligned} \tag{28}$$

This gives

$$\begin{aligned} u_0(x) &= \sin x + \cos x - \frac{\pi}{2}x \\ u_1(x) &= \int_0^{\frac{\pi}{2}} xtu_0(t)\,dt = \frac{\pi}{2}x - \frac{\pi^4}{48}x \end{aligned} \tag{29}$$

The noise terms $\mp\frac{\pi}{2}x$ appear in $u_0(x)$ and $u_1(x)$. Canceling this term from the zeroth component $u_0(x)$ gives the exact solution

$$u(x) = \sin x + \cos x \tag{30}$$

that justifies the integral equation. It is to be noted that the other terms of $u_1(x)$ vanish in the limit with other terms of the other components.

2.4 The Variational Iteration Method

The method provides rapidly convergent successive approximations to the exact solution if such a closed form solution exists [7], [9], [14]. We will apply the variational iteration method to handle Fredholm integral equation. The method works effectively if the kernel $K(x,t)$ is separable and can be written in the form $K(x,t) = g(x)h(t)$. This means that we should differentiate both sides of the Fredholm integral equation to convert it to an identical Fredholm integro-differential equation. It is important to note that integro-differential equation needs an initial condition that should be defined. In view of this fact, we will study only the cases where $g(x) = x^n,\ n \geq 1$. The standard Fredholm integral equation is of the form

$$u(x) = f(x) + \int_a^b k(x,t)u(t)\,dt \tag{31}$$

or equivalently

$$u(x) = f(x) + g(x)\int_a^b h(t)u(t)\,dt. \tag{32}$$

Recall that the integral at the right side represents a constant value. Differentiating both sides of (32) with respect to x gives

$$u'(x) = f'(x) + g'(x)\int_a^b h(t)u(t)\,dt. \tag{33}$$

The correction functional for the integro-differential equation (33) is

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\zeta)\left(u_n'(\zeta) - f'(\zeta) - g'(\zeta)\int_a^b h(r)\hat{u}_n(r)\,dr\right)d\zeta \tag{34}$$

As presented before, the variational iteration method is used by applying two essential steps. It is required first to determine the Lagrange multiplier $\lambda(\zeta)$ that can be identified optimally via integration by parts and by using a restricted variation. However,$\lambda(\zeta) = 1$ for first order integro-differential equations. Having determined λ, an iteration formula, without restricted variation, given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\zeta) - f'(\zeta) - g'(\zeta)\int_a^b h(r)\hat{u}_n(r)\,dr\right)d\zeta \tag{35}$$

is used for the determination of the successive approximations $u_{n+1}(x),\ n \geq 0$ of the solution $u(x)$. The zeroth approximation u0 can be any selective function. However, using the given initial value $u(0)$ is preferably used for the selective zeroth approximation u_0 as will be seen later. Consequently, the solution is given by

$$u(x) = \lim_{n\to\infty} u_n(x) \tag{36}$$

The variational iteration method will be illustrated by studying the following Fredholm integral equations.

Example 2.4. *Use the variational iteration method to solve the Fredholm integral equation*

$$u(x) = \sin x - x + x\int_0^{\frac{\pi}{2}} u(t)\,dt \tag{37}$$

Solution:

Differentiating both sides of this equation with respect to x gives

$$u'(x) = \cos x - 1 + \int_0^{\frac{\pi}{2}} u(t)\,dt \tag{38}$$

The correction functional for this equation is given by

$$u_{n+1}(x) = u_n(x) - \int_0^x \left(u_n'(\zeta) - \cos\zeta + 1 - \int_0^{\frac{\pi}{2}} u_n(r)\,dr\right) d\zeta \tag{39}$$

where we used $\lambda = 1$ for first-order integro-differential equations. The initial condition $u(0) = 0$ is obtained by substituting $x = 0$ into (37). We can use the initial condition to select $u_0(x) = u(0) = 0$. Using this selection into the correction functional gives the following successive approximations

$$\begin{aligned}
u_0(x) &= 0,\\
u_1(x) &= u_0(x) - \int_0^x \left(u_0'(\zeta) - \cos\zeta + 1 - \int_0^{\frac{\pi}{2}} u_0(r)\,dr\right) d\zeta\\
&= \sin x - x\\
u_2(x) &= u_1(x) - \int_0^x \left(u_1'(\zeta) - \cos\zeta + 1 - \int_0^{\frac{\pi}{2}} u_1(r)\,dr\right) d\zeta\\
&= (\sin x - x) + \left(x - \frac{\pi^2}{8}x\right)\\
u_3(x) &= u_2(x) - \int_0^x \left(u_2'(\zeta) - \cos\zeta + 1 - \int_0^{\frac{\pi}{2}} u_2(r)\,dr\right) d\zeta\\
&= (\sin x - x) + \left(x - \frac{\pi^2}{8}x\right) + \left(\frac{\pi^2}{8}x - \frac{\pi^4}{64}x\right)
\end{aligned} \tag{40}$$

and so on. Canceling the noise terms, the exact solution is given by

$$u(x) = \sin x \tag{41}$$

2.5 The Direct Computation Method

In this part, the direct computation method will be applied to solve the Fredholm integral equations [2], [3], [4]. The method approaches Fredholm integral equations in a direct manner and gives the solution in an exact form and not in a series form. It is important to point out that this method will be applied for the degenerate or separable kernels of the form

$$k(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t) \tag{42}$$

Examples of separable kernels are $x - t, xt, x^2 - t^2, xt^2 + x^2t$,etc.
The direct computation method can be applied as follows:

1. We first substitute (42) into the Fredholm integral equation the form

$$u(x) = f(x) + \int_a^b k(x,t)u(t)\,dt \tag{43}$$

2. This substitution gives

$$u(x) = f(x) + g_1(x)\int_a^b h_1(t)u(t)\,dt + g_2(x))\int_a^b h_2(t)u(t)\,dt + \cdots + g_n(x)\int_a^b h_n(t)u(t)\,dt \tag{44}$$

3. Each integral at the right side depends only on the variable t with constant limits of integration fort. This means that each integral is equivalent to a constant. Based on this, Equation (44) becomes

$$u(x) = f(x) + \lambda\alpha_1 g_1(x) + \lambda\alpha_2 g_2(x) + \cdots + \lambda\alpha_n g_n(x) \tag{45}$$

where

$$\alpha_i = \int_a^b h_i(t)u(t)\,dt \; 1 \le i \le n \tag{46}$$

4. Substituting (45) into (46) gives a system of n algebraic equations that can be solved to determine the constants $1 \leq i \leq n$. Using the obtained numerical values of α_i into (45), the solution $u(x)$ of the Fredholm integral equation (43) is readily obtained.

Example 2.5. *Solve the linear Fredholm integral equation*

$$u(x) = e^x - \frac{e}{2} + \frac{1}{2} + \frac{1}{2}\int_0^1 u(t)\,dt$$

Solution:

Let us set

$$\alpha = \int_0^1 u(t)\,dt$$

. Then we have

$$u(x) = e^x - \frac{e}{2} + \frac{1}{2} + \frac{\alpha}{2}$$

Replacing the value of $u(x)$ in the above integral yields

$$\begin{aligned}\alpha &= \int_0^1 \left(e^t - \frac{e}{2} + \frac{1}{2} + \frac{\alpha}{2}\right)dt \\ &= (e-1) + \left(\frac{1}{2} - \frac{e}{2}\right) + \frac{\alpha}{2}\end{aligned}$$

and this reduces to $\frac{\alpha}{2} = \frac{e}{2} - \frac{1}{2}$. Therefore, the solution is $u(x) = e^x$. This solution can be verified easily.

2.6 The Successive Approximation Method

The successive approximation method can be used for solving integral equations by finding successive approximations to the solution [5], [6]. It starts with an initial guess as $u_0(x)$ called the zeroth approximation, which can be any real valued function that will be used in a recurrence relation to determine the other approximations.

Let Fredholm integral equation of the second kind of the form

$$u(x) = F(x) + \int_a^b k(x,t)u(t)\,dt \tag{47}$$

Example 2.6. *Solve the Fredholm integral equation*

$$u(x) = 1 + \int_0^1 xu(t)\,dt \tag{48}$$

by using the successive approximation method.

Solution:

Let us consider the zeroth approximation is $u_0(x) = 1$, and then the first approximation can be computed as

$$\begin{aligned} u_1(x) &= 1 + \int_0^1 x u_0(t)\, dt \\ &= 1 + \int_0^1 x\, dt \\ &= 1 + x \end{aligned}$$

Proceeding in this manner, we find

$$\begin{aligned} u_2(x) &= 1 + \int_0^1 x u_1(t)\, dt \\ &= 1 + \int_0^1 x(1+t)\, dt \\ &= 1 + x\left(1 + \frac{1}{2}\right) \end{aligned}$$

Similarly, the third approximation is

$$\begin{aligned} u_3(x) &= 1 + x\int_0^1 \left(1 + \frac{3}{2}\right)\, dt \\ &= 1 + x\left(1 + \frac{1}{2} + \frac{1}{4}\right) \end{aligned}$$

Thus, we get

$$u_n(x) = 1 + x\left\{1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots + \frac{1}{2^{n-1}}\right\}$$

and hence

$$\begin{aligned} u(x) &= \lim_{n\to\infty} u_n(x) \\ &= 1 + \lim_{n\to\infty} x \sum_{k=0}^{n} \frac{1}{2^k} \\ &= 1 + x\left(1 - \frac{1}{2}\right)^{-1} \\ &= 1 + 2x \end{aligned}$$

This is the desired solution.

2.7 The Series Solution Method

A real function $u(x)$ is called analytic if it has derivatives of all orders such that the Taylor series at any point b in its domain

$$u(x) = \sum_{n=0}^{k} \frac{u^n(b)}{n!}(x-b)^n \tag{49}$$

converges t of(x) in a neighborhood of b. For simplicity, the generic form of Taylor series at $x = 0$ can be written as

$$u(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{50}$$

The series solution method that stems mainly from the Taylor series for analytic functions will be used for solving Fredholm integral equations [2], [4]. We will assume that the solution $u(x)$ of the Fredholm integral equations

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)\,dt \tag{51}$$

is analytic, and therefore possesses a Taylor series of the form given in (50), where the coefficients a_n will be determined recurrently. Substituting (50) into both sides of (51) gives

$$\sum_{n=0}^{\infty} a_n x^n = T(f(x)) + \lambda \int_a^b k(x,t) \left(\sum_{n=0}^{\infty} a_n t^n \right) dt \tag{52}$$

or for simplicity we use

$$a_0x^0 + a_1x^1 + a_2x^2 + \cdots = T(f(x)) + \lambda \int_a^b k(x,t) \left(a_0t^0 + a_1t^1 + a_2t^2 + \cdots \right) dt \tag{53}$$

where $T(f(x))$ is the Taylor series for $f(x)$. The integral equation (51) will be converted to a traditional integral in (52) or (53) where instead of integrating the unknown function $u(x)$, terms of the form t^n, $n \geq 0$ will be integrated. Notice that because we are seeking series solution, then if $f(x)$ includes elementary functions such as trigonometric functions exponential functions, etc., then Taylor expansions for functions involved in $f(x)$ should be used.

We first integrate the right side of the integral in (52) or (53), and collect the coefficients of like powers of x. We next equate the coefficients of like powers of x in both sides of the resulting equation to obtain a recurrence relation in a_j, $j \geq 0$. Solving the recurrence relation

will lead to a complete determination of the coefficients $a_j,\ j \geq 0$. Having determined the coefficients $a_j,\ j \geq 0$, the series solution follows immediately upon substituting the derived coefficients into (50). The exact solution may be obtained if such an exact solution exists. If an exact solution is not obtainable, then the obtained series can be used for numerical purposes. In this case, the more terms we evaluate, the higher accuracy level we achieve.
It is worth noting that using the series solution method for solving Fredholm integral equations gives exact solutions if the solution $u(x)$ is a polynomial. However, if the solution is any other elementary function such as $\sin x, e^x$, etc, the series method gives the exact solution after rounding few of the coefficients $a_j,\ j \geq 0$. This will be illustrated by studying the following example.

Example 2.7. *Solve the Fredholm integral equation by using the series solution method*

$$u(x) = -x^4 + \int_{-1}^{1} (xt^2 - x^2 t) u(t)\, dt \tag{54}$$

Solution:
Substituting $u(x)$ by the series

$$u(x) = \sum_{n=0}^{\infty} a_n x^n. \tag{55}$$

into both sides of Eq. (54) leads to

$$\sum_{n=0}^{\infty} a_n x^n = -x^4 + \int_{-1}^{1} (xt^2 - x^2 t) \left(\sum_{n=0}^{\infty} a_n t^n \right) dt \tag{56}$$

Evaluating the integral at the right side, and equating the coefficients of like powers of x in both sides of the resulting equation we find

$$\begin{gathered} a_0 = 0, \quad a_1 = -\tfrac{30}{133}, \quad a_2 = \tfrac{20}{133}, \quad a_3 = 0, \quad a_4 = -1 \\ a_n = 0, \qquad n \geq 5 \end{gathered} \tag{57}$$

Consequently, the exact solution is given by

$$u(x) = -\frac{30}{133} x + \frac{20}{133} x^2 - x^4. \tag{58}$$

Results

We found some results of analysis methods of the paper gives some analysis solutions related to Fredholm integral equations:

These solutions moved away from the pure theoretical side and abundant of practical examples without disturbing the scientific accuracy in order for the information to be easy to take, there is difference in the linear Fredholm integral equation and nonlinear, and we found that linear Fredholm integral equation has familiar general methods. As for nonlinear Fredholm integral equation, it does not have one form of solution and the new thing that has been added in this paper is to illustrate some examples from the analysis solutions of the nonlinear Fredholm integral equation to facilitate and clarify.

Conclusion

This paper content some analysis solution for solving Fredholm integral equations of second kind this equations has wide range of physical and engineering applications. we investegate some analysis solutions include the Adomian decomposition method, the Modified decomposition method, Noise terms phenomenon, the variational iteration method, the Direct competition method, Successive approximation method and the Series solution method, and we compare the three methods in terms of accuracy in the solution we have found that the Adomian method gives the more accurate solutions then the other two methods because the Adomian decomposition method (ADM) is an effective, convenient, and accurate method for finding analytical solutions.

References

[1] Abbass Hassan Taqi, "New Technique for Solution Linear Fredholm Integral Equation of The second Kind", *Journal of Kirkuk University-Scientific Studies* , vol.1, No.1 ,2006.

[2] Adomian, G. Solving Frontier Problem: The Decomposition Method, *Kluwer: Boston*, 1994.

[3] Adomian, G. Nonlinear Stochastic Operator Equations,*Academic Press:San Diego, CA*, 1986.

[4] Adomian, G. A review of the Decomposition Method and Some Recent Results for Nonlinear Equation,*Math.Comput. Modelling*, 13(7), pp. 17-43, 1992.

[5] Adomian, G. & Rach, R. Noise Terms in Decomposition Series Solution, *Computers Math. Appl.*,24(11), pp. 61-64, 1992.

[6] Cherruault, Y., Saccomandi, G. & Some, B., New Results for Convergence of Adomians Method Applied to Integral Equations, *Mathl. Comput. Modelling* 16(2), pp. 85-93, 1993.

[7] Cherruault, Y. & Adomian, G., Decomposition Methods: A new Proof of Convergence,*Mathl. Comput. Modelling*,18(12), pp. 103-106, 1993.

[8] Josef Dicka, Peter Kritzerb, Frances Y. Kuoc,∗, Ian H. Sloanc, " Lattice-Nystom Method for Fredholm Integral Equations of the Second Kind with Convolution Type Kernels", *Journal of Complexity*, 23, 2007.

[9] Lovitt, W.V. Linear Integral Equations, *Dover Publications, Inc.: New York*, 1950.

[10] M. Rahman, "Integral Equations and their Applications", *Library of Congress*, 2007.

[11] S. Saha Ray and P. K. Sahu, "Numerical Methods for Solving Fredholm Integral Equations of Second Kind", *Hindawi Publishing Corporation, Abstract and Applied Analysis*, Volume 2013.

[12] Wazwaz, A.M., Necessary Conditions for the Appearance of Noise Terms in Decomposition Solution Series,*Appl. Math. Comput.*, 81, 1997, 199-204.

[13] Wazwaz, A.M., Partial Differential Equations and Solitary Waves Theory, *HEP and Springer Beijing and Berlin*, 2009.

[14] Wazwaz, A.M., A First Course in Integral Equations, *World Scientific: Singapore*, 1997.

[15] http://www.internationaljournalssrg.org/ssrg-journals.html.